Fuzzy Facility Location Problem with Preference of Candidate Sites and Asymmetric A-Distance

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Received April 2010; revised August 2010

Abstract. We consider the asymmetric A-distance though it is not a metric due to the asymmetry in a rigorous sense. However, it reflects on an actual situation of urban areas that time from the demand point to the facility depends on the passing direction and in some places slopes and so it is asymmetric. Further, most residents in urban areas hope that even the city office should not be so near and so far from their homes. That implies the judge criteria existing in residents’ mind are conflicting, while the local government considers the preference of the facility location site, from the point of construction cost, land cost, etc. So, we propose the following model considering the satisfaction degree with respect to the asymmetric A-distance from the facility for each customer (residents) and preference of the site. 1) The region we consider here is the rectangular urban area and several demand points are set up in this area. 2) For each demand point, satisfaction degree with respect to the asymmetric A-distance to the facility is defined and it is denoted by the membership function. 3) The preference of the site as a candidate point of the facility is also given. The objective is to find the non-dominated sites of the facility maximizing both minimal satisfaction degree among all demand points and its preference.

Keywords: Facility location, Demand points, Asymmetric A-distance, Minimal satisfaction degree, Preference of candidate sites

1. Introduction. There exist huge number of facility location models and Hamacher et al. [1] tried to classify them using similar codes to queue and schedule. Widmayer et al. [2] have introduced A-distance that is a generalization of rectilinear distance used in the urban area. For rectilinear distance, we should refer to [3] as a classic but successful model and efficient algorithm due to geometrical approach. In order to use more realistic distance, we consider asymmetric A-distance though, in a rigorous sense, it is not a metric due to the asymmetry. Further, most residents in urban areas hope that even the city office should not be so near and so far from their homes. That implies the judge criteria existing in residents’ mind are conflicting. As a similar model from a facility type, E. Carrizosa and E. Conde [4] have considered a semi-desirable facility. However, most researches on a facility location we consider so far are classified the facility into either desirable facility (typical one is a public facility) or undesirable one (e.g., [5-7]), while local government considers the preference of the facility location site, from the point of construction cost, land cost, etc. So, we propose the following model considering the satisfaction degree
with respect to the asymmetric A-distance from the facility for each customer (residents) and preference of the site. The assumed conditions are set up as follows.

1) The region we consider here is the rectangular urban area and the asymmetric A-distance is adopted as the measurement of distance.

2) There exist demand points describing the center of each sub-region calculated by population based weighted average.

3) For each demand point, satisfaction degree with respect to the asymmetric A-distance to the facility is defined and this satisfaction degree is denoted by the membership function of the distance to the facility.

4) Preference of the site as a candidate point of the facility is represented by the membership function of the coordinate of the site. Land fee, safety of land and other factors are taken into this preference and mainly this preference may be considered from the construction side such as local government side.

5) The objective is to find the site of the facility maximizing both minimal satisfaction degree among all demand points and its preference.

Prominent features of above model are 2) and 3) and so it is suitable for analyzing the location problem of a disposal center, a shopping mall or crematory and so on. We propose an efficient algorithm to find the non-dominated sites of the facility after defining non-domination of sites since usually there exists no site optimizing both criteria at a time. Assumption 4) is also one of prominent features in our model. Though it is introduced first in our previous model (Ishii et al. [8]), and however, it treats the asymmetric rectilinear distance case. The asymmetric A-distance is another prominent feature and new notion is not considered so far. It is more realistic and practical than the asymmetric rectilinear distance. This paper consists of four sections including this introduction section. Section 2 formulates our problem and includes the definition of non-dominated sites. Section 3 proposes solution procedure to find some non-dominated sites. Section 4 concludes this paper.

2. Problem Formulation. First, we define and explain what the asymmetric A-distance is.

[Asymmetric A-distance]

There exists a set of directions \( A = \{\alpha_1, \alpha_2, \ldots, \alpha_a, \alpha_{a+1}, \alpha_{a+2}, \ldots, \alpha_{2a-1}, \alpha_{2a}\} \) where \( a \geq 2 \) and each \( \alpha_i, i = 1, 2, \ldots, a \), is an angle from \( x \) axis in an orthogonal coordinate such that \( 0^\circ \leq \alpha_1 < \alpha_2 < \cdots < \alpha_a < 180^\circ \) and \( \alpha_{a+k} = \alpha_k + 180^\circ, k = 1, 2, \ldots, a \). Hereafter, if no confusion occurs, directions \( \alpha_i, i = 1, 2, \ldots, 2a \) and angles \( \alpha_i, i = 1, 2, \ldots, 2a \) are used as the same meaning. Directions \( \alpha_j, \alpha_{j+1} \) are called neighboring, that is, \( \alpha_{2a+1} \) is interpreted as \( \alpha_1 \). Further, a half line and a line segment are called A-directional (or A-oriented) if their directions are ones of \( \alpha_1, \alpha_2, \ldots, \alpha_{2a} \). If this direction is one of \( \alpha_1, \alpha_2, \ldots, \alpha_{2a} \), then we call it positive. In the case of \( \alpha_{a+1}, \alpha_{a+2}, \ldots, \alpha_{2a} \), we call it negative. Then, the asymmetric A-distance \( d_{AS} \) from \( p^1 \) to \( p^2 \) is defined as follows.

\[
d_{AS}(p^1, p^2) = \begin{cases} 
    w_1d_2(p^1, p^2) & \text{if direction } p^1p^2 \text{ is positive} \\
    w_2d_2(p^1, p^2) & \text{if direction } p^1p^2 \text{ is negative} \\
    \min_{p^3 \in \mathbb{R}^2} \{d_{AS}(p^1, p^3) + d_{AS}(p^3, p^2)\} & \text{Otherwise}
\end{cases}
\]

where \( d_2(p^1, p^2) \) is the Euclidean distance between \( p^1 \) and \( p^2 \).
The demand points are denoted as \((p_i, q_i), i = 1, 2, \ldots, n\) describing the center of each sub-region calculated by population based weighted average in the urban rectangular region \(X = \{(x, y) | p_L \leq x \leq p_U, q_L \leq y \leq q_U\}\).

(2) For each demand point \(i\), satisfaction degree with respect to the asymmetric A distance \(d_{iAS}\) from it and satisfaction degree is defined by the following membership functions \(\mu_i(d_{iAS})\). That is, demand points are divided into 3 categories according to their attitudes to the facility, i.e., “not welcome group” \(J_1\), “welcome group” \(J_2\) and “not so far but not so near group” \(J_3\).

\[
\mu_i(d_{iAS}) = \begin{cases} 
0 & (d_{iAS} \leq a_i) \\
\frac{d_{iAS} - a_i}{b_i - a_i} & (a_i \leq d_{iAS} \leq b_i), \ i \in J_1 \\
1 & (b_i \leq d_{iAS}) 
\end{cases}
\]

\[
\mu_i(d_{iAS}) = \begin{cases} 
1 & (d_{iAS} \leq c_i) \\
1 - \frac{d_{iAS} - c_i}{e_i - c_i} & (c_i \leq d_{iAS} \leq e_i), \ i \in J_2 \\
0 & (e_i \leq d_{iAS}) 
\end{cases}
\]

\[
\mu_i(d_{iAS}) = \begin{cases} 
0 & (d_{iAS} \leq a_i) \\
\frac{d_i - a_i}{b_i - a_i} & (a_i \leq d_{iAS} \leq b_i) \\
1 & (b_i \leq d_{iAS} \leq c_i), \ i \in J_3 \\
1 - \frac{d_{iAS} - c_i}{e_i - c_i} & (c_i \leq d_{iAS} \leq e_i) \\
0 & (e_i \leq d_{iAS}) 
\end{cases}
\]

According to the result in T. Matutomi and H. Ishii [9], when \(\alpha_j < \alpha_{j+1}\) an angle of the line connecting demand point \(i\) with the facility site \((x, y)\) is \(\alpha_{j+1}\),

\[
0 \leq \alpha_j < \alpha_{j+1} < 180^\circ \Rightarrow d_{iAS} = \frac{\sqrt{w_1(m_1+m_2)(x-p_i)-(y-q_i)}}{m_2\sqrt{1+m_1^2+m_1\sqrt{1+m_2^2}}} \{(p_i - x) - (q_i - y)\}
\]

\[
\alpha_j < 180^\circ \leq \alpha_{j+1} \Rightarrow d_{iAS} = w_2M_1\{m_2(p_i-x)-(y-q_i)\} + w_1M_2\{m_1(x-p_i)-(y-q_i)\}
\]

where \(m_1 = \max(\tan \alpha_j, \tan \alpha_{j+1})\), \(m_2 = \min(\tan \alpha_j, \tan \alpha_{j+1})\), \(M_1 = \frac{\sqrt{1+m_1^2}}{m_1-m_2}\), \(M_2 = \frac{\sqrt{1+m_2^2}}{m_1-m_2}\).

Of course, when the line connecting demand point \(i\) with the facility site \((x, y)\) is positive \(A\)-oriented, say angle is \(0 \leq \alpha_j < 180^\circ\),

\[
d_{iAS} = w_1\sqrt{(p_i-x)^2 + (q_i-y)^2} = w_1\sqrt{1 + \tan^2 \alpha_j|x-p_i|}
\]

\[
= \begin{cases} 
w_1\sqrt{1 + \tan^2 \alpha_j(x-p_i)} & (0 \leq \alpha_j < 90^\circ) \\
w_1\sqrt{1 + \tan^2 \alpha_j(p_i-x)} & (90^\circ \leq \alpha_j < 180^\circ)
\end{cases}
\]

and negative oriented

\[
d_{iAS} = w_2\sqrt{(p_i-x)^2 + (q_i-y)^2} = w_2\sqrt{1 + \tan^2 \alpha_j|x-p_i|}
\]

\[
= \begin{cases} 
w_2\sqrt{1 + \tan^2 \alpha_j(x-p_i)} & (180^\circ \leq \alpha_j < 270^\circ) \\
w_2\sqrt{1 + \tan^2 \alpha_j(p_i-x)} & (270^\circ \leq \alpha_j)
\end{cases}
\]

(3) For each candidate site \(x = (x, y) \in X\) of the rectangular region \(X\), preference function described by the membership function \(\mu_F(x)\) is attached.
(4) The objective is to find the site of the facility in the rectangular region $X = \{(x, y)|p_L \leq x \leq p_U, q_L \leq y \leq q_U\}$ maximizing both the minimal satisfaction degree $\min\{\mu_i(d_i) i = 1, 2, \ldots, n\}$ among all demand points and preference function $\mu_F(x)$.

(5) Usually, the unique optimal site maximizing both the minimal satisfaction degree and the preference functions does not exist. So, we seek some non-dominated sites after defining the non-dominated site.

First, note that $i \in J_1 \cup J_3$, $0 < \alpha_i = \mu_i(d_{iAS}) \leq 1 \Rightarrow d_{iAS} \geq d_i^L$, $i \in J_2 \cup J_3$, $0 < \alpha_i = \mu_i(d_{iAS}) \leq 1 \Rightarrow d_{iAS} \leq d_i^R$ where $d_i^L = \alpha_i f_i + a_i$, $f_i = b_i - a_i$, $d_i^R = (1 - \alpha_i)g_i + c_i$ and $g_i = e_i - c_i$.

Then, our problem is transformed into the following problem $P$.

\[ P: \text{Maximize } \alpha \]
\[ \text{subject to } \alpha \leq \frac{d_i - a_i}{f_i}, \quad i \in J_1 \cup J_3, \quad \alpha \leq 1 - \frac{d_i - c_i}{g_i}, \quad i \in J_2 \cup J_3, \quad 0 < \alpha \leq 1, \]
\[ p_L \leq x \leq p_U, \quad q_L \leq y \leq q_U \]

(Non-dominated site) . . .

For the sites $x^1 = (x_1, y_1)$, $x^2 = (x_2, y_2)$ $\in X$, if $\mu_D(x^1) \geq \mu_D(x^2)$, $\mu_F(x^1) \geq \mu_F(x^2)$ and at least one inequality holds without equality, then we call $x^1$ dominates $x^2$ where $\alpha = \min\{\mu_i(d_i)|i = 1, 2, \ldots, n\}$ minimal satisfaction degree among all demand points with respect to the distance from the facility site $x$. $x$ is called non-dominated site if there exists no site dominating $x$.

3. Solution Procedure. First, let us define $X^\beta = \{x|\mu_F(x) \geq \beta, \ x \in X\}$, $1 \geq \beta \geq 0$ and introduce the following subproblem $P^\beta$ in order to solve $P$.

\[ P^\beta: \text{Maximize } \alpha \]
\[ \text{subject to } \alpha f_i + a_i \leq d_{iAS}, \quad i \in J_1 \cup J_3, \quad d_{iAS} \leq (1 - \alpha)g_i + c_i, \quad i \in J_2 \cup J_3, \]
\[ 0 < \alpha \leq 1, \quad p_L \leq x \leq p_U, \quad q_L \leq y \leq q_U, \quad x \in X^\beta. \]

Let an optimal solution and optimal value of $P^\beta$ be $x(\beta)$ and $\alpha(\beta)$ respectively if $P^\beta$ is feasible. Then, we have the following property.

**Property 3.1.** If $\alpha(\beta^1) = \alpha(\beta^2)$ for $\beta^1 > \beta^2$, then $x(\beta^2)$ is not a non-dominated solution.

Next, we consider the solution method for $P^\beta$. From each demand point $i$, $i = 1, 2, \ldots, n$, we draw all A lines passing through the demand point $i$. Then, the rectangular region is divided into sub-regions $X_1, X_2, \ldots, X_s$ by these lines where each $X_i$ is a polygon consisting of A-directional edges. Here, we introduce an auxiliary problem $P_\ell^\beta$, $\ell = 1, 2, \ldots, s$ of subproblem $P^\beta$.

\[ P_\ell^\beta: \text{Maximize } \alpha \]
\[ \text{subject to } \alpha f_i + a_i \leq d_{iAS}, \quad i \in J_1 \cup J_3, \quad d_{iAS} \leq (1 - \alpha)g_i + c_i, \quad i \in J_2 \cup J_3, \]
\[ 0 < \alpha \leq 1, \quad x \in X^\beta \cap X_\ell. \]

Each $P_\ell^\beta$ is a linear programming problem with three decision variables $\alpha$, $x$, $y$ since the condition $x \in X^\beta \cap X_\ell$ is denoted by the linear inequalities. Therefore, it can be solved efficiently by the algorithm due to N. Megiddo [10]. So, by implicitly solving all $P_\ell^\beta$, $\ell = 1, 2, \ldots, s$ and choosing the best one among all optimal solutions deriving $P_\ell^\beta$, $\ell = 1, 2, \ldots, s$, we can solve $P^\beta$.

(Solution procedure for $P$)

Step 1: Set $\beta = \varepsilon_0$ (small positive number), $DS = \emptyset$ and go to Step 2 ($DS$ is a set of some non-dominated sites).

Step 2: Solve $P^\beta$ and obtain $x(\beta)$ and $\alpha(\beta)$. If there exist no site $x \in DS$ dominating $x(\beta)$ then go to Step 3. Otherwise, go to Step 4.
Step 3: Update $\mathbf{DS} \leftarrow \mathbf{DS} \cup \{\mathbf{x}(\beta)\}$. If $\beta = 1$, terminating the solution procedure. Otherwise update $\beta \leftarrow \min\{\mu_F(\mathbf{x}(\beta)) + \varepsilon_\beta, 1\}$ ($\varepsilon_\beta$ is a suitable small positive number) and return to Step 2.

Step 4: If $\beta = 1$, terminating the solution procedure. Otherwise update $\min f \leftarrow F(\mathbf{x}(\beta)) + \varepsilon_\beta$ and return to Step 2.

It is difficult to determine which value of $\varepsilon_\beta$ is suitable in the general case of $\mu_F$. Therefore, we try to solve a special but important case of $\mu_F$, that is, block-wisely constant case. Further, note that it includes the finite possible site case.

(Case that the preference function is constant block-wisely)

$$
\mu_F(\mathbf{x}) = \begin{cases} 
  t_1 & x \in A_1 \\
  t_2 & x \in A_2 \\
  \vdots \\
  t_q & x \in A_q \\
  0 & x \in X - A_1 - A_2 - \cdots - A_q
\end{cases}
$$

where $A_i \subseteq X$, $A_i \cap A_j = \phi$ ($i \neq j$), $i, j = 1, \ldots, q$, each $A_i$, $i = 1, 2, \ldots, q$ is a polygon (we assume that it is, described by $O(n)$ number of linear inequalities) and $0 < t_1 < t_2 < \cdots < t_q \leq 1$. The region $X - A_1 - A_2 - \cdots - A_q$ represents the area that facility cannot be constructed.

In order to solve this case, we introduce subproblem $\bar{Q}_t^t$, $t = 1, 2, \ldots, q$

$$
\bar{Q}_t^t : \text{Maximize } \alpha \\
\text{subject to } \alpha f_i + a_i \leq d_i, \ i \in J_1 \cup J_3 \\
d_{iAS} \leq (1 - \alpha) g_i + c_i, \ i \in J_2 \cup J_3, \ 0 < \alpha \leq 1, \ x \in A_t
$$

Further, we define the corresponding auxiliary problem $\tilde{Q}_t^t$, $t = 1, 2, \ldots, s$

$$
\tilde{Q}_t^t : \text{Maximize } \alpha \\
\text{subject to } \alpha f_i + a_i \leq d_{iAS}, \ i \in J_1 \cup J_3 \\
d_{iAS} \leq (1 - \alpha) g_i + c_i, \ i \in J_2 \cup J_3, \ 0 < \alpha \leq 1, \ x \in A_t \cap X_t.
$$

Since $A_t \cap X_t$ is a polygon, it is denoted by some linear inequalities. Further, it has only three decision variables. That is, $\tilde{Q}_t^t$ is a linear programming problem with three variables. So, it can be solved in $O(n)$ computational time by using Megiddo algorithm [10]. Therefore, by using brute force calculations, time complexity of our algorithm is $O(snq)$ for this special case when $A_t$ is denoted by a small number of linear inequalities though we need not solve each auxiliary problem from scratch and some subproblems should not be solved.

4. Conclusions. Our solution procedure is straightforward and so more refinement is possible. Especially, geometric approach may be useful since our model is the extension of the rectilinear facility location model in H. Ishii et al. [8]. In a real case of urban area, barriers may exist. At that time, calculation of A-distance becomes complicated. Further usually there exist already old facilities and so we also consider the share of services among old ones and new facility to be located (refer to [11]). Moreover, we should consider a multi-criteria case since the current requests of residents and constructor of a facility are various and their concerning criteria of them are quite different [12,13]. These extensions are future but important problems and need to be investigated. Further, from the viewpoints of urban planning, we should also combine facility location problems with human factors using Multi-Agent and how to solve from multi-factors. As such approaches, we should refer to [14-17] and construct more realistic facility models.
REFERENCES