GENERALIZED EXPECTED VALUE MODEL BASED ON COMPOUND QUANTIFICATION AND ITS APPLICATION IN TRANSPORTATION PROBLEMS

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Received December 2009; revised April 2010

ABSTRACT. As a kind of particular programming, transportation problem draws much attention in many fields such as energy development, materials management, etc. In this paper, by analyzing the essence of stochastic programming and the deficiencies of existing methods, we propose a comparison method based on synthesizing effect for the standard to judge an objective value is good or not, and give the axiom system of stochastic synthesizing effect function, further, we introduce a quasi-linear pattern based on expectation and variance for the satisfaction of random constraints. Moreover, using synthesizing effect and quasi-linear pattern, we establish a stochastic programming pattern (generalized expected value model, and denoted by GEM for short) with good operability, and for the stochastic transportation problem, we also establish its corresponding generalized expected value model. Finally, we analyze the performance of GEM by combining with a transportation case under random environment. All these indicate that GEM includes the existing methods, and it can effectively solve the stochastic transportation problem under complex environment or with incomplete information, in addition, GEM can merge decision consciousness into solution through quantitative way.

Keywords: Stochastic programming, Chance-constrained, Synthesizing effect, Generalized expected value model, Reliability coefficient, Transportation problem

1. Introduction. Transportation problem, a typical linear programming, plays a key role in logistics activity, so reasonable transporting process has practical significance to the overall planning and management of the whole logistic activities. There have been many perfect methods (such as the tabular method) for numerical transportation problems, however, there is no general solution to non-numerical transportation problems.

In actual, the transportation problem, with objective uncertainties (production, market, transportation conditions and so on) and subjective uncertainties (the judging of products, the evaluating of social benefits and so on), is a optimization problem of complex system. Therefore, with the difference of strategy of processing uncertain information, the delivery schemes usually have some differences, sometimes greater. At present, stochastic mathematics theory and fuzzy set theory can often be used as processing uncertain information. Stochastic methods are suitable for the uncertainty caused by inadequate objective conditions, and fuzzy methods are suitable for the uncertainty caused by the
difference of subjective understanding. Therefore, uncertain transportation problems can be divided into stochastic and fuzzy two kinds.

For uncertain programming, many scholars have discussed the solution methods, for example, [1] proposed a two-stage genetic algorithm for solving the transportation problem with fuzzy demands and supplies; [2] proposed several non-dominated solutions by considering preference of arcs in a transportation route, flexibility of demand and supply quantities; [3] gave a mathematical model for transportation problem with uncertain demand by intervals; [4] considered the multi-objective transportation problem, whose cost factors, the supply and demand are all fuzzy numbers; [5,6] proposed fuzzy objective programming methods for stochastic transportation problems; [7] established the expected value model, chance-constrained programming and dependent-chance programming of transportation problems by using uncertain programming thoughts; [8] for the random assignment problem, established risk critical value models based on individual efficiency and objective benefit; [9] proposed the concept of synthesizing effect function, and established a general solution mode based on synthesizing effect function for fuzzy programming; for uncertain optimization problem, [10] used the interval-valued fuzzy sets to get a new method to solve the fuzzy linear programming.

Since the stochastic transportation problem is a kind of particular stochastic programming, and its solution is closely related to that of stochastic programming problem. At present, there are three basic methods for the stochastic programming problem: 1) Expected value model. Its basic idea is to convert the stochastic programming into an ordinary one by using expectation to describe random variable centrally. 2) Chance-constrained programming (see [11]). Its basic idea is to convert stochastic constraints and objective functions into ordinary ones through some reliability principles. 3) Dependent-chance programming (see [12]), its basic idea is to give the optimization decision by maximizing the chance of stochastic event under uncertain environment. Though the three methods have achieved good application results (for example, [13] established the expected value model on Sugeno measure space, [14] considered the optimal sizing of batteries of distributed power system by using the chance-constrained model, [15] established the bus dispatching model based on dependent-chance goal programming, etc.), they could not effectively solve stochastic programming problems under complicated environment, the deficiencies are listed as follows: 1) when the randomness tends to be greater, the expectation could not effectively represent and describe the random variable, then the reliability of expected value model cannot be guaranteed; 2) when the characteristic of randomness is too complex (that is, the distribution of the random variable is difficult to obtain), the computation complexity will become too high to establish feasible solution scheme for chance-constrained and dependent-chance programming.

Based on the above discussions, in this paper, on basic of analyzing the deficiencies of existing stochastic programming methods and combining the characteristic of transportation problems under random environment, the main jobs are as follows: 1) for random information, propose a description way based on compound quantification strategy and a kind of ordering method based on synthesizing effect; 2) give a simplified strategy by analyzing the essence of constrains of chance-constrained programming; 3) establish a generalized expected value model (GEM) for stochastic programming based on 1) and 2); 4) establish GEM of stochastic transportation problem, and then combing a specific transportation example in random environment, further analyze characteristics of GEM and its efficiency applying in transportation problems.

In what follows, for the random variable $\xi$ and event $A$ on probability space $(\Omega, \mathcal{F}, \text{Pr})$, let $E(\xi)$ and $D(\xi)$ be the mathematical expectation and variance of $\xi$, respectively, and $\text{Pr}(A)$ the probability of event $A$. 

2.1. Overview. Stochastic programming is the key point in many actual problems such as production planning and resources distributing, its general form can be stated as follows:

\[
\begin{align*}
&\text{max}(\text{min}) \ f(x, \xi), \\
&\text{subject to} \ g_j(x, \xi) \leq 0, \quad j = 1, 2, \ldots, m.
\end{align*}
\]

(1)

Here, \(x = (x_1, x_2, \ldots, x_n)\) is the decision variable, \(\xi = (\xi_1, \xi_2, \ldots, \xi_n)\) is a given random variable on space \((\Omega, \mathcal{B}, \text{Pr})\), \(f(x, \xi)\) and \(g_j(x, \xi)\) are random variable functions.

Because random variables could not be directly ordered, and \(g_j(x, \xi) \leq 0\) usually could not be completely satisfied, (1) is just a formal model and cannot be solved directly. Thus, it is necessary to transform the stochastic programming problem to a solvable one in some strategy. In the following, we will recall several common programming methods.

2.1.1. Expected value model. Mathematical expectation, with good theoretical properties, is a common way to centrally describe the possible values of a random variable. If we use the expected value to represent random variable approximately, then model (1) can be converted into the following one:

\[
\begin{align*}
&\text{max}(\text{min}) \ E(f(x, \xi)), \\
&\text{subject to} \ E(g_j(x, \xi)) \leq 0, \quad j = 1, 2, \ldots, m.
\end{align*}
\]

(2)

Generally, we call model (2) expected value model. When the variance of the random variable is much greater, the expected value will not describe the characteristics of variable effectively, so the optimal solution obtained by model (2) often cannot satisfy the requirement of decision.

2.1.2. Chance-constrained programming. The constraints of stochastic programming usually can not be satisfied absolutely, if we use some satisfaction degree (reliability) to deal with the constraints and objective functions, then model (1) can be converted into following one:

\[
\begin{align*}
&\text{max}(\text{min}) \ \tilde{f}(x), \\
&\text{subject to} \ \text{Pr}(f(x, \xi) \geq (\leq)\tilde{f}(x)) \geq \alpha, \\
&\quad \text{Pr}(g_j(x, \xi) \leq 0) \geq \beta_j, \quad j = 1, 2, \ldots, m.
\end{align*}
\]

(3)

Generally, we call model (3) chance-constrained programming model. Here, \(\beta_j, \alpha \in [0, 1]\) represents the reliability for the constraints and objective functions, respectively. Compared with model (2), model (3) has the ability of controlling decision quality beforehand, however, it still has two shortages as follows: 1) the computation complexity is higher; 2) it can’t forecast what range the \(\beta_j\) take value in to make (3) has solutions, that is, if \(\beta_j\) is not appropriate, (3) probably has not solutions.

2.1.3. Dependent-chance programming. For the resources allocation under random environment, the following model (4) can be established through maximizing the probability of dependent stochastic event:

\[
\begin{align*}
&\text{max} \ \text{Pr}(h_k(x, \xi) \leq 0, \ k = 1, 2, \ldots, q), \\
&\text{subject to} \ g_j(x, \xi) \leq 0, \quad j = 1, 2, \ldots, p.
\end{align*}
\]

(4)

Generally, (4) is called as dependent-chance programming. Here, \(\xi\) denotes given random environment, \(g_j(x, \xi) \leq 0\) are constrains related to \(\xi\), \(h_k(x, \xi) \leq 0\) are events related to \(\xi\). This model has good interpretability, but when the distribution of \(\xi\) is not known, it is difficult to realize the computation of \(\text{Pr}(h_k(x, \xi) \leq 0)\).
The above discussions show that existing stochastic programming models have their own limitations, in which establishing an operable method for processing objectives and constraints is the core to solve stochastic programming problem. For the evaluation of objective function value in the decision process, not only will we consider the size of objective function value, but also its uncertainty. Therefore, we can systematically consider the size and uncertainties of the objective function value by some way, for example, we can evaluate the objective function value by a certain synthesizing effect value. When processing constraints, we should try our best to simplify the decision method. Based on the above analysis, in the following, we separately give the synthesizing effect pattern of objective function and processing method of constraints.

2.2. The axiomatic system for random synthesizing effect function. Considering that the maximization decision problems can be transformed into the minimization ones, and the expectation and variance separately centrally describes the size and uncertainty of stochastic variable from different ways (that is, \((E(\xi), D(\xi))\) is a kind of compound quantification value of \(\xi\), \(E(\xi)\) is a main index describing the size of the value of \(\xi\) on the whole, \(D(\xi)\) is a secondary index reflecting whether \(E(\xi)\) can approximately represents \(\xi\), \(E(\xi)\) and \(D(\xi)\) complement and restrict each other). For simplicity, in the following, we mainly discuss the synthesizing effect strategy of expectation and variation on the minimization stochastic programming problem. According to the essential characteristic of the optimal decision, we should obey the following rules in seeking for decision scheme.

**Principle 1** When the expectation values are same, the smaller the variance is, the better the effect is;

**Principle 2** When the variances are same, the smaller the expectation is, the better the effect is.

If we abstractly regard the synthesizing effect strategy of expectation and variance as a function \(S(u, v)\) (here, \(u\) expresses the expectation value with the range of interval \(\Theta\), \(v\) expresses the variance with the range of interval \([0, +\infty)\)), then \(S(u, v)\) is a mapping from \(\Theta \times [0, +\infty)\) to \((-\infty, +\infty)\) satisfying:

**Condition 1** For any given \(u \in \Theta\), \(S(u, v)\) is monotone decreasing on \(v\);

**Condition 2** For any given \(v\), \(S(u, v)\) is monotone decreasing on \(u\);

**Condition 3** \(S(u, 0)\) is strictly monotone decreasing.

Obviously, Conditions 1 and 2 that must be obeyed separately corresponds to Principles 1 and 2, and Condition 3 shows that when the variance is 0 (that is, the uncertainty of constraints do not exist), objective effect value only depends on objective function value. For convenience, we call \(S(u, v)\) satisfying the above Conditions 1 \sim 3 stochastic synthesizing effect function on \(\Theta\).

By \(S(u, v)\), the random variable \(\xi\) can be centrally quantized a synthesizing effect value \(S(E(\xi), D(\xi))\) that reflects the size of \(\xi\), the bigger the \(S(E(\xi), D(\xi))\) is, the better the performance of \(\xi\) is. So \(\min f(x, \xi)\) in (1) can be converted into \(\max S(E(f(x, \xi)), D(f(x, \xi)))\).

According to above definition, it is not difficult to prove the following conclusions.

1) For any given \(a, b \in (-\infty, +\infty)\), \(a < b\), \(k\), \(\alpha, \beta \in [0, +\infty)\), both \(S_1(u, v) = (b - u)(b - a)^{-1}(1 + kv\alpha)^{-1}\) and \(S_2(u, v) = (a - u)(b - a)^{-1}(\beta + kv\alpha)^{-1}\) are stochastic synthesizing effect functions on \([a, b]\).

2) For any given \(b, k, \alpha \in [0, +\infty)\), \(\beta \in (0, +\infty)\), both \(S_3(u, v) = -(u + b)^{\beta}(1 + kv\alpha)\) and \(S_4(u, v) = [(1 + u)(1 + kv\alpha)]^{-1}\) are stochastic synthesizing effect functions on \([0, +\infty)\).

2.3. Quasi-linear model of chance-constrained. The core of the chance-constrained programming is to transform the random constraints into ordinary ones through a given reliability. Though it establishes a method of processing uncertain constraints, there exists
higher complexity degree so that it can’t realize the solution. In this section, we mainly discuss the simplification of chance-constraints.

For chance-constraint $\Pr(g(x, \xi) \leq 0) \geq \beta$, by

$$\Pr(g(x, \xi) \leq 0) \geq \beta \Leftrightarrow \Pr\left(\frac{g(x, \xi) - E(g(x, \xi))}{\sqrt{D(g(x, \xi))}} \leq \frac{-E(g(x, \xi))}{\sqrt{D(g(x, \xi))}}\right) \geq \beta, \quad (5)$$

we can know, if we set $\psi = [g(x, \xi) - E(g(x, \xi))]/\sqrt{D(g(x, \xi))}$ and let $\psi_\beta$ denote the $\beta$-quantile of $\psi$ (that is, $\Pr(\psi \leq \psi_\beta) = \beta$, $\Pr(\psi > \psi_\beta) = 1 - \beta$), then (5) can be stated

$$\Pr(g(x, \xi) \leq 0) \geq \beta \Leftrightarrow \frac{-E(g(x, \xi))}{\sqrt{D(g(x, \xi))}} \geq \psi_\beta \Leftrightarrow E(g(x, \xi)) + \psi_\beta \sqrt{D(g(x, \xi))} \leq 0. \quad (6)$$

By $E(\psi) = 0$, $D(\psi) = 1$, we know that $\psi_\beta$ is the standard $\beta$-quantile of $g(x, \xi)$. Usually, the value of $\psi_\beta$ has regularity (Table 1 lists the standard quantiles of common distributions). If we regard $\psi_\beta$ as constant $C$, then $\Pr(g(x, \xi) \leq 0) \geq \beta$ can be simplified as

$$E(g(x, \xi)) + C\sqrt{D(g(x, \xi))} \leq 0. \quad (7)$$

Obviously, the computation complexity of (7) is far lower than $\Pr(g(x, \xi) \leq 0) \geq \beta$, so using (7) to replace $\Pr(g(x, \xi) \leq 0) \geq \beta$ will greatly improve the operability of (3).

In Table 1, $N(\mu, \sigma^2)$ denotes the normal distribution with parameters $\mu$ and $\sigma^2$, $U(a, b)$ is the uniform distribution with parameters $a$ and $b$, $Exp(\lambda)$ is the exponential distribution with parameter $\lambda$, $t(n)$ is the $t$-distribution with freedom degree $n$, $X^2(n)$ is the $X^2$-distribution with freedom degree $n$, and $F(m, n)$ is the $F$-distribution with freedom degree $m$ and $n$.

**Remark 2.1.** The above discussion shows that constant $C$ in (7) is a parameter reflecting the satisfaction degree of constraint. The larger (smaller) $C$ is, the higher (lower) the attention degree of constraints will be.

**Remark 2.2.** From Table 1, we know that: 1) the standard $\beta$-quantile of normal distribution and $t$-distribution are almost the same; 2) besides $F$-distribution, the standard 0.9-quantiles of other distributions are almost the same; 3) when $\beta$ takes values from 0.5 to 0.75, the standard $\beta$-quantile has significant differences for different distributions; 4) the standard 0.95-quantile for all those distributions does not exceed 2; 5) the standard 0.75-quantiles of normal distribution, $t$-distribution and $X^2$-distribution are all between 0.5 and 0.7, and that of $F$-distribution with $m < n$ and exponential distribution are between 0.35 and 0.45, and that of uniform distribution is significantly larger than that of other distributions; 6) when $\psi_\beta$, that is $C$, takes values from 0.87 to 1.12, whatever distribution, its reliability $\beta$ will not be lower than 0.75.

The above analysis indicates that, in condition of exact distribution of $g(x, \xi)$ is unknown: 1) when the distribution characteristics of $g(x, \xi)$ can not be determined, decision-maker can give $C$ according to Remark 2.2, and then get the rough reliability of $g(x, \xi) \leq 0$; 2) if we select $C$ combining with the distribution characteristics of $g(x, \xi)$, the decision will be more exact, for details please see Section 2.5. In the following, to emphasize the role of $C$, we call $C$ quasi-linear reliability coefficient (reliability coefficient in short) of constraint $g(x, \xi) \leq 0$.

### 2.4. Generalized expected value model.

Combing with the above discussions, if both the objective function and the constraints contain random variables, we will use synthesizing effect function to describe the objective function, and satisfaction of constrains can
Table 1. The standard $\beta$-quantiles of several common distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Expectation</th>
<th>Variance</th>
<th>Parameter values</th>
<th>$\beta$</th>
<th>$\psi_\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(\mu, \sigma^2)$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
<td>$\mu, \sigma^2$</td>
<td>0.75</td>
<td>0.68</td>
</tr>
<tr>
<td>$U(a, b)$</td>
<td>$a + b$</td>
<td>$(a + b)^2$</td>
<td>$a, b$</td>
<td>0.75</td>
<td>0.87</td>
</tr>
<tr>
<td>$Exp(\lambda)$</td>
<td>$\lambda^{-1}$</td>
<td>$\lambda^{-2}$</td>
<td>$\lambda$</td>
<td>0.75</td>
<td>0.39</td>
</tr>
<tr>
<td>$t(n)$</td>
<td>0</td>
<td>$n/(n - 2)$</td>
<td>$n = 8$</td>
<td>0.75</td>
<td>0.61</td>
</tr>
<tr>
<td>$\chi^2(n)$</td>
<td>$n$</td>
<td>$2n$</td>
<td>$n = 8$</td>
<td>0.75</td>
<td>0.55</td>
</tr>
<tr>
<td>$F(m, n)$</td>
<td>$\frac{n}{n - 2}$</td>
<td>$\frac{2n^2(m + n - 2)}{m(n - 2)(n - 4)(n &gt; 4)}$</td>
<td>$m = 4$</td>
<td>0.95</td>
<td>1.86</td>
</tr>
</tbody>
</table>

be described as in (7). Then model (1) can be converted into the following one:

$$\left\{ \begin{array}{l}
\max \ S(E(f(x, \xi)), D(f(x, \xi))) , \\
\text{s.t.} \ E(g_j(x, \xi)) + C_j \sqrt{D(g_j(x, \xi))} \leq 0, \quad j = 1, 2, \cdots, m .
\end{array} \right. \quad (8)$$

**Remark 2.3.** We call (8) generalized expected value model, which is denoted by GEM. Obviously, when $S(u, v) = -u$, $C_j \equiv 0$, (8) is the expected value model (2); when $f(x, \xi)$ and $g_j(x, \xi)$ have no randomness (that is, $D(f(x, \xi)) = 0, D(g_j(x, \xi)) = 0$), (8) has the same solution as corresponding ordinary programming; when $f(x, \xi)$ or $g_j(x, \xi)$ has randomness, the optimal scheme by (8) varies with $S(u, v)$ and $C_j$, thus how to systematically select $S(u, v)$ and $C_j$ is the key of (8).
2.5. The determination strategy of reliability coefficient. Combining with the discussion in Section 2.3, we can determine the reliability coefficient $C$ of $g(x, \xi) \leq 0$ according to the following steps:

**Step 1** Determine the rough probability distribution characteristics of $g(x, \xi)$ through certain way such as theoretical analysis or statistical analysis.

**Step 2** Combine with the probability distribution characteristics of $g(x, \xi)$, Table 1 and the risk demand for decision process to determine the reliability coefficient $C$. Here, Figure 1 gives several common probability density curves.

3.1. Formalized description for transportation problems. The general transportation problem can be expressed in general ways: For a given goods, there are $m$ sources $A_1, A_2, \ldots, A_m$ with outputs $a_1, a_2, \ldots, a_m$, respectively, and $n$ destinations $B_1, B_2, \ldots, B_n$ with sales $b_1, b_2, \ldots, b_n$, and the freight of unit goods from $A_i$ to $B_j$ is $c_{ij}$, try to determine the delivery scheme such that the total freight is the lowest.

If $x_{ij}$ represents the number of goods delivered from source $A_i$ to destination $B_j$, then the mathematical model of transportation problem can be expressed as:

$$
\begin{align*}
\min z &= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}, \\
\text{s.t.} \sum_{j=1}^{n} x_{ij} &\leq a_i, \quad i = 1, 2, \ldots, m, \\
\sum_{i=1}^{m} x_{ij} &\geq b_j, \quad j = 1, 2, \ldots, n, \\
x_{ij} &\geq 0, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n.
\end{align*}
$$

(9)

For model (9), when $a_i$, $b_j$ and $c_{ij}$ are real variables, we call (9) numerical transportation problem, and if they have some uncertainty we call (9) uncertain transportation problem. In particular, when $a_i$, $b_j$ and $c_{ij}$ have randomness (fuzziness), we call model (9) stochastic (fuzzy) transportation problem. For the numerical transportation problem, the optimal delivery scheme could be obtained by the tabular method. For the uncertain one, there does not exist order relationship among uncertain information, therefore, (9) is only a formal model, and cannot be solved directly. In what follows, we only focus on the stochastic transportation problem.

3.2. Generalized expected value model for stochastic transportation problem. Using model (8), we can convert (9) into the following one:

$$
\begin{align*}
\max z^* &= S(E(z), D(z)), \\
\text{s.t.} \sum_{j=1}^{n} x_{ij} - E(a_i) + C^{(1)}_i \sqrt{D(a_i)} &\leq 0, \quad i = 1, 2, \ldots, m, \\
- \sum_{i=1}^{m} x_{ij} + E(b_j) + C^{(2)}_j \sqrt{D(b_j)} &\leq 0, \quad j = 1, 2, \ldots, n, \\
x_{ij} &\geq 0, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n.
\end{align*}
$$

(10)

Remark 3.1. $z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$, $C^{(1)}_i$, $C^{(2)}_j$ respectively denote the reliability coefficients of supply constrains and demand constrains, and we call (10) GEM of stochastic transportation problem. Model (10) takes into account both size and uncertainty of freight, and when the outputs and sales are random variables, however, their exact distributions are unknown, we can determine reliability coefficients $C^{(1)}_i$, $C^{(2)}_j$ according to Section 2.5, moreover, model (10) is an ordinary numerical programming problem, which can be solved by conventional methods.
Table 2. Output, sales and freight in every source and destination

<table>
<thead>
<tr>
<th></th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>B4</th>
<th>output</th>
</tr>
</thead>
<tbody>
<tr>
<td>sales</td>
<td>N(2,1)</td>
<td>U(1,3)</td>
<td>U(6,10)</td>
<td>N(9,4)</td>
<td>N(3,2)</td>
<td>N(4,3)</td>
<td>N(7,3)</td>
<td>13</td>
</tr>
<tr>
<td>sales</td>
<td>N(10,1)</td>
<td>N(4,12)</td>
<td>N(4,6)</td>
<td>N(8,12)</td>
<td>N(5,5)</td>
<td>N(3,1)</td>
<td>U(4,6)</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 3. Demand for each destination in recent 20 weeks

<table>
<thead>
<tr>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>B4</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>18</td>
<td>19</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

4. Case Analysis. In this part, combining a transportation problem with random demand and freight, we consider how to establish a generalized expected value model.

Example 4.1. Given goods with three sources A1, A2, A3 and four destinations B1, B2, B3, B4. The corresponding outputs, sales and freights are shown in Table 2. Here, 1) Since there are many uncontrollable factors in transportation (such as weather forms, traffic size, the change of transit routing, adding to urgent jobs, etc.), thus, the freight $c_{ij}$ has uncertainty to some extent, and we suppose $c_{ij}$ is a random variable with a certain random distribution and mutually independent; 2) Because the sale environment is uncertain, the demands $\xi_1, \xi_2, \xi_3, \xi_4$ are random, Table 3 gives the demands of each destinations in recent 20 weeks. Try to determine the optimal delivery scheme with lowest total freight.

Obviously, this is a stochastic transportation problem, however, the exact distribution of $\xi_1, \xi_2, \xi_3, \xi_4$ is unknown, thus the problem can not be directly solved by using model (2) − (4), in the following, we consider the transportation problem through model (10). If we regard the values in Table 3 as a group of sample values of $\xi_1, \xi_2, \xi_3, \xi_4$, and their probability density histograms are shown as Figure 2, and their approximations of mathematical expectation and standard deviation are $E(\xi_1) \approx 4, \sqrt{D(\xi_1)} \approx 1.414; E(\xi_2) \approx 6, \sqrt{D(\xi_2)} \approx 1.265; E(\xi_3) \approx 4.95, \sqrt{D(\xi_3)} \approx 1.285; E(\xi_4) \approx 8.25, \sqrt{D(\xi_4)} \approx 2.407$.

Refer to common probability density curves in Figure 1, the rough probability distribution of $\xi_1, \xi_2, \xi_3, \xi_4$ in Figure 2 are uniform distribution, normal distribution, normal distribution and $\chi^2$-distribution, respectively, then we can determine the values of $C$ combining Table 1 and decision risk demand. For example, if the decision-maker demands that the satisfaction for $B_3$ should be not less than 0.95, and the reliability of the other three destinations be not less than 0.75, then the values of $C_i$ can be determined as follows: $C_1 = 0.75, C_2 = 0.8, C_3 = 1.65, C_4 = 0.9$; if the decision-maker demands that the satisfaction for $B_1$ and $B_3$ should be not less than 0.95, and the reliability of other destinations be not less than 0.75, then we can use $C_1 = 1.56, C_2 = 0.8, C_3 = 1.65, C_4 = 0.9$.

If we take

$$S(u, v) = -u \left( 1 + \left[ H((\sqrt{v} - b)/k) \right]^a \right)$$

(11)

as the synthesizing effect function to process objective (here, $\alpha \in [1, +\infty), k, b \in (0, +\infty), H(x)$ satisfies: 1) $H(x) = 0$ for each $x < 0$; 2) $H(x) = x$ for each $x \geq 0$), then we can
establish the mathematical model of this transportation problem by the model (10):

\[
\begin{align*}
& \text{max } z^* = -E(z) \left( 1 + [H((\sqrt{D(z)} - b)/k)]^\alpha \right), \\
& \text{s.t. } x_{11} + x_{12} + x_{13} + x_{14} - 13 \leq 0, \\
& \quad x_{21} + x_{22} + x_{23} + x_{24} - 9 \leq 0, \\
& \quad x_{31} + x_{32} + x_{33} + x_{34} - 11 \leq 0, \\
& \quad x_{11} + x_{21} + x_{31} - 4 - 1.414C_1 \geq 0, \\
& \quad x_{12} + x_{22} + x_{32} - 6 - 1.265C_2 \geq 0, \\
& \quad x_{13} + x_{23} + x_{33} - 4.95 - 1.285C_3 \geq 0, \\
& \quad x_{14} + x_{24} + x_{34} - 8.25 - 2.407C_4 \geq 0, \\
& \quad x_{ij} \geq 0, \quad i = 1, 2, 3, \quad j = 1, 2, 3, 4.
\end{align*}
\]

(12)

Here, \( z = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \), \( E(z) = 2x_{11} + 9x_{12} + 10x_{13} + 7x_{14} + 2x_{21} + 3x_{22} + 4x_{23} + 5x_{24} + 8x_{31} + 4x_{32} + 3x_{33} + 5x_{34} \), \( D(z) = x_{11}^2 + 4x_{12}^2 + 4x_{13}^2/3 + 3x_{14}^2 + x_{21}^2/3 + 2x_{22}^2 + 4x_{23}^2 + 5x_{24}^2 + 4x_{31}^2/3 + 3x_{32}^2 + x_{33}^2 + x_{34}^2/3 \).

Obviously, the bigger the synthesizing effect value of (12) shows that the better the solution scheme is. Model (11) is a kind of synthesizing effect pattern with punishment.
Table 4. The computation results for \( b = 12, k = 10 \) and EVM

<table>
<thead>
<tr>
<th>EVM</th>
<th>( C_j )</th>
<th>Optimal solution</th>
<th>E.V.</th>
<th>( E(z) )</th>
<th>( \sigma(z) )</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( 0 )</td>
<td>((3.82, 0, 0, 0.18, 6, 0, 2.42, 0, 0, 4.95, 5.83))</td>
<td>---</td>
<td>82.10</td>
<td>12.31</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>( 0.4 )</td>
<td>((3.31, 0, 0, 0.69, 6, 0, 2.2, 0, 0, 4.93, 6.05))</td>
<td>--21</td>
<td>82.10</td>
<td>12.00</td>
<td>2</td>
</tr>
<tr>
<td>( \alpha = 2 )</td>
<td>( 0.8 )</td>
<td>((4.56, 0, 0, 1.04, 6.50, 0.68, 1.81, 0, 0, 4.78, 6.21))</td>
<td>--93</td>
<td>93.14</td>
<td>12.76</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>( 1.2 )</td>
<td>((5.13, 0, 0, 2.87, 0.6, 2.13, 1.35, 0, 0, 4.62, 5.56))</td>
<td>--108</td>
<td>106.59</td>
<td>13.39</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>( 1.6 )</td>
<td>((5.69, 0.95, 0.02, 3.74, 0.56, 1.87, 1.50, 0, 0, 4.59, 5.47))</td>
<td>--128</td>
<td>123.13</td>
<td>14.15</td>
<td>5</td>
</tr>
<tr>
<td>( \alpha = 3 )</td>
<td>( 0 )</td>
<td>((3.23, 0, 0, 0.77, 6, 0, 2.23, 0, 0, 4.95, 6.02))</td>
<td>--82</td>
<td>82.10</td>
<td>12.00</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>( 0.4 )</td>
<td>((4.56, 0, 0, 1.04, 6.50, 0.249, 0, 0, 5.46, 5.53))</td>
<td>--92</td>
<td>92.46</td>
<td>13.41</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>( 0.8 )</td>
<td>((5.13, 0, 0, 2.88, 0.7, 0.1, 0.48, 1.51, 0, 0, 5.50, 5.50))</td>
<td>--106</td>
<td>104.90</td>
<td>14.21</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>( 1.2 )</td>
<td>((5.69, 0.22, 0, 4.49, 0.63, 1.36, 1.32, 0, 0.99, 5.12, 4.88))</td>
<td>--122</td>
<td>119.62</td>
<td>15.04</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>( 1.6 )</td>
<td>((6.26, 1.32, 0.51, 4.72, 0.579, 1.75, 1.46, 0, 0.92, 4.74, 5.34))</td>
<td>--144</td>
<td>138.86</td>
<td>15.42</td>
<td>11</td>
</tr>
<tr>
<td>( \alpha = 4 )</td>
<td>( 0 )</td>
<td>((3.23, 0, 0, 0.77, 6, 0, 2.23, 0, 0, 4.95, 6.02))</td>
<td>--82</td>
<td>82.10</td>
<td>12.00</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>( 0.4 )</td>
<td>((4.56, 0, 0, 1.04, 6.50, 0.249, 0, 0, 5.46, 5.53))</td>
<td>--92</td>
<td>92.46</td>
<td>13.41</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>( 0.8 )</td>
<td>((5.13, 0, 0, 2.88, 0.7, 0.1, 0.48, 1.51, 0, 0, 5.50, 5.50))</td>
<td>--106</td>
<td>104.90</td>
<td>14.21</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>( 1.2 )</td>
<td>((5.69, 0.22, 0, 4.49, 0.63, 1.36, 1.32, 0, 0.99, 5.12, 4.88))</td>
<td>--122</td>
<td>119.62</td>
<td>15.04</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>( 1.6 )</td>
<td>((6.26, 1.32, 0.51, 4.72, 0.579, 1.75, 1.46, 0, 0.92, 4.74, 5.34))</td>
<td>--144</td>
<td>138.86</td>
<td>15.42</td>
<td>16</td>
</tr>
</tbody>
</table>

Table 5. The computation results for \( b = 16, 20, k = 10 \)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Optimal solution</th>
<th>E.V.</th>
<th>( E(z) )</th>
<th>( \sigma(z) )</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_j )</td>
<td>( 0.4 )</td>
<td>((4.56, 0, 0, 1.04, 6.351, 0, 2.49, 0, 0, 5.26, 5.54))</td>
<td>--92</td>
<td>92.46</td>
<td>13.41</td>
</tr>
<tr>
<td>( \alpha = 1, b=16 )</td>
<td>( 1 )</td>
<td>((5.13, 0, 0, 2.88, 0.7, 0.1, 0.48, 1.51, 0, 0, 5.50, 5.50))</td>
<td>--104</td>
<td>104.42</td>
<td>14.60</td>
</tr>
<tr>
<td>( \alpha = 1, b=20 )</td>
<td>( 0.4 )</td>
<td>((4.56, 0, 0, 1.04, 6.51, 0, 2.49, 0, 0, 5.46, 5.54))</td>
<td>--92</td>
<td>92.46</td>
<td>13.41</td>
</tr>
<tr>
<td>( \alpha = 1, b=20 )</td>
<td>( 0.8 )</td>
<td>((5.13, 0, 0, 2.88, 0.7, 0.1, 0.48, 1.51, 0, 0, 5.50, 5.50))</td>
<td>--104</td>
<td>104.42</td>
<td>14.60</td>
</tr>
<tr>
<td>( \alpha = 1, b=20 )</td>
<td>( 1.2 )</td>
<td>((5.13, 0, 0, 2.88, 0.7, 0.1, 0.48, 1.51, 0, 0, 5.50, 5.50))</td>
<td>--104</td>
<td>104.42</td>
<td>14.60</td>
</tr>
<tr>
<td>( \alpha = 1, b=20 )</td>
<td>( 1.6 )</td>
<td>((5.13, 0, 0, 2.88, 0.7, 0.1, 0.48, 1.51, 0, 0, 5.50, 5.50))</td>
<td>--104</td>
<td>104.42</td>
<td>14.60</td>
</tr>
<tr>
<td>( \alpha = 1, b=20 )</td>
<td>( 1.6 )</td>
<td>((5.69, 0, 0, 4.71, 0, 0, 0.48, 0, 0, 6.49, 4.51))</td>
<td>--116</td>
<td>116.38</td>
<td>14.49</td>
</tr>
<tr>
<td>( \alpha = 2, b=20 )</td>
<td>( 0.4 )</td>
<td>((5.69, 0, 0, 4.71, 0, 0, 0.48, 0, 0, 6.49, 4.51))</td>
<td>--116</td>
<td>116.38</td>
<td>14.49</td>
</tr>
</tbody>
</table>

A feature for objective function. Here, 1) \( b \) is a threshold parameter that can be considered as completely acceptable uncertainty of decision (that is, standard deviation of objective function) during decision process, this parameter reflects the endurable capacity to uncertainty from macro facet, the larger (smaller) the \( b \), the higher (lower) the capacity. 2) \( k \) is a kind of flexible parameter to process uncertainty, it can be intuitively understood as the basic amount that the uncertainty of decision-making exceeds the threshold \( b \). 3) \( \alpha \) is a parameter, based on \( b \) and \( k \), reflecting the emphasis on the uncertainty. 4) When \( \sqrt{v} > b + k \), \( S(u, v) \) will decrease with regard to \( u \) in exponential form, it indicates that there is no the optimal solution of (12) when \( \sqrt{D(z)} > b + k \).

Based on above analysis, the transportation problems is solved and tested using expected value model and model (12) with different parameters. The results are shown as Tables 4 – 9, here, \( j = 1, 2, 3, 4 \), \( \sigma(z) = \sqrt{D(z)} \), E.V. expresses the effect value, and EVM represents expected value model.

The above discussions and results indicate that: 1) model (10) has good interpretability and adaptability, and its computational complexity is much lower than that of chance-constrained programming and dependent-chance programming; 2) model (10) can solve the transportation problem, in which the random variable of constraints with unknown
exact probability distribution, however, models (3) and (4) only solve the problem with exact probability distribution; 3) from Case 2, we can see that model (10) includes expected value model, and its standard deviation $\sqrt{D(z)}$ of objective function is smaller than that of Case 1, which shows the superiority of (10); 4) the change of reliability coefficient $C$ has evident effect on optimal decision, the effect value decreases with the increasing of $C$, which shows that rising demand to constraints will increase freight (such as Cases 2–6, Cases 7–11, Cases 12–16); 5) when the uncertainty threshold $b$ is smaller (that is, the basic requirement to uncertainty of objective function is higher), corresponding decision will change with $\alpha$ in condition of the same $C$ (that is, the power of punishment varies), such as Cases 5, 10, 15 and Cases 6, 11, 16, however, when $b$ is larger, the result of decision-making mainly depends on reliability coefficient $C$ (such as Table 5), so the decision-maker can select parameters flexibly according to own condition and competitors’ situation.

5. Conclusions. In this paper, by analyzing the characteristics and deficiencies of existing stochastic programming methods, and for stochastic transportation problems we have the following contributions: 1) Take mathematical expectation and variance as compound quantitative description, quasi-linear pattern of random constrains and random comparison pattern based on synthesizing effect are established. 2) We propose the axiom system of stochastic synthesizing effect function, and then give an operable stochastic programming model GEM, further apply it in stochastic transportation problems, establishing GEM of stochastic transportation problems. 3) Combing with a transportation example that sale and transportation environment exists randomness, the performance of GEM is discussed from different levels. The results indicate that GEM can effectively solve the stochastic transportation problem under complex environment or with unknown random distributions. This model not only covers existing relevant discussion, but also has many advantages, such as simpler operability, better explanatory, and lower computational complexity, etc. And it can effectively incorporate uncertainty into the decision, so the discussions in this paper enriches the existing stochastic programming theory and methods, laying the foundation for further establishing optimization methods under complex environment.

Acknowledgment. This research is supported by the National Natural Science Foundation of China (71071049, 70871036).

REFERENCES


