International Journal of Innovative
Computing, Information and Control
Volume 8, Number 5(B), May 2012

ICIC International ©2012 ISSN 1349-4198
pp. 3743–3754

LINEAR QUADRATIC OPTIMAL CONTROL BASED ON DYNAMIC COMPENSATION

Guoshan Zhang[1], Lei Liu[1] and Wanquan Liu[2]

[1]School of Electrical Engineering and Automation
Tianjin University
No. 92, Weijin Road, Tianjin 300072, P. R. China
zhanggs@tju.edu.cn; liulei_sophia@163.com

[2]Department of Computing
Curtin University
WA, 6102, Australia
w.liu@curtin.edu.au

Received October 2010; revised March 2011

Abstract. *The linear-quadratic (LQ) optimal problem based on dynamic compensation is considered for a general quadratic performance index in this paper. First, it is shown that there exists a dynamic compensator with a proper dynamic order such that the closed-loop system is asymptotically stable and its associated Lyapunov equation has a symmetric positive-definite solution. Then, the quadratic performance index is derived to be a simple expression related to the symmetric positive-definite solution and the initial value of the closed-loop system. In order to solve the optimal control problem for the system, the proposed Lyapunov equation is transformed into a Bilinear Matrix Inequality (BMI) and a corresponding path-following algorithm to minimize the quadratic performance index is proposed in which an optimal dynamic compensator can be obtained. Finally, several numerical examples are provided to demonstrate the effectiveness and feasibility of the proposed approach.*

Keywords: Optimal control, Dynamic compensation, Linear system, Path-following algorithm, Bilinear matrix inequality (BMI)

1. Introduction.

Optimal control theory has attracted considerable attention for many years owing to its comprehensive practical applications. Many fruitful results have been achieved for the linear-quadratic (LQ) optimal control problem, which is for normal linear systems [1-12] and discrete-time stochastic linear systems [13]. Most of the results are focused on state feedback control and output feedback control. For some types of systems, it is not possible or economically feasible to measure all state variables in applications, so the designer wants to control the system based on directly observed outputs of the system, like static output feedback [5] or dynamic compensation [6,7]. In [5], an algebraic necessary condition for minimizing the performance index is given and an algorithm for computing optimal feedback gain is presented. In [6], the optimal control problem by the dynamic compensator with fixed dimension is also considered. [7] has studied the design method of a dynamic compensator only associated with the derivatives of the output signals. Moreover, [8] has given a survey of static output feedback control problems and presented that the dynamic compensator can be brought back to the static output feedback case, and provided some useful methods for the research of this paper. Also some researchers have studied the relationship between state feedback control and static output feedback control, and the optimal output feedback control law is derived from the corresponding state feedback problem, see [9-12].

As we know, the LQ optimal control problem can be transformed into a problem of solving Riccati equations/inequalities, and there have been some algorithms to solve these nonlinear matrix equations/inequalities. [5,6] have given an algorithm, named Levine-Athans algorithm, for solving algebraic Riccati equation. [14-16] have introduced some iterative algorithms for solving matrix equations. [17,18] have proposed a simple matrix transformation for turning Nonlinear Matrix Inequatility (NLMI) to Linear Matrix Inequatility (LMI), while [19-22] have investigated how to convert BMI problem into an iterative LMI algorithm. However, these algorithms cannot be applied directly to solve the LQ optimal problem based on dynamic compensation. Among these algorithms, path-following algorithm [22] is verified to be much more effective since it is to linearize the BMI using a first order perturbation approximation and then iteratively compute a perturbation that "slightly" improves the controller performance by solving a semidefinite program (SDP). It is easy to implement in control theory and engineering.

Dynamic compensation plays an indispensable role in controller design, since static output feedback may not be able to stabilize the system in some cases even if the system is controllable and observable; however, such stabilization problem can be solved by dynamic compensation with same conditions. Furthermore, dynamic compensation has already been applied in H_∞ control [23], quadratic zero-sum game [24] and so on. LQ optimal control problem based on dynamic compensation, which represents a promising research area, is not well studied yet. So in this paper, we will present the LQ optimal problem for a general quadratic performance index based on dynamic compensation. First, we will show that there exists a dynamic compensator with a proper dynamic order such that the closed-loop system is asymptotically stable, and its associated Lyapunov equation has a symmetric positive-definite solution. Then, the given quadratic performance index can be derived to be a simple expression related to the solution of the Lyapunov equation and the initial value of the closed-loop system. An iterative algorithm is proposed in the light of the path-following algorithm to solve the optimization problem for the given quadratic performance index. By applying this algorithm, we can achieve an optimal dynamic compensator and the minimum value of the quadratic performance index. Finally, several numerical examples are provided to demonstrate the effectiveness and feasibility of the proposed approach.

2. Problem Statement and Preliminaries.

Consider the following linear time-invariant system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \\ x(0) = x_0 \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the input vector, $y(t) \in \mathbf{R}^r$ is the output vector, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{r\times n}$ are constant matrices. We assume that the realization $\{A, B, C\}$ is both controllable and observable.

Consider the following performance index with the linear quadratic form

$$J = \frac{1}{2}\int_0^\infty \left[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\right] \mathrm{d}t \tag{2}$$

where $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{r\times r}$ are weight matrices which are symmetric and positive-definite.

For the system (1), we consider the dynamic compensator

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c y(t) \\ u(t) = C_c x_c(t) + D_c y(t) \\ x_c(0) = x_{c0} \end{cases} \tag{3}$$

where $x_c(t) \in \mathbf{R}^l$ ($l \leq n$) is the state vector of the compensator, $A_c \in \mathbf{R}^{l\times l}$, $B_c \in \mathbf{R}^{l\times r}$, $C_c \in \mathbf{R}^{m\times l}$ and $D_c \in \mathbf{R}^{m\times r}$ are matrices of the dynamic compensator which are to be determined.

The purpose of this paper is to design dynamic compensator (3) with proper dynamic order l for the system (1) such that the closed-loop systems is asymptotically stable and the linear quadratic performance index (2) is minimized.

Remark 2.1. *The literature [6] considered the compensator (3) but the quadratic performance index is special and the given algorithm is not suitable to solve the LQ optimal problem of this paper. [7] used the performance index (2) but the dynamic compensator is only associated with the derivative of the output signals and the obtained performance index is inferior to the performance index based on state feedback. So the problem considered in this paper is more general.*

3. **Main Results.**

3.1. **Optimal control based on dynamic compensation.**

The resultant closed-loop system from system (1) and its dynamic compensator (3) is

$$\dot{\xi}(t) = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix} \xi(t) \tag{4}$$

where

$$\xi(t) = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}.$$

We define

$$\bar{A} := \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}$$

and let

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad K = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$$

and $\bar{A} = \hat{A} + \hat{B}K\hat{C}$, then the closed-loop system with a dynamic compensator of order l can be written as a static output feedback controller structure.

Here, the quadratic performance index (2) is described by

$$J = \frac{1}{2}\int_0^\infty \left[\xi^{\mathrm{T}}(t)\bar{Q}\xi(t)\right] \mathrm{d}t \tag{5}$$

where

$$\bar{Q} = \hat{Q} + \hat{C}^{\mathrm{T}}K^{\mathrm{T}}\hat{R}K\hat{C}, \quad \hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}.$$

Remark 3.1. *Applying $\hat{Q}$, $\hat{R}$, K and C to $\bar{Q}$, we obtain*

$$\bar{Q} = \begin{bmatrix} Q + C^{\mathrm{T}}D_c^{\mathrm{T}}RD_cC & C^{\mathrm{T}}D_c^{\mathrm{T}}RC_c \\ C_c^{\mathrm{T}}RD_cC & C_c^{\mathrm{T}}RC_c \end{bmatrix}.$$

Apparently, $\bar{Q}$ is symmetric and positive-semidefinite.

Lemma 3.1. [8] *Let the time-invariant system be defined as in (1), there exists dynamic compensator (3) with dynamic order $l \geq n - \max\{m, r\}$, such that the closed-loop system (4) is asymptotically stable.*

The next result characterizes linear quadratic optimal control in terms of Lyapunov equation of the closed-loop system (4).

Theorem 3.1. *Consider the time-invariant system (1), if there exists a dynamic compensator (3) with dynamic order* $l \geq n - \max\{m, r\}$ *such that the closed-loop system (4) is asymptotically stable and* $\bar{Q} > 0$. *Then the following Lyapunov Equation (6)*

$$\bar{A}^{\mathrm{T}} P + P\bar{A} + \bar{Q} = 0 \tag{6}$$

has symmetric positive-definite solution P, *and the performance index* $J = \frac{1}{2}\xi^T(0)P\xi(0)$.

Proof: From Lemma 3.1 we know that there exists dynamic compensator (3) with order $l \geq n - \max\{m, r\}$ such that the closed-loop system (4) is asymptotically stable and the Lyapunov Equation (6) has the symmetric positive-definite solution P since $\bar{A}$ is stable and $\bar{Q} > 0$. So we choose a Lyapunov function as

$$V(\xi, t) = \xi^{\mathrm{T}}(t)P\xi(t) > 0. \tag{7}$$

It is obvious that $V(\xi, t)$ is positive-definite, and the time-derivative of $V(\xi, t)$ along the solution of (4) is given by

$$\dot{V}(\xi, t) = 2\dot{\xi}^{\mathrm{T}}(t)P\xi(t) = \xi^{\mathrm{T}}\left[\bar{A}^{\mathrm{T}}P + P\bar{A}\right]\xi(t). \tag{8}$$

Then from (6),

$$\dot{V}(\xi, t) = -\xi^{\mathrm{T}}(t)\bar{Q}\xi(t) \tag{9}$$

is negative-definite. From (7) and (9), we obtain

$$\xi^{\mathrm{T}}(t)\bar{Q}\xi(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\xi^{\mathrm{T}}(t)P\xi(t)\right]. \tag{10}$$

Substituting (10) into (5),

$$\begin{aligned} J &= \frac{1}{2}\int_0^\infty \xi^{\mathrm{T}}(t)\bar{Q}\xi(t)\mathrm{d}t = -\frac{1}{2}\xi^{\mathrm{T}}(t)P\xi(t)\Big|_0^\infty \\ &= -\frac{1}{2}\xi^{\mathrm{T}}(\infty)P\xi(\infty) + \frac{1}{2}\xi^{\mathrm{T}}(0)P\xi(0). \end{aligned} \tag{11}$$

Because the poles of the closed-loop system are in the open left-half-plane, $Re[\lambda(\bar{A})] < 0$ and $\xi(\infty) \to 0$, the following (12) can be obtained.

$$J = \frac{1}{2}\xi^{\mathrm{T}}(0)P\xi(0). \tag{12}$$

The matrix P is the solution of the Lyapunov Equation (6).

In order to obtain the optimal performance index, we should solve the minimization problem described by

$$\begin{aligned} &\min J = \tfrac{1}{2}\xi^{\mathrm{T}}(0)P\xi(0), \\ &\text{s.t.} \begin{cases} \bar{A}^{\mathrm{T}}P + P\bar{A} + \bar{Q} = 0, \\ P > 0, \quad \bar{Q} > 0. \end{cases} \end{aligned} \tag{13}$$

In next section, we will give an iterative algorithm based on the path-following method [22] to solve the problem (13). The algorithm demonstrates to be promising.

3.2. **Solution of the minimization problem.** In order to solve that Lyapunov equation conveniently, we should add a small positive slack factor $\epsilon_1 > 0$ and transform that Lyapunov Equation (6) into the following inequality

$$\left|\bar{A}^{\mathrm{T}}P + P\bar{A} + \bar{Q}\right| < \epsilon_1 I \tag{14}$$

where I is an identity matrix, which has the same dimension as of $\bar{A}$, and $|X| < \epsilon_1 I$ is interpreted as $-\epsilon_1 I < X < \epsilon_1 I$.

Equation (14) represents

$$\bar{A}^{\mathrm{T}}P + P\bar{A} + \bar{Q} = M$$

where $|M| < \epsilon_1$. That is

$$\bar{A}^{\mathrm{T}}P + P\bar{A} + \bar{Q} - M = 0.$$

Let $\bar{Q}_{approx} = \bar{Q} - M$, then $\bar{Q} = \bar{Q}_{approx} + M$, the corresponding performance index is

$$2J = \int_0^\infty \xi^{\mathrm{T}}(t)\left(\bar{Q}_{approx} + M\right)\xi(t)\mathrm{d}t = \xi^{\mathrm{T}}(0)(P_{approx} + M)\xi(0) \approx \xi^{\mathrm{T}}(0)P_{approx}\xi(0).$$

So, by adding a small positive slack factor $\epsilon_1 > 0$ to transform Equation (6) into inequality (14), one can see that the performance index is approximately obtained. If ϵ_1 is small enough, then the approximation is the performance index to be expected.

As K in $\bar{A}$ and P are both unknown matrix variables, the inequality (14) is actually BMI, which cannot be solved directly by LMI. In the following, we present a path-following method for solving BMI (14). In fact, this BMI is linearized by using a perturbation approximation, and then it becomes a LMI. The detailed algorithm is as follows.

Algorithm 3.1

- Step 1: Let $j = 1$. Select an initial feedback gain K_j satisfying that $\bar{A}$ is stable.
- Step 2: Solve the following LMI problem

$$\begin{aligned} &\min J_j = \xi_0^{\mathrm{T}} P_j \xi_0, \\ &\text{s.t.} \begin{cases} |(\hat{A} + \hat{B}K_j\hat{C})^{\mathrm{T}}P_j + P_j(\hat{A} + \hat{B}K_j\hat{C}) \\ +\hat{Q} + \hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{R}K_j\hat{C}| < \epsilon_1 I, \\ P_j > 0. \end{cases} \end{aligned} \tag{15}$$

  We obtain P_j and J_j. If this LMI optimal problem has solution go to Step 3. Otherwise, go to Step 1.
- Step 3: Substituting $P_j = P_j + \delta P$, $K_j = K_j + \delta K$ into (15), one can assume that δP and δK are small and therefore by neglecting the second order terms we can obtain the following optimization problem:

$$\begin{aligned} &\min \delta J = \xi_0^{\mathrm{T}} \delta P \xi_0, \\ &\text{s.t.} \begin{cases} |\hat{A}^{\mathrm{T}}P_j + P_j\hat{A} + P_j\hat{B}K_j\hat{C} + \hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{B}^{\mathrm{T}}P_j \\ +P_j\hat{B}\delta K\hat{C} + \hat{C}^{\mathrm{T}}\delta K^{\mathrm{T}}\hat{B}^{\mathrm{T}}P_j + \delta P\hat{A} + \hat{A}^{\mathrm{T}}\delta P \\ +\delta P\hat{B}K_j\hat{C} + \hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{B}^{\mathrm{T}}\delta P + \hat{Q} \\ +\hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{R}K_j\hat{C} + \hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{R}\delta K\hat{C} \\ +\hat{C}^{\mathrm{T}}\delta K^{\mathrm{T}}\hat{R}K_j\hat{C}| < \epsilon_2 I. \end{cases} \end{aligned} \tag{16}$$

  Note that the constraints of δP and δK are

$$|\delta P| < I, \tag{17}$$

$$\delta K^{\mathrm{T}}\delta K < I. \tag{18}$$

  Suppose that ϵ_2 is a small positive scalar, then we can obtain δP and δK. If this LMI problem has a solution, go to Step 4. Otherwise, go to Step 1.
- Step 4: Let $j = j+1$, $P_j = P_{j-1} + \delta P$, $K_j = K_{j-1} + \delta K$, compute $J_j = \frac{1}{2}\xi_0^{\mathrm{T}}P_j\xi_0$. If $J_j < J_{j-1}$, $J_{j-1} - J_j > \epsilon_3$ and $j < N$ (ϵ_3 is a given small positive scalar, N is the upper bound for the iteration number), then go back to Step 2. Otherwise, stop. Then the optimal performance index is obtained.

This iterative algorithm ends until a desired performance is achieved, or the performance cannot be improved further. The choice of initial values of K_1 is important for convergence to an acceptable solution [22]. The numerical examples are shown that as long as we can find a K_1 such that the closed-loop system is asymptotically stable, we conclude that K_1

can be adjusted iteratively using the free variable δK and the optimal performance index J can be obtained accordingly.

The algorithm above can also be applied to solve the problem of optimal control based on static output feedback by solving the BMI. Compared with results in the existing literatures, the proposed method is more general. The numerical examples in next section demonstrate the effectiveness of the proposed approach.

4. **Numerical Examples.** Since the performance index is related to the initial value of the closed-loop system, we set $x_{c0} = 0$ in the following examples so as to obtain the minimal index for convenient comparisons.

Example 4.1. [4] *Consider a linear system (1) and its quadratic performance index (2) with the the following given parameters*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$R = 1$, *and the initial value of the state vectors* $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}$.

(1) We first design a state feedback controller:

$$u(t) = Kx(t) \tag{19}$$

where K is the state feedback controller gain.

Using the lqr() function in MATLAB, we can obtain the optimal controller gain $K = [-1\ -1]$ and the minimal performance index $J^* = 1$. The trajectories of the state vectors are shown in Figure 1.

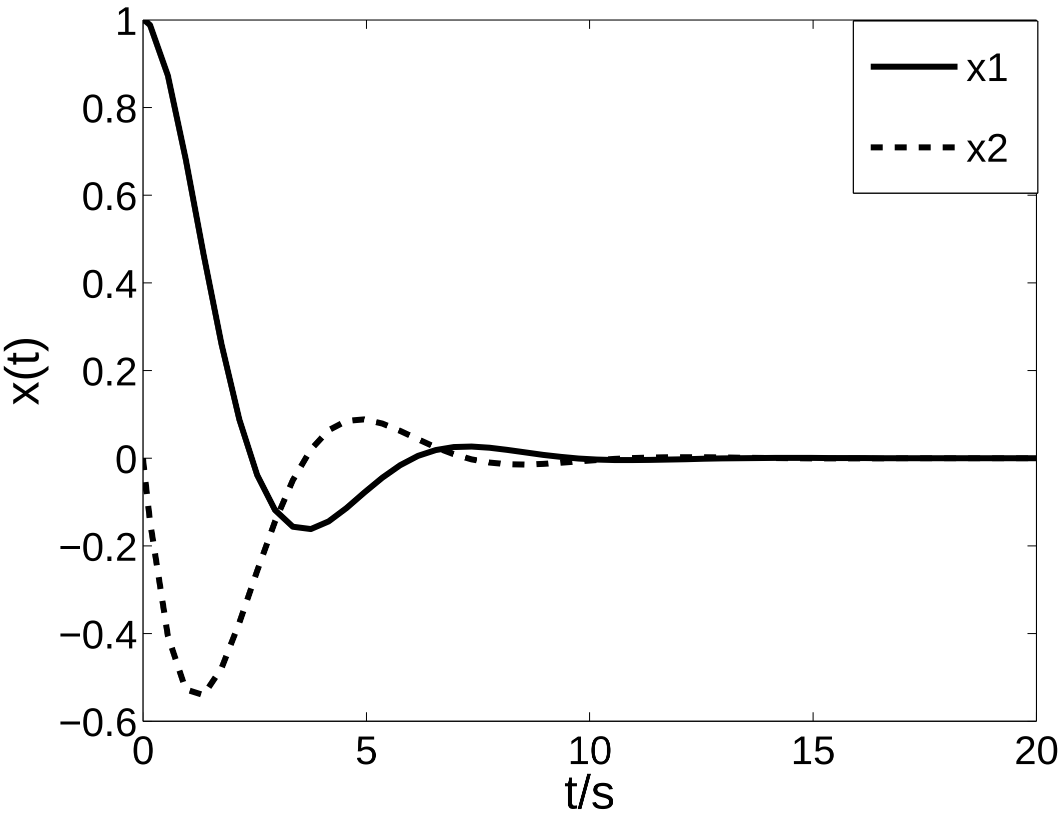

FIGURE 1. Optimal state curves of state feedback

(2) Secondly, we design a static output feedback controller:

$$u(t) = Ky(t) \tag{20}$$

where K is the static output feedback controller gain.

In [4], one can obtain the optimal controller gain $K = -0.565$ and the minimal performance index $J^* = 1.0548$. Now simulation curves of the state vectors are shown in Figure 2.

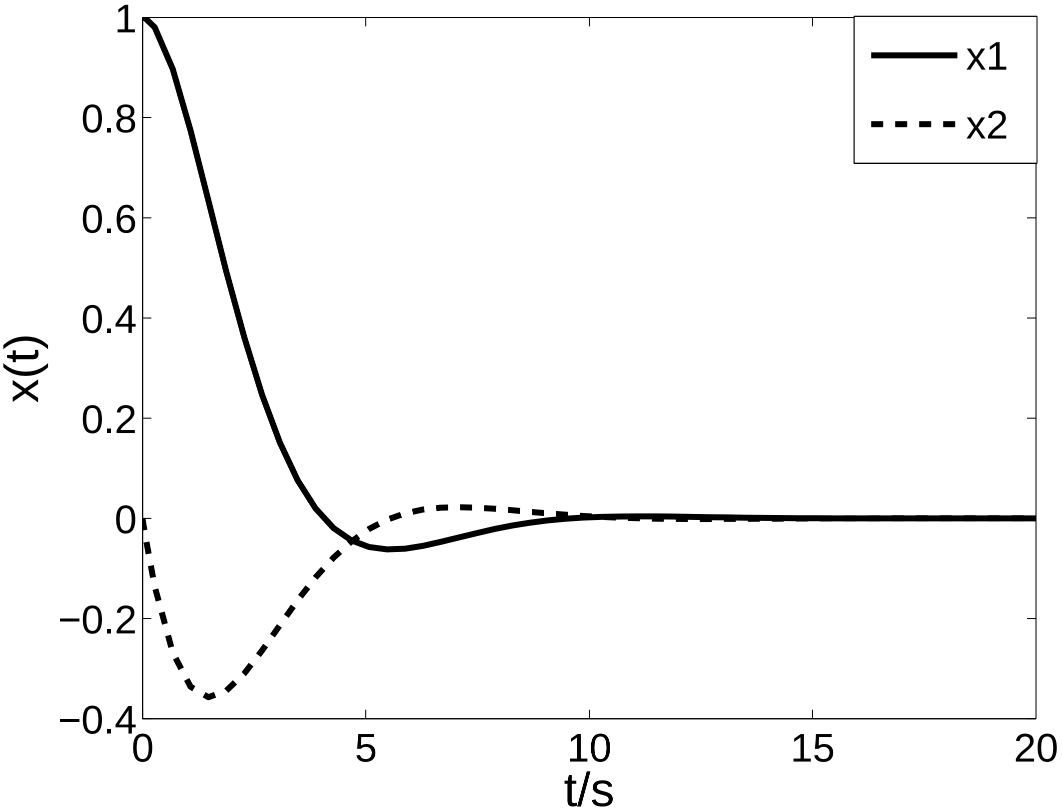

FIGURE 2. Optimal state curves of static output feedback

(3) Finally, we design a dynamic compensator (3) with first order, that is $x_c(t) \in \mathbf{R}^1$. In light of Algorithm 3.1 and MATLAB LMI Toolbox, we can obtain the optimal dynamic compensator.

Let

$$K_1 = \begin{bmatrix} -3 & 1 \\ 1 & -7 \end{bmatrix},$$

after 34 steps iteration, we can obtain the optimal controller gain

$$K = \begin{bmatrix} -1.3667 & 1.3833 \\ 3.7448 & -6.7164 \end{bmatrix}$$

and the minimal performance index $J^* = 0.8219$. Now the trajectories of the state vectors are shown in Figure 3.

Example 4.2. [10] *Consider a linear system (1) and its quadratic performance index (2) with the the following given parameters*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

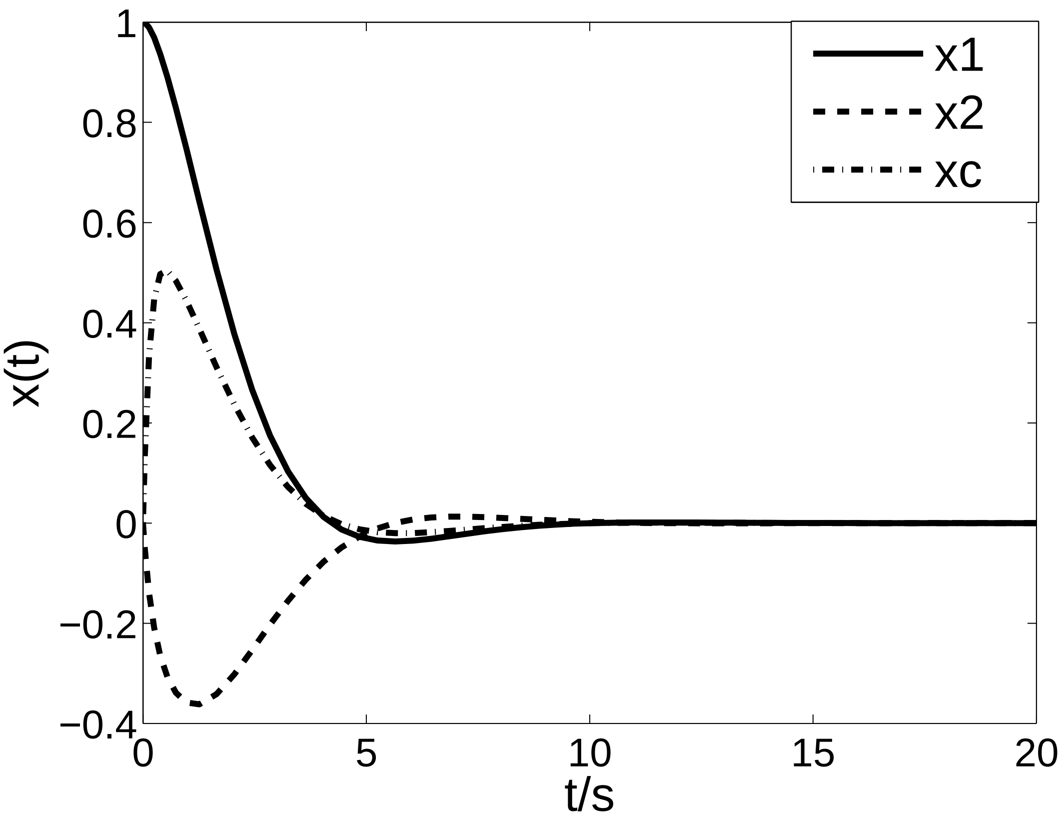

FIGURE 3. Optimal state curves of dynamic compensator

$R = 1$, *and the initial value of the state vectors is a random vector with* $E\{\xi(0)\xi(0)^{\mathrm{T}}\} = I$.

We first design a state feedback controller (19). By using lqr() function in MATLAB, we can obtain the optimal controller gain $K = [-0.4142 \ \ -1.3522]$ and the minimal performance index $J^* = 1.6323$.

[10] claimed that it can be easily verified that this system cannot even be stabilized by linear static output feedback control, and hence it does not admit a LQ optimal output feedback. Compared with [10], designing a dynamic compensator with order one, we will obtain the optimal controller gain as follows:

$$K = \begin{bmatrix} 0.1815 & 0.1446 \\ 3.9474 & -0.8421 \end{bmatrix}$$

and the minimal performance index is $J^* = 1.3670$. It is smaller than the performance index based on optimal state feedback.

Example 4.3. [11] *Consider a linear system (1) and its quadratic performance index (2) with the the following given parameters*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$R = 1$, *and the initial value of the state vectors is a random vector with* $E\{\xi(0)\xi(0)^{\mathrm{T}}\} = I$.

In [11], one can easily obtain the optimal state feedback controller gain is $K = [-0.4142 \ -1.3522]$, and the minimal performance index is $J^* = 1.6323$. Moreover, one can find an unique static output feedback gain $K = -1.1877$ such that $J = 1.9789$ is minimized.

Designing a dynamic compensator with order one, we will achieve the optimal controller gain is

$$K = \begin{bmatrix} -1.1217 & 0.0320 \\ -0.3557 & -1.0408 \end{bmatrix}$$

and the minimal performance index is $J^* = 1.4230$. It is smaller than the performance index based on state feedback case and static output feedback.

Example 4.4. *Consider a linear system (1) and a quadratic performance index (2) with the following parameters*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$R = 1$, *and the initial value of the state vectors* $x(0) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$.

(1) Similarly, we first design state feedback controller (19). By using the lqr() function in Matlab, we can obtain the optimal controller gain

$$K = [-2.4142 \ -1.8521 \ -2.1689]$$

and the minimal performance index $J^* = 12.0309$.

Although this system is controllable and observable, one can verify that this system can not be stabilized by a static output feedback as well as by the first order dynamic compensator. From [8], we know that dynamic compensator with the order $l \geq n - \max\{m, r\}$ can make the system stable theoretically, so we design dynamic compensator with order two.

(2) We then design a dynamic compensator (3). For the controller with second order, that is $x_c(t) \in \mathbf{R}^2$. In light of Algorithm 3.1 and MATLAB LMI Toolbox, we can obtain the optimal dynamic compensator.

Let

$$K_1 = \begin{bmatrix} -1.1 & 0 & -1 \\ -1 & -1 & -0.5 \\ -0.1 & -0.3 & -0.5 \end{bmatrix},$$

after 3 steps iteration, we can obtain the optimal controller gain

$$K = \begin{bmatrix} -1.2049 & -0.2782 & -1.1769 \\ -1.1769 & -0.7623 & -0.4556 \\ -0.0770 & -0.4207 & -0.7704 \end{bmatrix}$$

and the minimal performance index $J^* = 5.0144$.

Example 4.5. *Consider a linear system (1) and a quadratic performance index (2) with the following parameters*

$$A = \begin{bmatrix} -2 & -2.5 & -0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1.5 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$R = 1$, *and the initial value of the state vectors* $x(0) = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^{\mathrm{T}}$.

If $K = 0$, we can obtain the performance index $J^* = 18.4444$.

(1) We first design a state feedback controller (19). By using lqr() function in MATLAB, we can obtain the optimal controller gain

$$K = [-0.5761 \ -0.8182 \ -0.6180]$$

and the minimal performance index $J^* = 6.4438$.

(2) We then design static output feedback controller (20). By implementing Algorithm 3.1, we can obtain the optimal feedback controller by MATLAB LMI Toolbox.

Let $K_1 = -1$, after 43 steps iteration, we can obtain the optimal controller gain $K = -0.5399$ and the minimal performance index $J^* = 6.5517$.

(3) Finally, we design a dynamic compensator (3). For the controller with first order, that is $x_c(t) \in \mathbf{R}^1$. In light of Algorithm 3.1 and MATLAB LMI Toolbox, we can obtain the optimal dynamic compensator.

Let

$$K_1 = \begin{bmatrix} -0.9501 & 0.6068 \\ 0.2311 & -0.4860 \end{bmatrix},$$

after 4 steps iteration, we obtain the optimal controller gain

$$K = \begin{bmatrix} -0.5713 & 0.4997 \\ 1.2157 & -0.5463 \end{bmatrix}$$

and the minimal performance index $J^* = 2.1578$.

Proceeding in this way, for the controller with second order, we can obtain the optimal dynamic compensator.

Let

$$K_1 = \begin{bmatrix} -1.5495 & 3.0845 & 0.3377 \\ 0.0872 & -0.1524 & 1.2497 \\ -0.4150 & -0.9583 & -0.9734 \end{bmatrix},$$

after 10 steps iteration, we can obtain the optimal controller gain

$$K = \begin{bmatrix} -0.8355 & 2.6966 & 1.0080 \\ -0.0554 & -0.5650 & 2.1049 \\ 0.3623 & -0.9350 & -0.9440 \end{bmatrix}$$

and the minimal performance index $J^* = 1.6511$.

For the controller with third order, we can obtain the optimal dynamic compensator.

Let

$$K_1 = \begin{bmatrix} -1.2 & 2 & 1.5 & -0.5 \\ 0.5 & -1 & 1.24 & -0.6 \\ -0.1 & -1.4 & -0.97 & 0.01 \\ 0.1 & 2.5 & 1.5 & -1 \end{bmatrix},$$

after 8 steps iteration, we can obtain the optimal controller gain

$$K = \begin{bmatrix} -0.3958 & 1.8714 & 2.2275 & -0.4953 \\ 0.4773 & 0.3807 & 1.4127 & -0.8622 \\ -0.0614 & -0.3818 & -0.8328 & -0.0580 \\ -0.5852 & 3.0625 & 2.6635 & -1.1452 \end{bmatrix}$$

and the minimal performance index $J^* = 0.6730$.

We can also design a controller with fourth order, and the minimal performance index can be obtained, that is $J^* = 0.0811$.

Remark 4.1. *The comparisons of the performance indices under different controllers are shown in Table 1.*

Remark 4.2. *From the performance indices of feedback forms with different orders, we find that the dynamic compensator with higher order can achieve smaller performance index. So we conjecture that the performance index will approach to zero with the increase of the compensator's dynamic order if the system is controllable and observable.*

TABLE 1. Performance index comparison

Examples	Controller form	Index
Example 4.1	State Feedback	1.0000
	Static Output Feedback	1.0875
	Dynamic Compensator with order 1	0.8219
Example 4.2	State Feedback	1.6323
	Dynamic Compensator with order 1	1.3670
Example 4.3	State Feedback	1.6323
	Static Output Feedback	1.9787
	Dynamic Compensator with order 1	1.4230
Example 4.4	State Feedback	12.0309
	Dynamic Compensator with order 2	5.0144
Example 4.5	Uncontrolled	18.4444
	State Feedback	6.4438
	Static Output Feedback	6.5517
	Dynamic Compensator with order 1	2.1578
	Dynamic Compensator with order 2	1.6511
	Dynamic Compensator with order 3	0.6730
	Dynamic Compensator with order 4	0.0811

Remark 4.3. *Although the linear quadratic optimal control with higher order compensator can achieve better performance, the dynamic compensator with lower order can be implemented easily and economically. So in engineering practice, one needs to consider the trade-off between the performance index and the order of the compensator to reach a balance.*

5. **Conclusions.** In this paper, we have considered the optimal control problem in terms of dynamic compensation for linear system with a general quadratic performance index. It is shown that if there exists a dynamic compensator with proper dynamic order such that the closed-loop system is asymptotically stable, then a Lyapunov equation has a symmetric positive-definite solution and the given quadratic performance index can be expressed as a simple form. By transforming the Lyapunov equation into a BMI, a new algorithm has further been proposed in terms of path-following algorithm, then the optimal dynamic compensator and the minimum value of the quadratic performance index can be obtained by MATLAB LMI Toolbox. Finally, several numerical examples are given to show that dynamic compensators with higher order can achieve better performance by comparison of the quadratic performance indices.

Acknowledgment. The authors thank the two reviewers and the associate editor for their invaluable comments on this paper and the quality of this paper is improved significantly after revision based on these comments. Also this work was supported by the National Natural Science Foundation of China with Grant No. 60674019, 61074088.

REFERENCES

[1] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley-Interscience, USA, 1972.
[2] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*, Prentice Hall, UK, 1990.
[3] K. L. Teo, C. J. Goh and K. H. Wang, *A Unified Computational Approach to Optimal Control Problems*, Longman Scientific and Technical, USA, 1991.
[4] S. S. Hu, Z. Q. Wang and W. L. Hu, *Optimal Control Theory and Systems*, Sciencep, China, 1994.

[5] W. S. Levine and M. Athans, On the determination of the optimal constant output feedback gains for linear multivariable systems, *IEEE Transactions on Automatic Control*, vol.15, no.1, pp.44-48, 1970.
[6] W. S. Levine, T. L. Johnson and M. Athans, Optimal limited controllers state variable feedback for linear systems, *IEEE Transactions on Automatic Control*, vol.16, no.6, pp.785-793, 1971.
[7] S. S. Hu and L. X. Gao, A design method of optimal regular using dynamic output feedback, *Journal of Nanjing Aeronautical Institute*, vol.20, no.4, pp.61-68, 1988.
[8] V. L. Syrmos, C. T. Abdallah, P. Dorato and K. Grigoriadis, Static output feedback – A survey, *Automatica*, vol.33, no.3, pp.125-137, 1997.
[9] L. Huang and Z. Li, Fundamental theorem for optimal output feedback problem with quadratic performance criterion, *International Journal of Control*, vol.50, no.6, pp.2341-2347, 1989.
[10] G. X. Gu, On the existence of linear optimal control with output feedback, *SIAM Journal of Control and Optimization*, vol.28, no.3, pp.711-719, 1990.
[11] F. Gao, W. Q. Liu, V. Sreeram and K. L. Teo, Characterization and selection of global optimal output feedback gains for linear time-invariant systems, *Optimal Control Applications and Methods*, vol.21, no.5, pp.195-209, 2000.
[12] E. Jacob and W. Arie, A result on output feedback linear quadratic control, *Automatica*, vol.44, no.1, pp.265-271, 2008.
[13] V. Dragan and T. Morozan, The linear quadratic optimization problem for a class of discrete-time stochastic linear systems, *International Journal of Innovative Computing, Information and Control*, vol.4, no.9, pp.2127-2137, 2008.
[14] C. H. Guo and A. J. Laub, On the iterative solution of a class of nonsymmetric algebraic Riccati equations, *SIAM Journal on Matrix Analysis and Applications*, vol.22, no.2, pp.376-391, 2000.
[15] F. Ding and T. W. Chen, On iterative solutions of general coupled matrix equations, *SIAM Journal of Control and Optimization*, vol.44, no.6, pp.2269-2284, 2006.
[16] F. Ding, P. X. Liu and J. Ding, Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle, *Applied Mathematics and Computation*, vol.197, no.1, pp.41-50, 2008.
[17] Y. Y. Cao, J. Lam and Y. X. Sun, Static output feedback stabilization: An ILMI approach, *Automatica*, vol.34, no.12, pp.1641-1645, 1998.
[18] C. Lin, Q. G. Wang and T. H. Lee, Robust normalization and stabilization of uncertain descriptor systems with norm-bounded perturbations, *IEEE Transactions on Automatic Control*, vol.50, no.4, pp.515-520, 2005.
[19] J. G. Van Antwerp and R. D. Braatz, A tutorial on linear and bilinear matrix inequalities, *Journal of Process Control*, vol.10, no.4, pp.363-385, 2000.
[20] S. Kanev, C. Schere, M. Verhaegen and B. De Schutter, Robust output-feedback controller design via local BMI optimization, *Automatica*, vol.40, no.7, pp.1115-1127, 2004.
[21] D. B. Fernando, J. M. Ricardo and F. C. Carlos, Multivariable PID control with set-point weighting via BMI optimization, *Automatica*, vol.44, no.2, pp.472-478, 2008.
[22] A. Hassibi, J. How and S. Boyd, A path-following method for solving BMI problems in control, *Proc. of American Control Conference*, San Diego, CA, USA, pp.1385-1389, 1999.
[23] F. Long, C. L. Li and C. Z. Cui, Dynamic output feedback H_∞ control for a class of switched linear systems with exponential uncertainty, *International Journal of Innovative Computing, Information and Control*, vol.6, no.4, pp.1727-1736, 2010.
[24] X. Zhang, H. Zhang, X. Wang and Y. Luo, A new iteration approach to solve a class of finite-horizon continuous-time nonaffine nonlinear zero-sum game, *International Journal of Innovative Computing, Information and Control*, vol.7, no.2, pp.597-608, 2011.