APPLICATION OF GENETIC ALGORITHM ON OBSERVER-BASED D-STABILITY CONTROL FOR DISCRETE MULTIPLE TIME-DELAY SINGULARLY PERTURBATION SYSTEMS

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ABSTRACT. This study proposes a Genetic Algorithm (GA) application for the observer-based controller design for discrete multiple time-delay, singularly perturbed systems. The corresponding slow and fast subsystems of the original system are first determined. The GA then derives the observer-based controllers for the D-stability of the slow and the fast subsystems, and a composite observer-based controller for the original system synthesized from the designed subsystems controllers. This study proposes a stability condition dependent upon the singular perturbation parameter $\varepsilon$, to guarantee the stability of the original system under the composite observer-based controller. This study finds the stability criteria of the original uncontrolled system by establishing the stability criteria for its corresponding slow and fast subsystems. If any of the criteria conditions is satisfied, this study uses the condition to find the upper bound $\varepsilon^*$ of $\varepsilon$ and can guarantee the stability of the original system by examining the stability of corresponding subsystems, if $\varepsilon \in [0, \varepsilon^*)$. Finally, an illustrative example demonstrates the efficiency of the proposed controller.

Keywords: Genetic algorithm, Composite observer-based controller, Multiple time-delay singularly perturbation systems, D-stability

1. Introduction. The existence of small time-constants always increases the order of the dynamical system and makes the analysis and control of the system complicated. Some research techniques exist to resolve this problem. The most popular and effective is the singular perturbation method, of which characters are time-scale separation and reduction in order. For controller design by the singular perturbation method, divide the original system into two lower order subsystems called the slow and fast subsystems, and a controller then designed for each subsystem. This study obtains the full-order controller by combining the two reduced-order controllers. This simplifies the procedure of controller design and the stability of system can be inferred from the stabilities of the reduced-order subsystems (Kokotovic et al. [1], Khalil [2], Hsiao et al. [3,4]). Recently, Yao et al. explored an important class of stochastic singularly perturbed systems [5-7]. The controller design in typical singular perturbation methods applies traditional optimal control laws, and always needs to solve Riccati equations. The Riccati equations are complicated and difficult to solve and system processing is slow.
Genetic algorithm (GA) a parallel research method is an optimization technique inspired by the laws of natural selection [8,9]. GA has found application resolving many optimization problems, such as the traveling salesman problem, scheduling problems, transportation problems, image processing, etc. [10-14]. An advantage of using GA to solve optimization problems is that it is not necessary to solve complex equations. Hence, in this paper, GA is applied to resolve the observer-based controller design problem of discrete multiple time-delay singularly perturbation systems. It is reasonable to assume there will be a considerable reduction in the complexity of observer-based controller design.

For GA application in optimization problems, represent each candidate solution for the problem as a chromosome of one individual and evaluate it using a fitness function. GA randomly generates the chromosomes, then by means of a natural selection process, the crossover of genes between individuals and random gene mutation, the algorithm finds optimal solutions. In this paper, GA designs the observer-based D-stability controller. Combining the subsystem controllers creates the composite controller for the original full-order system.

This paper is structured as follows. Section 2 describes the discrete multiple time-delay singularly perturbation systems and corresponding subsystems. Section 3 describes the derivation of the observer-based subsystem controllers. Section 4 derives the composite observer-based controller designs for the original systems. Section 5 proposes an observer-based controller design algorithm, based on the results from previous sections. Section 6 provides simulation results and demonstrates the ability of GA to solve this kind of control problem.

2. Problem Formulation. The following equations describe a class of C-mode [15] discrete singular perturbed system with multiple time delays [3]:

\[
x(k + 1) = \sum_{i=0}^{n} A(\varepsilon, i)x(k - i) + Bu(k) \\
y(k) = Cx(k)
\] (2.1)

where \(x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \) and \(A(\varepsilon, i) = \begin{bmatrix} A_{11} & \varepsilon \hat{A}_{11} \\ A_{21} & \varepsilon \hat{A}_{21} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^T \) are matrices with appropriate dimensions. This study obtains system (2.1) from the sampled data control of a singular perturbed continuous time system [16]. Such systems often occur naturally. Examples of these systems include communication systems, nuclear systems, flight control systems and power systems [5]. The small positive scalar \(\varepsilon\) is a singular perturbation parameter and normally exists in various physical systems. For examples, in biochemical models, it may indicate a small quantity of an enzyme. In nuclear reactor models, it is due to fast neutrons.

**Definition 2.1.** A feedback control system is \(D(\alpha, r)\)-stable if all poles of the system are within a specific radius \(D(\alpha, r)\) centered on \((\alpha, 0)\) with radius \(r\). In other words, the solutions of its characteristic equation satisfy

\[|(z - \alpha)/r| < 1\] (2.2)

in which \(r > 0\) and \(|\alpha| + r < 1\).

Our aim is to design an observer-based controller so that the system (2.1) is \(D(\alpha, r)\)-stable. An adapted genetic algorithm can provide the required controller gains. Unlike the work reported in paper [3], in which the time-delay system, reformulated as a system without time-delay, produces a massive increase in the dimension of the system.
matrices, observer-based controller derives directly from the time-delay system (2.1) with no increase in the dimension of the system matrices. Moreover, considering the slow and fast subsystems actually provides a reduction in complexity and enhances in performance. This is possible because the system matrices of slow and fast subsystems are much smaller than matrices of the full-order system (2.1). According to the quasi-steady-state approach [17,18], the observer-based controller design for the slow and fast subsystems of the original system (2.1) can then be derived [3].

3. GA on the Observer-Based Controller Design of the Subsystems. The literature contains some observer-based controller designs for stabilizing discrete two-time-scale systems [18,19]. This section outlines the separate design of the observer-based controllers for the slow and fast subsystems, such that both subsystems are stable.

3.1. Controller design for the slow subsystem. This study rewrites the slow subsystem of the original system (2.1) as [3]:

\[
\begin{align*}
x_s(k+1) &= \sum_{i=0}^{n} A_{si} x_s(k-i) + B_s u_s(k) \quad (3.1a) \\
y_s(k) &= \sum_{i=0}^{n} C_{si} x_s(k-i) + D_s u_s(k) \quad (3.1b)
\end{align*}
\]

where

\[
\begin{align*}
A_{si} &= A_{1i} + \varepsilon \left( \sum_{j=0}^{n} \hat{A}_{1j} \right) \left( I - \varepsilon \sum_{j=0}^{n} \hat{A}_{2j} \right)^{-1} A_{2i}, \\
B_s &= B_1 + \varepsilon \left( \sum_{i=0}^{n} \hat{A}_{1i} \right) \left( I - \varepsilon \sum_{i=0}^{n} \hat{A}_{2i} \right)^{-1} B_2, \\
C_{si} &= C_2 \left( I - \varepsilon \sum_{j=0}^{n} \hat{A}_{2j} \right)^{-1} A_{2i}, \\
D_s &= C_2 \left( I - \varepsilon \sum_{i=0}^{n} \hat{A}_{2i} \right)^{-1} B_2.
\end{align*}
\]

**Assumption 3.1.** The slow subsystem (3.1) is stabilizable and detectable.

The observer-based controller for the slow subsystem (3.1) is:

\[
\begin{align*}
\hat{x}_s(k+1) &= \sum_{i=0}^{n} A_{si} \hat{x}_s(k-i) + B_s u_s(k) + F_s \left[ y_s(k) - \hat{y}_s(k) \right], \\
\hat{y}_s(k) &= \sum_{i=0}^{n} C_{si} \hat{x}_s(k-i) + D_s u_s(k) \quad (3.2) \\
u_s(k) &= -\sum_{i=0}^{n} K_{si} \hat{x}_s(k-i)
\end{align*}
\]

where the matrices \( K_{si} \) and \( F_s \) are controller gains. Let the observer error \( e_s(k) = x_s(k) - \hat{x}_s(k) \), have

\[
\begin{align*}
x_s(k+1) &= \sum_{i=0}^{n} (A_{si} - B_s K_{si}) x_s(k-i) + B_s \sum_{i=0}^{n} K_{si} e_s(k-i) \quad (3.3a) \\
e_s(k+1) &= x_s(k+1) - \hat{x}_s(k+1) = \sum_{i=0}^{n} (A_{si} - F_s C_{si}) e_s(k-i) \quad (3.3b)
\end{align*}
\]
Rewrite (3.3a) and (3.3b) in matrix form as below:

\[
\begin{bmatrix}
x_s(k+1) \\
e_s(k+1)
\end{bmatrix} = \sum_{i=0}^{n} \begin{bmatrix}
A_{si} - B_sK_{si} & B_sK_{si} \\
0 & A_{si} - F_sC_{si}
\end{bmatrix} \begin{bmatrix}
x_s(k-i) \\
e_s(k-i)
\end{bmatrix},
\]

or \( \tilde{X}_s(k+1) = \sum_{i=0}^{n} M_{si} \tilde{X}_s(k-i) \) where \( \tilde{X}_s(k) = \begin{bmatrix} x_s(k) \\ e_s(k) \end{bmatrix}, M_{si} = \begin{bmatrix} A_{si} - B_sK_{si} & B_sK_{si} \\
0 & A_{si} - F_sC_{si} \end{bmatrix} \).

Taking the \( z \)-transform of the slow subsystem (3.3), have

\[
\begin{bmatrix}
x_s(z) \\
e_s(z)
\end{bmatrix} = \begin{bmatrix} c_s(z) & c_s(z) \left( B_s \sum_{i=0}^{n} K_{si} z^{-i} \right) o_s(z) \\
0 & 0
\end{bmatrix} \begin{bmatrix} z \tilde{x}_s(0) \\
z \tilde{e}_s(0)
\end{bmatrix}
\]

where

\[
o_s(z) = \left[ zI - \sum_{i=0}^{n} (A_{si} - F_sC_{si}) z^{-i} \right]^{-1}
= \left[ \left( I - f(z) o_{a0}^{-1}(z) \right) o_{a0}(z) \right]^{-1}
= o_{a0}^{-1}(z) \left[ \left( I - f(z) o_{a0}^{-1}(z) \right) \right]^{-1}
= o_{a0}^{-1}(z) \left[ \left( I - \Delta_{oa}(z) \right) \right]^{-1}
\]

with \( \Delta_{oa}(z) = f(z) o_{a0}^{-1}(z), f(z) = \sum_{i=0}^{n} (A_{si} - F_sC_{si}) z^{-i}, o_{a0}(z) = zI - (A_{a0} - F_sC_{a0}) \),

\[
c_s(z) = \left[ zI - \sum_{i=0}^{n} (A_{si} - B_sK_{si}) z^{-i} \right]^{-1} = c_{a0}^{-1}(z) \left( I - \Delta_{ks}(z) \right)^{-1} \text{ with } \Delta_{ks}(z) = k_s(z) c_{a0}^{-1}(z),
\]

\[
k_s(z) = \sum_{i=0}^{n} (A_{si} - B_sK_{si}) z^{-i}, c_{a0}(z) = \left[ zI - (A_{a0} - B_sK_{a0}) \right].
\]

**Lemma 3.1.** [20] For any matrix \( M \in R^{nxm}, \) if \( \rho[M] < 1, \) then \( |\det(I + M)| > 0. \)

**Lemma 3.2.** [3] Suppose that a singular perturbation parameter \( \varepsilon \) satisfying \( \varepsilon \sum_{i=0}^{n} \|A_{2i}\| < 1, \) then the matrix \( \left( I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i} \right) \) is nonsingular.

For the \( D(\alpha, r)\)-stability of the closed-loop slow subsystems (3.1) and (3.2), select \( F_s \) such that \( o_{a0}^{-1}(z) \) is \( D(\alpha, r) \)-stable and

\[
|\det[I - \Delta_{oa}(z)]| > 0, \quad \forall \left| \frac{z - \alpha}{r} \right| \geq 1 \tag{3.5a}
\]

to guarantee that all poles of \( o_a(z) \) are within the disk \( D(\alpha, r) \) select \( K_{a0} \) such that \( c_{a0}^{-1}(z) \) is \( D(\alpha, r) \)-stable, and

\[
|\det[I - \Delta_{ks}(z)]| > 0, \quad \forall \left| \frac{z - \alpha}{r} \right| \geq 1 \tag{3.5b}
\]

to guarantee that all poles of \( [I - \Delta_{ks}(z)]^{-1} \) are within the disk \( D(\alpha, r) \).

**Lemma 3.3 (Maximum Modulus Theorem [21]).** If \( f(z) \) is analytic in a bounded domain \( D \) and continuous in \( D \) (i.e., the closure of \( D \)), then \( |f(z)| \) takes its maximum on the boundary of \( D \).
Theorem 3.1. Choose $F_s$ so that $o_{s0}^{-1}(z)$ is $D(\alpha, r)$-stable and $\rho \left[ \hat{\Delta}_{os}(e^{j\theta}) \right] < 1$, $\forall \theta \in [0, 2\pi)$.

Moreover, choose $K_{si}$ so that $c_{s0}^{-1}(z)$ is $D(\alpha, r)$-stable and $\rho \left[ \hat{\Delta}_{ks}(e^{j\theta}) \right] < 1$, $\forall \theta \in [0, 2\pi)$. The closed-loop systems (3.1) and (3.2) is $D(\alpha, r)$-stable.

Proof: Let $z_d = \frac{z_{d0}}{r}$, if $|\text{det} [I - \Delta_{os}(z_d)] | > 0$, $\forall |z_d| \geq 1$ and $|\text{det} [I - \Delta_{ks}(z_d)] | > 0$, $\forall |z_d| \geq 1$, then the system is $D(\alpha, r)$-stable. Let $z_d = \frac{z}{r}$ and by the definition of $\Delta_{os}(z)$ and $\Delta_{ks}(z)$, then $[I - \Delta_{os}(z_d)]$ and $[I - \Delta_{ks}(z_d)]$ can be rewritten as $[I - f(z^{-1})o_{s0}^{-1}(z^{-1})]$ becomes $[I - k_s(z^{-1})c_{s0}^{-1}(z^{-1})]$.

Moreover,

$$[I - f(z^{-1})o_{s0}^{-1}(z^{-1})] = \left[ I - \left( \sum_{i=1}^{n} (A_{si} - F_sC_{si})z^i \right) \right] (z^{-1}I - (A_{s0} - F_sC_{s0}))^{-1}$$

$$= [I - \hat{\Delta}_{os}(\frac{z}{r})]$$

$$[I - k_s(z^{-1})c_{s0}^{-1}(z^{-1})] = \left[ I - \left( \sum_{i=1}^{n} (A_{si} - B_sK_{si})z^i \right) \right] (z^{-1}I - (A_{s0} - B_sK_{s0}))^{-1}$$

$$= [I - \hat{\Delta}_{ks}(\frac{z}{r})].$$

Therefore, the examination of (3.4a) and (3.4b) is equivalent to investigating the following inequalities:

$$|\text{det} [I - \hat{\Delta}_{os}(\frac{z}{r})] | > 0, \forall |\frac{z}{r}| \leq 1 \quad (3.6a)$$

$$|\text{det} [I - \hat{\Delta}_{ks}(\frac{z}{r})] | > 0, \forall |\frac{z}{r}| \leq 1 \quad (3.6b)$$

According to Lemma 3.1, (3.6a) and (3.6b) are satisfied if $\rho[\hat{\Delta}_{os}(\frac{z}{r})] < 1$, $\forall |\frac{z}{r}| \leq 1$ and $\rho[\hat{\Delta}_{ks}(\frac{z}{r})] < 1$, $\forall |\frac{z}{r}| \leq 1$. Since $\rho[\hat{\Delta}_{os}(\frac{z}{r})]$ and $\rho[\hat{\Delta}_{ks}(\frac{z}{r})]$ are closed, bounded and analytic on the domain $|\frac{z}{r}| \leq 1$, $\rho[\hat{\Delta}_{os}(\frac{z}{r})]$ and $\rho[\hat{\Delta}_{ks}(\frac{z}{r})]$ will take a maximum on the boundary of $|\frac{z}{r}| \leq 1$ according to Lemma 3.3. Therefore, $\rho[\hat{\Delta}_{os}(\frac{z}{r})] < 1$ and $\rho[\hat{\Delta}_{ks}(\frac{z}{r})] < 1$ if $\rho \left[ \hat{\Delta}_{os}(e^{j\theta}) \right] < 1$, $\forall \theta \in [0, 2\pi)$ and $\rho \left[ \hat{\Delta}_{ks}(e^{j\theta}) \right] < 1$, $\forall \theta \in [0, 2\pi)$. Hence, the closed-loop systems (3.1) and (3.2) is $D(\alpha, r)$-stable. That completes the proof.

3.2. GA design algorithm. The procedure for genetic algorithm generation of observer and controller gains in the slow subsystem is as follows.

Define the chromosomes of each generation as $X_{sij} = [F_s^{ij}, K_{s0}^{ij}, K_{s1}^{ij}, K_{s2}^{ij}]$, $i = 0, 1, 2, \ldots$ $l$, $j = 0, 1, 2, \ldots, m$ as $i$th chromosome in the $j$th generation for the slow subsystem. This paper’s aim is to search for $X_{sij}$ with greatest fitness function such that the slow subsystems are $D(\alpha, r)$-stable.

Before proceeding to find $F_s$ and $K_{si}$, $i = 0, 1, 2, \ldots, l$, by GA, first define fitness function

$$f(X_{sij}) = \frac{1}{1 + \text{MSE}(X_{sij})}$$

for the selection procedure, where $\text{MSE}(X_{sij}) = \frac{\sum_{k=1}^{N} e^2(k)}{N}$ is the mean square observer error of a closed-loop slow subsystem based on $X_{sij}$, $N$ is the number of sampling points. Accordingly, select the fittest chromosomes $X_{sij}$ at $j$th generation and pass it to $(j + 1)$th generation.

Step 1: Generate a number of initial chromosomes with uniform random elements.
Step 2: Query each chromosome as to whether or not it satisfies the stability criteria given by Theorem 3.1. If stability is satisfied, then select the chromosome one of the 0th generations, select another chromosome and repeat the process. Choose a set of chromosomes as the 0th generations such that each chromosome makes the closed-loop subsystem $D(\alpha, r)$-stable.

Step 3: For the selection phase, utilize a fitness function to find corresponding fitness values for each chromosome identified in Step 2. From each chromosome, choose those chromosome-fragments possessing the greatest fitness for genetic algorithm processing.

Step 4: During the crossover phase, randomly select a pair of chromosomes from the previous generation’s set of chromosomes and summate them after multiplying each with a uniform random number to obtain two new chromosomes at each cycle. When assembly of a new set of chromosomes completes, move to Step 5. The crossover equation is as follows.

$$
X_{si(j+1)} = r_1X_{si(j+1)} + (1 - r_1)X_{s(i+1)(j+1)},
$$
$$
X_{s(i+1)(j+1)} = r_2X_{si(j+1)} + (1 - r_2)X_{s(i+1)(j+1)},
$$

$r_1$, $r_2$: uniform random number.

Step 5: Introduce chromosome mutation to just one chromosome following the crossover procedure. A uniform random number is generated for each chromosome in turn, and compare it with the preset probability number to decide whether or not the mutation procedure is fired.

Step 6: Check to see if any one of the preset conditions is satisfied. If so, then stop the procedure, otherwise return to Step 2. The preset conditions in this example are the decrease in rate of mean squared observer error, or the number of evolution generations.

3.3. Controller design for the fast subsystem. Derive the fast subsystem as follows [3]:

$$
x_f(k + 1) = \varepsilon \sum_{i=0}^{n} \hat{A}_{2i} x_f(k - i) + B_2 u_f(k),
$$
$$
y_f(k) = C_2 x_f(k).
$$

(3.7)

Assumption 3.2. The fast subsystem (3.7) is stabilizable and detectable.

The observer-based controller for the fast subsystem (3.7) is:

$$
\hat{x}_f(k + 1) = \varepsilon \sum_{i=0}^{n} \hat{A}_{2i} \hat{x}_f(k - i) + B_2 u_f(k) + F_f [y_f(k) - \hat{y}_f(k)]
$$
$$
\hat{y}_f(k) = C_2 \hat{x}_f(k)
$$
$$
u_f(k) = - \sum_{i=0}^{n} K_{fi} \hat{x}_f(k - i).
$$

(3.8)

where the matrices $K_{fi}$ are controller gain for delay states and $F_f$ is observer gain.

For the $D(\alpha, r)$-stability of the closed-loop fast subsystems (3.7) and (3.8), select $F_f$ such that $o_f^{-1}(z)$ be $D(\alpha, r)$-stable and $|\det[I - \Delta_{of}(z)]| > 0$, $\forall \left|\frac{z - a}{r}\right| \geq 1$ to guarantee that all poles of $o_f(z)$ are within the disk $D(\alpha, r)$ and then select $K_{f0}$ such that $c_{f0}^{-1}(z)$ is stable, and $|\det[I - \Delta_{kf}(z)]| > 0$, $\forall \left|\frac{z - \alpha}{r}\right| \geq 1$ to guarantee that all poles of $[I - \Delta_{kf}(z)]^{-1}$
are within the disk $D(\alpha, r)$. The design procedure is similar to that in slow subsystem
and not repeated here.

**Theorem 3.2.** Choose $F_f$ so that $o_f^{-1}(z)$ is $D(\alpha, r)$-stable and $\rho \left[ \tilde{\Delta}_f(e^{i\theta}) \right] < 1$, $\forall \theta \in [0, 2\pi)$. Moreover, choose $K_{fi}$ so that $c_i^{-1}(z)$ is $D(\alpha, r)$-stable and $\rho \left[ \tilde{\Delta}_{fi}(e^{i\theta}) \right] < 1$, $\forall \theta \in [0, 2\pi)$. The closed-loop system (3.9) is $D(\alpha, r)$-stable.

**Proof:** The proof is similar to that in Theorem 3.1 and not repeated here.

The operational procedures of GA for the design of observer and controller gains in the
fast subsystem follow those of the slow subsystem.

4. **Composite Observer-Based Controller Design for the Original System.** In
this section, propose a composite observer-based controller in the following Theorem so
that the original system (2.1) is $D(\alpha, r)$-stable for a sufficiently small value of $\varepsilon$.

The proposed composite observer-based controller is

$$\hat{x}(k + 1) = \sum_{i=0}^{n} A_i \hat{x}(k - i) + B \hat{u}(k) + F [y(k) - \hat{y}(k)],$$
$$\hat{y}(k) = C \hat{x}(k),$$
$$\hat{u}(k) = - \sum_{i=0}^{n} K_i \hat{x}(k - i)$$

(4.1)

where

$$\hat{x}(k) = \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & \varepsilon \tilde{A}_{1i} \\ A_{21} & \varepsilon \tilde{A}_{2i} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad K_i = \begin{bmatrix} K_{1i} \\ K_{2i} \end{bmatrix},$$

(4.2)

and

$$K_{1i} = K_{si} - K_{fi} \left( I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i} \right)^{-1} \left( \sum_{i=0}^{n} A_{1i} - B_2 \sum_{i=0}^{n} K_{si} \right),$$
$$K_{2i} = K_{fi}, \quad i = 0, 1, 2, \ldots, n.$$ (4.3a)

and

$$F_1 = F_s \left[ I + C_2 \left( I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i} \right)^{-1} F_f \right] - \left( \varepsilon \sum_{i=0}^{n} \tilde{A}_{1i} \right) \left( I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i} \right)^{-1} F_f,$$
$$F_2 = F_f.$$ (4.3b)

Propose a criterion for estimating the upper bound of the $\varepsilon$ in Theorem 4.1 that is
effective only if $\varepsilon$ is sufficient small.

**Theorem 4.1.** If the gains $(F_s, K_{si})$ and $(F_f, K_{fi})$ are designed such that the systems
(3.4) and (3.10) are both $D(\alpha, r)$-stable, then the original system (2.1) with the composite
observer-based controller (4.1) is $D(\alpha, r)$-stable if the singular perturbation parameter $\varepsilon$
satisfies:

$$\min \left\{ \rho \left[ \tilde{\Gamma}_1 (\varepsilon, e^{i\theta}) \right], \rho \left[ \tilde{\Gamma}_2 (\varepsilon, e^{i\theta}) \right] \right\} < 1, \quad \forall \theta \in [0, 2\pi)$$

(4.4)
where
\[
\hat{\Gamma}_1(\varepsilon, e^{j\theta}) = \left[ e^{-j\theta}I - \hat{Q}_s(\varepsilon, e^{j\theta}) \right]^{-1} Q_2(\varepsilon, e^{j\theta}) \left[ \eta(\varepsilon, e^{j\theta}) - \eta(\varepsilon, 1) \right] Q_3(\varepsilon, e^{j\theta})
\] (4.5a)
and
\[
\hat{\Gamma}_2(\varepsilon, e^{j\theta}) = \left[ \eta(\varepsilon, e^{j\theta}) - \eta(\varepsilon, 1) \right] Q_3(\varepsilon, e^{j\theta}) \left[ e^{-j\theta}I - \hat{Q}_s(\varepsilon, e^{j\theta}) \right]^{-1} Q_2(\varepsilon, e^{j\theta})
\] (4.5b)
with
\[
\eta(\varepsilon, \hat{z}) = \left[ (\hat{z}^{-1}I - Q_4(\varepsilon))^{-1} \right], \quad Q_2 = \sum_{i=0}^{n} M_{2i}(\varepsilon) \hat{z}^i|_{\hat{z}^{i}=e^{j\theta}},
\]
\[
Q_3 = \sum_{i=0}^{n} M_{3i}(\varepsilon) \hat{z}^i|_{\hat{z}^{i}=e^{j\theta}}, \quad Q_4 = \sum_{i=0}^{n} M_{4i}(\varepsilon) \hat{z}^i|_{\hat{z}^{i}=e^{j\theta}},
\]
\[
\hat{Q}_s(\varepsilon, e^{j\theta}) = \sum_{i=0}^{n} \left[ \begin{array}{cc} (A_{si} - B_s K_{si}) & B_s \hat{K}_s \\ 0 & (A_{si} - F_s C_{si}) \end{array} \right] \hat{z}^i|_{\hat{z}^{i}=e^{j\theta}},
\]
\[
\hat{K}_s = K_{si} + \Lambda - K_{fi} \left[ I - \varepsilon \sum_{i=0}^{n} \hat{A}_{2i} + B_2 \sum_{i=0}^{n} K_{fi} \right]^{-1} \left( \sum_{i=0}^{n} A_{2i} + B_2 \Lambda \right),
\]
\[
\Lambda = \sum_{i=0}^{n} K_{fi} \left[ I - \varepsilon \sum_{i=0}^{n} \hat{A}_{2i} + T(i) F_f C_2 \right]^{-1} \left( \sum_{i=0}^{n} A_{2i} - T(i) F_f C_1 \right).
\]

**Proof:** The proof is similar to that of Theorem 4.1 in [3] and not repeated here.

In order to make the proposed strategy for the observer-based controller design practical, a design procedure outline follows.

**Observer-Based Controller Design Procedure:**

For a discrete multiple time-delay singular perturbed system in $C$-mode as Equation (2.1), the following steps outline the process of designing an observer-based controller.

**Step 1:** Derive the slow subsystem (3.1) of the original system (2.1).

**Step 2:** Check whether the slow subsystem is stabilizable and detectable. If the slow subsystem is not stabilizable or detectable, then stop.

**Step 3:** Derive the fast subsystem (3.7) of the original system (2.1).

**Step 4:** Check whether the fast subsystem is stabilizable and detectable. If the slow subsystem is not stabilizable or detectable, then stop.

**Step 5:** Design the observer-based controller (3.2) for the slow subsystem by using GA in Section 3.2.

**Step 6:** Design the observer-based controller (3.8) for the fast subsystem by using GA in Section 3.2.

**Step 7:** Derive the composite observer-based controller (4.1) based on the controller designed in Steps 5 and 6.

**Step 8:** Check whether the singular perturbation parameter $\varepsilon$ satisfies the inequality (4.4).

If it succeeds then end the design procedure, else go to Step 5.

**Remark 4.1.** Obtain the stability bound $\varepsilon^*$ by finding the upper bound of $\varepsilon$ that satisfies the criterion (4.4). This will allow us to investigate the stability of the original system by its corresponding slow and fast subsystems for all $\varepsilon \in [0, \varepsilon^*]$.

The following remark outlines the advantages of results reported in this paper compared to methods reported in the literature.
Remark 4.2. In the paper [3], rearranging the multiple-time delay systems as non-delay systems comes at the cost of enormous increase in system order. This increases computation time in the design of the controller and hence the design method in [3] becomes impractical. Moreover, are concerned with designing the D(α, r)-stability of the systems, while the goal of controller design in paper [3] is the asymptotic stability of the systems. Asymptotic stability is a special case of the results in this paper, found by letting α = 0 and r = 1. Another advantage of paper is that it does not need to solve the Riccati equation for the observer-based control, since; use the genetic algorithm (GA) to find the controller. In paper [3], the authors design the observer-based control either by solving the Riccati equation or by trial and error. Overall, the proposed control design methodology in this paper is novel and practical compared to that of paper [3].

5. Example. From discussions in Sections 3 and 4, illustrate the design procedures as follows.

In this section, provide an example to illustrate the observer-based controller design procedure derived from Sections 3 and 4. The following equations describe a discrete multiple time-delay singularly perturbed system [3]:

\[ x(k + 1) = \sum_{i=0}^{2} A(\varepsilon, i)x(k - i) + Bu(k) \]
\[ y(k) = Cx(k) \]  \hspace{1cm} (5.1)

where

\[
A(\varepsilon, i) = \begin{bmatrix}
A_{1i} & \varepsilon\tilde{A}_{1i} \\
A_{2i} & \varepsilon\tilde{A}_{2i}
\end{bmatrix},
B = \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix},
C = \begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
\text{ and}
\]

\[
A_{10} = \begin{bmatrix}
0.4 & 0.3 \\
0.22 & 0.35
\end{bmatrix},
A_{11} = \begin{bmatrix}
0.33 & 0 \\
0.3 & 0.25
\end{bmatrix},
A_{12} = \begin{bmatrix}
0.25 & 0.33 \\
0 & 0.32
\end{bmatrix},
\]

\[
\tilde{A}_{10} = \begin{bmatrix}
0.03 & 0.02 \\
0.03 & 0.01
\end{bmatrix},
\tilde{A}_{11} = \begin{bmatrix}
0.024 & 0 \\
0.036 & 0.037
\end{bmatrix},
\tilde{A}_{12} = \begin{bmatrix}
0.0175 & 0 \\
0.028 & 0.038
\end{bmatrix},
\]

\[
A_{20} = \begin{bmatrix}
-0.15 & 0.1 \\
0.12 & -0.15
\end{bmatrix},
A_{21} = \begin{bmatrix}
-0.12 & 0.12 \\
-0.1 & 0.8
\end{bmatrix},
A_{22} = \begin{bmatrix}
-0.11 & 0 \\
0 & 0.1
\end{bmatrix},
\]

\[
\tilde{A}_{20} = \begin{bmatrix}
0.075 & 0.0091 \\
0.04 & 0.06
\end{bmatrix},
\tilde{A}_{21} = \begin{bmatrix}
0.06 & 0.0065 \\
0.07 & 0.042
\end{bmatrix},
\tilde{A}_{22} = \begin{bmatrix}
0.056 & 0.009 \\
0.067 & 0.032
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
0.1 \\
0.25
\end{bmatrix},
B_2 = \begin{bmatrix}
0.008 & 0.012
\end{bmatrix},
C_1 = \begin{bmatrix}
0.5 & 0.23
\end{bmatrix},
C_2 = \begin{bmatrix}
0.21 & 0.9
\end{bmatrix}.
\]  \hspace{1cm} (5.2)

The \( \varepsilon \) value is supposed to be 0.02. The eigenvalues of the original system matrix are then 1.2721, 0.7482, -0.4751 ± 0.4771, 0.4433, -0.2498 ± 0.4059, -0.3055, -0.0600 ± 0.4195i, 0.1484 ± 0.2928i. It is apparent that the original system (5.1) which lacks control is unstable. Our aim in this example, is to design an observer-based controller, to control the system to be D(α, r)-stable with α and r set to 0 and 1. Follow the procedure for composite observer-based controller design for the system (5.1) as below:

Step 1: Controller Design in the Slow Subsystem. The system matrices in the slow subsystem (3.1) are

\[
A_{s0} = \begin{bmatrix}
0.3998 & 0.3001 \\
0.2199 & 0.3499
\end{bmatrix},
A_{s1} = \begin{bmatrix}
0.3298 & 0.0005 \\
0.2996 & 0.2561
\end{bmatrix},
\]

\[
A_{s2} = \begin{bmatrix}
0.2498 & 0.33 \\
-0.0002 & 0.3202
\end{bmatrix},
B_s = \begin{bmatrix}
0.1081 \\
0.2621
\end{bmatrix}.
\]  \hspace{1cm} (5.3)
Figure 1. The poles of slow subsystem with controller

\[ C_s^0 = \begin{bmatrix} 0.5762 & 0.116 \\ -0.116 & 0.748 \end{bmatrix}, \quad C_s^1 = \begin{bmatrix} -0.0235 & 0.0903 \end{bmatrix}, \quad D_s = 0.0125. \]

The eigenvalues of the original slow subsystem matrix are 1.26, 0.6452, \( j0.4523 \), \( -j0.4582 \), \( -0.1279 \pm j0.456 \). The slow subsystem (3.1) without control is unstable. According to the GA algorithm in Section 3.2, derive the observer gain and controller gain of the slow subsystem as

\[ F_s = \begin{bmatrix} 0.6873 & 0.7248 \end{bmatrix}^T, \quad K_{s0} = \begin{bmatrix} 0.4641 & 0.697 \end{bmatrix}, \]
\[ K_{s1} = \begin{bmatrix} 0.7031 & 0.2088 \end{bmatrix}, \quad K_{s2} = \begin{bmatrix} 0.7852 & 0.5486 \end{bmatrix}, \]

so that the eigenvalues of closed-loop slow subsystem, (3.1) and (3.2) are all inside unity. Figure 1 shows the poles of the closed-loop slow subsystem.

The GA parameters are:

(i). number of generators: 500,
(ii). populations: 30,
(iii). number of best individuals passed on to the next generation: 1,
(iv). number of crossover sites: random,
(v). crossover probability: 100%,
(vi). mutation probability: 8%.

Figure 2 depicts the convergence trend for the mean-square-error. The GA algorithm produced a mean-square-error of \( 8.23 \times 10^{-3} \).

Step 2: Controller Design in the Fast Subsystem. The eigenvalues of the original fast subsystem matrix are 0.1401, \(-0.0692 \pm j0.1127\), 0.0448 \( \pm j0.0761\), \(-0.0886\). Again, applying the GA algorithm from Section 3.2, derived observer gain and controller gain of
Figure 2. Convergence of the GA learning

so that the eigenvalues of the closed-loop fast subsystem (3.7) all fall inside a unit circle. Figure 3 shows the poles of the closed-loop fast subsystem.

Step 3: Composite Controller Design for the Original System. According to (4.3), the gains of composite observer-based controller (4.1) are

\[ K_{10} = \begin{bmatrix} -0.575 & -0.5685 \end{bmatrix}, \quad K_{11} = \begin{bmatrix} -1.0528 & 0.0033 \end{bmatrix}, \]
\[ K_{12} = \begin{bmatrix} -1.0493 & 0.4554 \end{bmatrix}, \quad K_{20} = \begin{bmatrix} -0.2428 & -0.1824 \end{bmatrix}, \]
\[ K_{21} = \begin{bmatrix} -0.875 & 0.1953 \end{bmatrix}, \quad K_{22} = \begin{bmatrix} -0.6105 & -0.8897 \end{bmatrix}, \]

and

\[ F_1 = \begin{bmatrix} 1.1992 & 0.8928 \end{bmatrix}^T, \quad F_2 = \begin{bmatrix} 0.9397 & 0.6124 \end{bmatrix}^T. \]

The eigenvalues of the closed-loop systems (2.1) and (4.1) are 0.9686, 0.9686, 0.8845, 0.3366, 0.3366, 0.6046, 0.6046, 0.6284, 0.6284, 0.5654, 0.5654 and 0.9225, all inside unit circle. Therefore, the system (2.1) is stable under the observer-based controller (4.1) with the GA determined gains (5.6).

Table 1 derives from Equations (5.1) – (5.6). The table shows that system order and the order of the observer-based controller, using proposed design methodology, is 2 while paper [3] reports a value of 6. It is apparent that the order designed by the proposed GA method yields a much-reduced value than that previously reported. Hence, the proposed design method offers substantial savings in computation time in the design and control of the closed-loop systems.

We plot the functions \( \rho \left( \hat{\Gamma}_1 (\varepsilon, e^{j\theta}) \right) \) and \( \rho \left( \hat{\Gamma}_2 (\varepsilon, e^{j\theta}) \right) \) versus \( \theta \) as shown in Figure 4, which demonstrates that controller design meets the condition (4.4), then conclude that
Figure 3. The poles of fast subsystem with controller

Table 1. Comparison of the system and control order

<table>
<thead>
<tr>
<th>methods</th>
<th>The control design method in paper [3]</th>
<th>The proposed control design method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>system order</td>
<td>control order</td>
</tr>
<tr>
<td>parameters</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>orders</td>
<td>6×6</td>
<td>6×1</td>
</tr>
</tbody>
</table>

under the observer-based controller (4.1), the original system (5.1) or equivalently (2.1) is stable. Figure 5 plots functions $\rho \left[ \tilde{\Gamma}_1 (\varepsilon, e^{i\theta}) \right]$ and $\rho \left[ \tilde{\Gamma}_2 (\varepsilon, e^{i\theta}) \right]$ in the range $\theta = [0, 2\pi)$, against $\varepsilon$. The figure shows that $\varepsilon^*$, has an upper bound, $\varepsilon^* = 1.05$, so that the closed-loop systems (2.1) and (4.1) is D(1, 0)-stable, if $\varepsilon \in [0, \varepsilon^*)$.

6. Conclusion. In this paper, a genetic algorithm is applied to D-stability observer-based control design, for discrete multiple time-delay, singularly perturbed systems. The GA does not need to solve the complicated Riccati equations to determine the observer gains. Initially, time-scale separation is used to derive the corresponding slow and fast subsystems of the original system. The observer-based controllers are separately designed for the slow and fast subsystems using GA. Finally, a composite observer-based controller for the original system is synthesized from the observer-based controllers.

The stability of discrete multiple time-delay, singularly perturbed systems can be investigated by establishing the corresponding slow and fast subsystems for a sufficiently small value of $\varepsilon$. We propose a stability criterion for the original system under the composite observer-based controller. If the criterion is satisfied, then stabilities of the slow and fast subsystems guarantee that the controlled system will be stable. The stability criterion
Figure 4. The value of $\rho \left( \Gamma_1 (\varepsilon, e^{j\theta}) \right)$ and $\rho \left( \Gamma_2 (\varepsilon, e^{j\theta}) \right)$ in the range $\theta = [0, 2\pi)$ with $\varepsilon = 0.02$.

Figure 5. Functions of $\rho \left( \Gamma_1 (\varepsilon, e^{j\theta}) \right)$ and $\rho \left( \Gamma_2 (\varepsilon, e^{j\theta}) \right)$ in (4.4) can be used to find the stability upper bound $\varepsilon^*$ of the singular perturbation parameter $\varepsilon$. A worked example at the end of the paper demonstrates the efficiency of the proposed controller.
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