SLIDING MODE ROBUST CONTROL FOR TWO-WHEELED MOBILE ROBOT WITH LOWER CENTER OF GRAVITY

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ABSTRACT. Two-wheeled mobile robots (2WMRs) with lower center of gravity exhibit more practicality and operability because their mass center locates on the bottom of the configuration center. By analyzing motion behaviors of the vehicle body, we present the underactuated property and give the dynamics equation of this kind of robot. The system matrices and nonholonomic constraint change drastically because of the unavoidable oscillation accompanying with underactuated vehicle body. Considering this fact, we propose a sliding mode robust controller based on the nominal models of the system matrices to realize the trajectory planning. The tracking performances can be achieved by adaptively regulating the coefficients of this control law. Simulation results verify the proposed controller is simple, practical and effective for the 2WMRs with lower gravity of center.

Keywords: 2WMRs, Lower gravity of center, Underactuated, Sliding mode robust controller

1. Introduction. The 2WMRs demonstrate more superiority to some other robots with three, four or more wheels. For example, 2WMRs can perform steering motion with zero radiuses easily; without traditional chassis and brake, their vehicle bodies become light weighted and their configurations become slim; with lower energy consumption, relative small wheel-terrain contact area assures long-time detection operation. Therefore, 2WMRs have potential application prospects in many areas especially in complex unstructured circumstances, such as seismic ruins, battle field and even planetary surface.

The most common 2WMR with mass center on the top of the configuration center, which is initially used to verify the self-balanced control techniques, is named as two-wheeled self-balanced robot \cite{1-3}. However, because the robot is naturally similar to an instable inverted pendulum system with multivariable, nonlinear and strong-coupling characteristics, it is difficult to apply this kind of robot in complicated, changeable and unstructured environment \cite{4, 5}.

In contrast, the 2WMRs, whose mass center locates on the bottom of its configuration center, have aroused attentions of some researchers \cite{6-8}. In this case, the system of this kind of robot changes into a stable one. With this structure improvement, the 2WMRs significantly enhance the autonomous capability and maneuverability, which will make the robots more suitable to explore the unstructured environments. Considering these benefits, some researchers had proposed the two-wheeled robot based on lower center of gravity structures. For example, Abbott \cite{9, 10} had designed a new 2WMR with these special characters and its control algorithms about its rolling ahead motion had been investigated; unfortunately, the steering motion had not been mentioned which is necessary for the robot’s trajectory planning. Salerno \cite{11-13} had developed a two-wheeled robot...
named Quasimoro. Note that the quasi-holonomy (QH) concept has been proposed to reduce the complex of dynamic mechanism and the control system. Deng [14,15] had also proposed a 2WMR with lower gravity center as a lunar rover, which owns high efficiency and has wide development propensity for planet exploration.

However, there are unavoidable drawbacks with the lower gravity center structure. It should been noted that the motion of the vehicle body, driven by reactionary torques of the two wheel motors, is essentially in an underactuated state. This underactuated behavior causes the position of the mass center changing drastically when the robot operates. This fact leads the robot system matrices to be more sophisticate through the strong nonlinear changeable pure rolling nonholonomic constraint, resulting in the disadvantageous oscillation of the vehicle body and the less compliance with the robot movement. Without consideration of this unexpected dynamic behavior, the robot could not realize the path-following accurately.

Many researchers devote to investigate powerful control algorithms to suppress this unexpected behavior, because eliminating oscillation by adding auxiliary devices will increase the complex mechanical structure greatly. Note that there are many control techniques [16-19] to hold the vehicle body on the unstable equilibrium state, nevertheless, they are all proposed for two-wheeled self-balance robots. Such results could not be applied to the 2WMRs with lower gravity center. Fortunately, we observe that the system matrices change slightly around the stable equilibrium point of vehicle body when the mass proportion of the vehicle body to whole robot is larger enough. Based on this equilibrium state, we introduce the normal models of the system matrices. In this study, we focus on designing the robust controller which uses sliding mode strategy to suppress the oscillation of the vehicle body based on the above-mentioned normal models.

This paper is organized as follows: section 2 presents the movement characters of the underactuated vehicle body driven by the two motors; then the system dynamic behavior of the 2WMR system considered time-vary mass center of the vehicle body are described in Section 3; subsequently Section 4 discusses the control problem and gives design technology of the sliding mode controller; In Section 5, we analyze the validity of proposed controller through simulation; finally, the concluding remarks will be addressed in the last section.

2. Dynamic Behaviors of the Vehicle Body. The vehicle body of the 2WMRs with lower gravity center is generally fixed with the motors’ stators, as well as the wheels are fastened with the motors’ rotors. According to this structure character, the vehicle body, driven by the reaction torques of the driving motors, exhibits underactuated character and appears auxiliary oscillation when the mobile robot actuates.

The motion of the vehicle body is similar to that of a pendulum, as shown in Figure 1. Let \( \tau_1, \tau_r \) represent output torques of left and right motors respectively, and the resultant torque \( \tau = \tau_1 + \tau_r \) can be considered as an equivalent torque acting on the vehicle body. Hence, the inclination of the vehicle body, denoted by the pendulum angle \( \theta \), can be obtained by

\[
ml^2\ddot{\theta} + \zeta \dot{\theta} + mgl \sin \theta = \tau
\]

where \( g \) is the acceleration of gravity; \( l \) is the displacement of the mass center of the vehicle body to the configuration center of the mobile robot; \( \zeta \) represents the viscous damping associated with the revolute joint.

Note that the parameter \( d(\theta) \), the vertical projection displacement of the mass center point \( C \) to the configuration centre \( P \), is changeable and can be expressed by

\[
d(\theta) = l \sin \theta(t)
\]
The oscillation of the vehicle body increases damage probabilities for the hardware, reduces compliance of the movement and decreases the tracking accuracy of the mobile robot. In order to suppress this oscillation, we can increase the mass proportion of the vehicle body in the mobile robot. Although this strategy could not eliminate the swing up motion absolutely, it can restrict the displacement $d(\theta)$ in a small domain. The equilibrium of the domain is stable state of the pendulum system, where we can define the normal models of the system matrices.


3.1. System parameters definition. The position of a 2WMR can be described by inertial Cartesian frame $Oxy$ as shown in Figure 2. Here, the vector $q = [x, y, \phi]^T$ is introduced to represent the whole postures of the robot on the $x - y$ plane, where $(x, y)$ are the coordinates of the mass center of the vehicle body, and $\phi$ is the orientation of the movement velocity relative to horizontal direction. Meanwhile, the configuration center $O$ of the mobile robot can be expressed by the coordinates $(x_O, y_O)$.

Suppose that the masses of vehicle body and a single wheel are represented as $m_c$ and $m_w$ respectively, and the total mass of the robot can be given by

$$m = m_c + 2m_w.$$
According to parallel axis theorem, the inertia moment of whole robot about the point \( O \) can be defined by

\[
I = I_c + 2I_w + 2m_wL^2 + m_c d^2(\theta).
\]

where, \( I_c \) is rotational inertia of the vehicle body about the vertical axis through the point \( O \); \( I_m \) is the rotational inertia of each wheel about wheel axis; \( 2L \) is the distance between the two wheels.

### 3.2. System description.

1. A typical WMRs system.

The mathematic model of a 2WMR is similar to that of the WMRs, thus we put forward the dynamics of the robot by taking account of WMRs.

A typical WMRs has a \( n \)-dimensional configuration space with generalized coordinates \( (q_1, q_2, \ldots, q_n) \) and subjects to \( m \) constraints, which can be described by Equation (3):

\[
M(q)\dot{q} + V_m(q, \dot{q})\dot{q} + G(q) + \tau_d = B(q)\tau - A^T(q)\lambda
\]

where, \( M(q) \in \mathbb{R}^{n\times n} \) is a symmetric, positive definite inertia matrix; \( V_m(q, \dot{q}) \in \mathbb{R}^{n\times n} \) is the centripetal and coriolis matrix; \( G(q) \in \mathbb{R}^n \) is the gravitational vector; \( \tau_d \) is the bounded unknown disturbances including unstructured unmodeled dynamics; \( B(q) \in \mathbb{R}^{n\times r} \) is the input transformation matrix; \( \tau \in \mathbb{R}^r \) is the input vector; \( A^T(q) \in \mathbb{R}^{m\times n} \) is the matrix associated with the constraints, and \( \lambda \in \mathbb{R}^m \) is Lagrange multiplier associated with the constraints.

All equality constraints based on kinematics are independent of time, and can be expressed as follows

\[\begin{align*}
A(q)\dot{q} &= 0
\end{align*}\]  

Then, by spanning the null space of \( A(q) \), a set of smooth and linearly independent vector field \( s_1(q), \ldots, s_{n-m}(q) \) can be obtained. If let \( S(q) = [s_1(q), \ldots, s_{n-m}(q)] \) be the full rank matrix consisting of these vectors, the following equation is governed by

\[\begin{align*}
S^T(q)A^T(q) &= 0
\end{align*}\]

According to (4) and (5), it is possible to find an auxiliary vector \( \nu(t) \in \mathbb{R}^{n-m} \) such that

\[\begin{align*}
\dot{q} &= S(q)\nu(t).
\end{align*}\]

2. Dynamic matrices.

Defining velocity vector \( v = [v, \omega]^T \), where \( v \) represents linear velocity and \( \omega \) denotes steering velocity, the relationship between the differential posture \( \dot{q} \) and the velocity vector can be governed by

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\phi}
\end{bmatrix} = S(q)v = 
\begin{bmatrix}
\cos \varphi & -d(\theta) \sin \varphi \\
\sin \varphi & d(\theta) \cos \varphi \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
v \\
\omega
\end{bmatrix}
\]

This is a typical nonholonomic constraint. Considering the dynamic model of the WMRs, the system matrices of a 2WMR with lower gravity center can be listed as

\[
M(q) = 
\begin{bmatrix}
m & 0 & 0 \\
0 & m & -md(\theta) \cos \varphi \\
md(\theta) \sin \varphi & -md(\theta) \cos \varphi & I
\end{bmatrix};
\]

\[
V_m(q, \dot{q}) = 
\begin{bmatrix}
0 & 0 & md(\theta) \dot{\varphi} \cos \varphi \\
0 & 0 & md(\theta) \dot{\varphi} \sin \varphi \\
0 & 0 & 0
\end{bmatrix};
\]
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\[ B(q) = \frac{1}{R} \begin{bmatrix} \cos \phi & \cos \phi \\ \sin \phi & \sin \phi \end{bmatrix}; \quad \tau = \begin{bmatrix} \tau_r \\ \tau_t \end{bmatrix}; \]

\[ A^T(q) = \begin{bmatrix} -\sin \phi \\ \cos \phi \\ -d(\theta) \end{bmatrix}; \quad \lambda = -m(\dot{x}_c \cos \phi + \dot{y}_c \sin \phi) \dot{\phi}. \]

It should be noted that the parameter \( d(\theta) \) in these matrices is a time-vary variable, which is different from the traditional models of WMR.


Although the nonholonomic system has been widely investigated, very few efforts about trajectory planning have been made by considering the underactuated vehicle body. Here, we also use Lagrange multiplier method to deal with the underactuated problem.

Differentiating Equation (6), substituting its result in system (3), and multiplying by \( S^T \), one can eliminate the constraint matrix \( A^T(q) \lambda \) from the dynamic system. Then, the complete equation of the mobile platform are derived as

\[ S^T M S \dot{v} + S^T (M \dot{S} + V_m S) v + S^T G + S^T \tau_d = S^T B \tau. \] (8)

where \( v(t) \in R^{n-m} \) is a new vector about practical velocity and it can decrease the system order by the matrix \( S(q) \). For clarity, one could define a series of auxiliary matrices, and the system (8) can be rewritten as follows:

\[ \tilde{M}(q) \dot{\tilde{v}} + \tilde{V}_m(q, \dot{q}) \dot{v} + \tilde{G}(q) + \tilde{\tau}_d = \tilde{\tau} \] (9)

where, \( \tilde{M}(q) = S^T M S; \tilde{V}_m(q, \dot{q}) = S^T (M \dot{S} + V_m S); \tilde{G}(q) = S^T G; \tilde{\tau}_d = S^T \tau_d; \tilde{\tau} = S^T B \tau. \)

Here, \( \tilde{M}(q) \in R^{r \times r} \) is a symmetric and positive definite inertia matrix; \( \tilde{V}_m(q, \dot{q}) \in R^{r \times r} \) is the centripetal and coriolis matrix; \( \tilde{G}(q) \in R^{r \times 1} \) is the gravitational vector; \( \tilde{\tau}_d \) represents bounded unknown disturbances including unstructured unmodeled dynamics and the outside boundary disturbance; \( \tilde{B}(q) = S^T B \in R^{r \times r} \) is an input matrix depends on the structure parameters of the mobile robot. In ref. [20], it can be known that the matrices of \( \tilde{M}(q), \tilde{V}_m(q, \dot{q}) \) and \( \tilde{\tau}_d \) are all boundedness.


4.1. Velocity controller. The control structure of the 2WMR is made up of two closed-loop subsystems: the velocity controller based on kinematics as well as the sliding mode robust controller based on dynamics. The velocity controller is to realize the tracking function. Here, we introduce a virtual robot with the desired posture \( \mathbf{q}_r = [x_r, y_r, \phi_r]^T \). If the current 2WMR is expressed as \( \mathbf{q} = [x, y, \phi]^T \), the tracking problem is just to find an appropriate controller such that \( \mathbf{q} \rightarrow \mathbf{q}_r \) as \( t \rightarrow \infty \).

For simplicity, a tracking error space \( \mathbf{q}_e = [e_1, e_2, e_3]^T \), defined between the actual and desired postures, ought to be introduce as shown in Figure 3.

Due to the geometric relation, the error states can be derived as:

\[ \mathbf{q}_e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \phi_r - \phi \end{bmatrix} \] (10)
Differentiating (10), and substituted by (7), it can be obtained that

$$\dot{q}_e = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} v_r \cos e_3 + \omega e_2 - u \\ v_r \sin e_3 - \omega e_1 \\ \omega_r - \omega \end{bmatrix}$$

where $v_r$, $\omega_r$ are the desired linear and angular velocities respectively. Equation (11) is called the steering system of the mobile robot. Most researchers considered this steering system but neglected the dynamics of the mobile robot. Using tracking error (10) and the controller (12) to the steering system (11), the system is asymptotically stable, i.e., $e_1, e_2,$ and $e_3$ go to zero as $t \to 0$ [20,21].

$$\dot{\mathcal{I}} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} v_r \cos e_3 + k_1 e_1 \\ \omega_r + k_2 v_r e_2 + k_3 v_r \sin e_3 \end{bmatrix}$$

where $k_1, k_2,$ and $k_3$ are positive constants and $v_r > 0$.

### 4.2. Sliding mode robust controller.

Let $v_c = [v, \omega]^T$ represents the desired velocity vector calculated by the Equation (12). Then, $e = v_c - v$ represents tracking error related to auxiliary states $v$ and $\omega$. Here, we construct the sliding surface as follows:

$$\mathcal{I} = [s_1 \\ s_2]^T = C e, \quad C = diag(e_1, e_2), \quad e_i > 0.$$  

Suppose that $M_0, V_{m0}, G_0, \tau_{d0}$ represent the nominal forms of $M, V_m, G, \tau_d$ respectively. So, the change ranges of the system matrices can be expressed by $\Delta M = M - M_0, \Delta V_m = V_m - V_{m0}, \Delta G = G - G_0, \Delta \tau_d = \tau_d - \tau_{d0}$. Based on these normal models, a sliding mode controller can be given by

$$\tau = M_0 \dot{v}_d + V_{m0} v_d + G_0 + \tau_{d0} + C^{-1} \Gamma sgn(\mathcal{I}) + C^{-1} \varepsilon sgn(\mathcal{I})$$

where $\varepsilon = [\varepsilon_1 \varepsilon_2]^T$, and $\varepsilon_i$ is a positive constant; $\Gamma = diag(\gamma_1, \gamma_2)$, and $\gamma_i$ satisfies the condition as follow:

$$\gamma_i > |\Delta M|_{max} |C \dot{v}_d| + |\Delta V_m|_{max} |C v| + C |\Delta G|_{max} + C |\Delta \tau_d|_{max}$$

Generally, the smaller of $\gamma_i$ causes slower speed rate, whereas the larger of $\gamma_i$ results in greater speed rate but the chattering phenomenon will increase simultaneously. Additionally, the coefficients of $\Gamma = diag(\gamma_1, \gamma_2)$ in controller (14) are adaptively changing, and the vector $\varepsilon$ is an exponent reaching term by which it can guarantee the system states reaching the slide surface with a fast rate and decrease while approaching the surface.

Based on the preliminary theory above, we can give the theorem as follows:
Theorem 4.1. The velocity tracking error $e = v_c - v$ with uncertainties and disturbances will asymptotically converge to zero, if the controller (14) is applied to system (9).

To prove this theorem, we should introduce a lemma firstly.

Lemma 4.1. The matrix $\dot{M} - 2\dot{V}_m$ is a skew symmetric.

Proof: The derivative of the inertia matrix $M$ is
\[ \dot{M} = \dot{S}^TMS + S^T\dot{M}S + S^T\dot{S}. \]

Considering $M - 2V_m$ is a skew-symmetric matrix, and substituting the $\dot{V}_m$, it can also be straightforward to show that $\dot{M} - 2\dot{V}_m$ is a skew-symmetric matrix.

The lemma has been proved.

The lemma illustrates the system (9) still has the common skew-symmetric character of the classical mobile robot which would be used to drive the controller of the 2WMR.

Proof: Supposing the Lyapunov function is
\[ V = \frac{1}{2} \dot{\Theta}^T \dot{M} \Theta \]

Clearly, $V \geq 0$. Considering the Lemma above, and substituting by Equation (14), the differentiation of the $V$ is obtained by
\[ \dot{V} = \frac{1}{2} \dot{\Theta}^T \dot{M} \Theta + \dot{\Theta}^T \dot{M} \Theta = \frac{1}{2} \dot{\Theta}^T (\dot{M} - 2\dot{V}_m) \Theta + \dot{\Theta}^T \dot{V}_m \Theta + \dot{\Theta} \Theta \]
\[ = \dot{\Theta}^T (\dot{V}_m \dot{C} \dot{e} + \dot{M} \dot{C} \dot{e}) \]
\[ = \dot{\Theta}^T (\dot{M} \dot{C} \dot{v}_d + \dot{V}_m \dot{C} \dot{v}_d - C \dot{\tau} + C \dot{\tau}_d) \]
\[ = \dot{\Theta}^T (\Delta \dot{M} \dot{C} \dot{v}_d + \Delta \dot{V}_m \dot{C} \dot{v}_d + C \Delta \dot{G} + C \Delta \dot{\tau}_d - \Gamma \text{sgn}(\Theta) - \varepsilon \text{sgn}(\Theta)) \]
\[ \leq -\Gamma |\Theta| - \varepsilon |\Theta| < 0 \]

Obviously, $V$ is a Lyapunov function, and the Theorem is proved.

In order to reduce the chattering phenomenon, the method used in this case is to utilize the saturation function to replace the sign one. Thus, replacing $\text{sgn}(\Theta)$ by $\text{sat}(\Theta, \eta)$ in (14) implies that
\[ \tau = \bar{M}_0 \dot{v}_d + \bar{V}_m \dot{v}_d + \bar{G}_0 + \bar{\tau}_d + \bar{\tau}_d \]
\[ \text{sat}(\Theta, \eta) = [\text{sat}(s_1, \eta)\text{sat}(s_2, \eta)]^T, \text{and its concrete definition is as follows} \]
\[ \text{sat}(s, \eta) = \begin{cases} \text{sgn}(s), & |s| > \eta > 0 \\ \frac{s}{\eta}, & |s| \leq \eta \end{cases}; \quad i = 1, 2. \]

Here, $\eta$ is a small positive constant.

5. Simulation Results. To verify the effectiveness of the proposed controller, we choose Matlab platform to perform simulation study. The parameters of the mobile robot can be set as $L = 0.25m$, $R = 0.2m$, $m_c = 5Kg$, $m_w = 3Kg$, $I_m = 0.55Kgm^2$, $l = 0.15m$, and $[x_0 \ y_0 \ \phi_0]^T = [1.50 \ 0 \ 0]^T$ is the initial position. The auxiliary variables of the reference trajectory that the 2WMR should track is given by $v_r = 2m/s$, $\omega_r = 1rad/s$. Thereby we can derive the desired states as $x_r = 2 \cos t$, $y_r = 2 \sin t$. In this case, suppose that the $\tau_d$ is considered as a bound-limited white noise whose power and sample time are 0.01W and 0.5sec. Owing to the limited scope of the inclination angle of the vehicle body, the gravitational vector $G(q)$ is also limited to a small domain; therefore, we consider gravitational vector $G(q)$ as a periodical part of the disturbance term and it can be simulated as
a sinusoid with an amplitude of 0.01 m and a frequency of 1 rad/s. Meanwhile, the coefficients of the proposed control algorithms are supposed as $k_1=15$, $k_2 = k_3=20$; $c_1 = c_2=5$; $\varepsilon_1 = \varepsilon_2=2$, due to the anticipant control performances.

The tracking results are shown in Figure 4. The results demonstrate that after a transient adjusting time near initial point, the practical mobile robot could track the plan trajectory promptly.

![Figure 4. Practical trajectory to a desired circle in $x - y$ plane](image)

The results in Figure 5 illustrate the changeable curves of the system states $q = [x \ y \ \phi]^T$, and the solid and dash lines represent the practical and desired paths of the robot respectively. It can be seen that by the control effect of the proposed sliding mode robust controller, the tracking errors will be converged to zero rapidly although there are somewhat chattering behaviors on the sliding surface. In this case, we adopt saturation function to decrease these oscillation phenomena and the $\eta = 0.02$ in $sat(3, \eta)$ can be further regulated according to the actual requirement of the system.

![Figure 5. The tracking curves of each state of the 2WMR](image)

Figure 6 exhibits the tracking error of every state $q_e = [e_1 \ e_2 \ e_3]^T$ that is used in velocity controller (12). Note that this tracking error vector is different from $q - q_r$, but it can reflect the tracking effects in another way.

Therefore, one can conclude that by the proposed control approach, the tracking issue of the robot can be solved successfully, and this control strategy is also suitable for any other lower gravity mobile robot similarly.
6. **Conclusions.** The vehicle body of the 2WMR with lower center of gravity, whose mass center of the vehicle body below the wheel axis, exhibits oscillation phenomena due to the underactuated character. With normal model of the system matrices, we proposed a sliding mode controller to suppress the swing up behavior, assure the tracking accuracy and enhance the robustness of this kind of mobile robot, all of which have been illustrated and verified by the simulation results.

Although the normal model used in the proposed controller simplifies the control algorithm and occupies less calculated amount, it still introduces somewhat conservativeness for the controlling system. On the other hand, the chattering phenomena could not be eliminated completely because of the existence of the switching function. Both of the problems restrict the application of the controller; however, we can alleviate such disadvantages by regulating the control parameters. If the control performances are not required strictly, the sliding mode robust control is effective and powerful for the 2WMR. Further work will devote to the development of adaptive robust controller to solve the present drawbacks and verify the effectiveness of the derived control strategies by experiments.

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