MAXIMUM ECONOMIC REVENUE FOR FUZZY PRICE IN FUZZY SENSE

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Abstract. We present a fuzzy optimization problem in business and economics. In this problem, a fuzzy price is determined by using a linear two degree demand functions. The objective is to find the optimal fuzzy revenue, which is derived from the fuzzy price. We use level (λ, 1) interval-valued fuzzy numbers to consider the fuzzy price and the fuzzy revenue. Using signed distance to defuzzify, we can obtain the demand function and revenue function in the fuzzy sense. What follows is that we can find the maximum revenue in the fuzzy sense.

Keywords: Fuzzy price, Fuzzy revenue, Fuzzy demand, Interval-valued fuzzy set

1. Introduction. We study a fuzzy optimization problem in business and economics [11-13]. The traditional methods for solving this problem involve using the extension principle and genetic algorithms [8-10]. However, in [1,5-7,15], triangular fuzzy numbers were used and the extension principle is applied to derive the defuzzified membership function using the centroid method to obtain the optimal solution in the fuzzy sense. In [14], the triangular fuzzy number was used to estimate the missing value in the fuzzy sense. From past experience, we know that deriving the membership function is a very tedious and difficult task. In this paper, we consider a fuzzy price optimization problem in business and economics. We use level (λ, 1) interval-valued fuzzy numbers to consider the fuzzy price and the fuzzy revenue. Signed distance is used to defuzzify to obtain the demand function and revenue function in the fuzzy sense.

The demand function is a two degree polynomial \( P(x) = a - bx + cx^2 \) and the revenue function is \( R(x) = xP(x) = ax - bx^2 + cx^3 \). We can find the value of \( x \) to maximize \( R(x) \). The demand function \( P(x) = a - bx + cx^2 \) coefficients \( a, b \) and \( c \) are fixed in a planning period for a monopoly market. However, for a perfect competitive market, \( a, b \) and \( c \) will vary according to the economic situation. It is reasonable to fuzzify them. If we fuzzify \( a \) into a triangular fuzzy number \( 0 < \Delta_1 < a, 0 < \Delta_2, \tilde{a} = (a - \Delta_1, a, a + \Delta_2) \), all is equal to 1 in the planning period the membership grade for \( \tilde{a} \) at point \( a \). This is not reasonable. For this reason, we consider that the membership grade of point \( a \) will lie in the interval \( [\lambda, 1] \), \( 0 < \lambda < 1 \). Therefore, we fuzzify \( a, b \) and \( c \) to a level (λ, 1) fuzzy numbers \( \tilde{a} = [\tilde{a}^L, \tilde{a}^U] = [(a - \Delta_3, a, a + \Delta_4; \lambda), (a - \Delta_1, a, a + \Delta_2)], \tilde{b} = [\tilde{b}^L, \tilde{b}^U] = [(b - \Delta_7, b, b + \Delta_8; \lambda), (b - \Delta_5, b, b + \Delta_6)] \) and \( \tilde{c} = [\tilde{c}^L, \tilde{c}^U] = [(c - \Delta_{11}, c, c + \Delta_12; \lambda), (c - \Delta_9, c, c + \Delta_{10})] \) to consider the problem in the fuzzy sense. Section 2 presents the preliminaries for Section 3. In Section 3, we consider the demand function to be the second degree polynomial \( P(x) = a - bx + cx^2 \) in the fuzzy sense. We give an example in Section 4. Section 5 presents the conclusion.
2. Preliminaries. We need the following definitions to use interval-valued fuzzy sets to consider the fuzzy price and fuzzy revenue for finding the maximum revenue in the fuzzy sense.

Definition 2.1. A fuzzy set \( \tilde{a} \) defined on \( \mathbb{R} \) is called a fuzzy point if its membership function is
\[
\mu_{\tilde{a}}(x) = \begin{cases} 
1, & \text{if } x = a \\
0, & \text{if } x \neq a
\end{cases}
\]

Definition 2.2. A fuzzy set \([a, b; \alpha]\) defined on \( \mathbb{R} \), \( a < b \), \( 0 \leq \alpha \leq 1 \) is called level \( \alpha \) fuzzy interval if its membership function is
\[
\mu_{[a, b; \alpha]} = \begin{cases} 
\alpha, & \text{if } a \leq x \leq b \\
0, & \text{otherwise}
\end{cases}
\]

Definition 2.3. A fuzzy set \( \tilde{C} \) defined on \( \mathbb{R} \) is called level \( \lambda \) triangular fuzzy number if its membership function is
\[
\mu_{\tilde{C}}(x) = \begin{cases} 
\lambda \frac{(x-a)}{b-a}, & a \leq x \leq b \\
\lambda \frac{(c-x)}{c-b}, & b \leq x \leq c \\
0, & \text{otherwise}
\end{cases}
\]

We denote it as \( \tilde{C} = (a, b, c; \lambda), a < b < c, 0 < \lambda \leq 1 \).

The family of all level \( \lambda \) triangular fuzzy numbers will be denoted by \( F_{\lambda}(\lambda) \).

Definition 2.4. ([2]) An interval-valued fuzzy set \( \tilde{A} \) (i-v fuzzy set for short) defined on \( \mathbb{R} \) is given by \( \tilde{A} = \{(x, [\mu_{\tilde{A}}(x), \mu_{\tilde{A}^U}(x)]) | x \in \mathbb{R}, \mu_{\tilde{A}}(x), \mu_{\tilde{A}^U}(x) \in [0, 1]\} \) and \( \mu_{\tilde{A}^L}(x) \leq \mu_{\tilde{A}^U}(x) \) for all \( x \in \mathbb{R} \). We denote it as \( \tilde{A} = [\tilde{A}^L, \tilde{A}^U] \).

\[\begin{align*}
\text{Figure 1. Interval-valued fuzzy } \tilde{A} \\
\text{If } 0 \leq \alpha < \lambda, \text{ the } \alpha \text{-level set of } \tilde{A} \text{ is defined as} \\
\{x | \mu_{\tilde{A}^V}(x) \geq \alpha\} - \{x | \mu_{\tilde{A}}(x) > \alpha\} = [\tilde{A}_{\lambda}^V(\alpha), \tilde{A}_{\lambda}^L(\alpha)] \cup [\tilde{A}_{\lambda}^L(\alpha), \tilde{A}_{\lambda}^U(\alpha)] \quad \text{(See Figure 1)} \tag{1}
\end{align*}\]

If \( \lambda \leq \alpha \leq 1 \), the \( \alpha \)-level set of \( \tilde{A} \) is defined as
\[
\{x | \mu_{\tilde{A}}(x) \geq \alpha\} = [\tilde{A}_{\lambda}^U(\alpha), \tilde{A}_{\lambda}^L(\alpha)] 
\tag{2}
\]

For any \( \alpha \in [0, \lambda) \), the following have a one-to-one correspondence
\[
[\tilde{A}_{\lambda}^V(\alpha), \tilde{A}_{\lambda}^L(\alpha)] \leftrightarrow [\tilde{A}_{\lambda}^V(\alpha), \tilde{A}_{\lambda}^L(\alpha); \alpha] \\
[\tilde{A}_{\lambda}^L(\alpha), \tilde{A}_{\lambda}^U(\alpha)] \leftrightarrow [\tilde{A}_{\lambda}^L(\alpha), \tilde{A}_{\lambda}^U(\alpha); \alpha] \quad \text{\tag{3}}
\]
For any $\alpha \in [\lambda, 1]$, the following has a one-to-one correspondence

$$[\overline{A}^U_{r}(\alpha), \overline{A}^U_{r}(\alpha)] \leftrightarrow [\overline{A}^L_{r}(\alpha), \overline{A}^U_{r}(\alpha); \alpha]$$

From Figure 1, (1)-(4) and decomposition theory, we get

$$\overline{A} = \bigcup_{0 \leq \alpha \leq \Lambda} ([\overline{A}^L_{r}(\alpha), \overline{A}^U_{r}(\alpha); \alpha] \cup [\overline{A}^L_{r}(\alpha), \overline{A}^U_{r}(\alpha); \alpha]) \cup \bigcup_{\Lambda \leq \alpha \leq 1} [\overline{A}^U_{r}(\alpha), \overline{A}^U_{r}(\alpha); \alpha]$$

Suppose that $\overline{B}^L = (a, b, c; \lambda) \in F_N(\lambda)$ and $\overline{B}^U = (p, b, q) \in F_N(1)$, where $p < a < b < q$ and $0 < \lambda \leq 1$. Then, $\overline{B} = [\overline{B}^L, \overline{B}^U] = ([a, b, c; \lambda), (p, b, q)]$ is called a level $(\lambda, 1)$ i-v fuzzy number. We denote the family of all levels $(\lambda, 1)$ i-v of fuzzy numbers as $F_{IV}(\lambda)$. (See Figure 2)

![Figure 2. Level $(\lambda, 1)$ i-v fuzzy numbers $\overline{B}$](image)

From Definition 2.3, the $\alpha$-level set of $\overline{B}$ is $\overline{B}^L(\alpha) = [\overline{B}^L_{r}(\alpha), \overline{B}^U_{r}(\alpha)]$, where

$$\overline{B}^L_{L}(\alpha) = a + (b - a) \frac{\alpha}{\Lambda} \quad \text{and} \quad \overline{B}^U_{L}(\alpha) = c - (c - b) \frac{\alpha}{\Lambda}$$

The $\alpha$-level set of $\overline{B}^U$ is $\overline{B}^U(\alpha) = [\overline{B}^U_{r}(\alpha), \overline{B}^U_{r}(\alpha)]$, where

$$\overline{B}^U_{L}(\alpha) = p + (b - p) \alpha \quad \text{and} \quad \overline{B}^U_{r}(\alpha) = q - (q - b) \alpha$$

Using (5)-(7), we have

$$\overline{B} = \bigcup_{0 \leq \alpha \leq \Lambda} ([\overline{B}^U_{r}(\alpha), \overline{B}^U_{r}(\alpha); \alpha] \cup [\overline{B}^L_{r}(\alpha), \overline{B}^U_{r}(\alpha); \alpha]) \cup \bigcup_{\Lambda \leq \alpha \leq 1} [\overline{B}^U_{r}(\alpha), \overline{B}^U_{r}(\alpha); \alpha]$$

$$= \bigcup_{0 \leq \alpha \leq \Lambda} ([p + (b - p) \alpha, a + (b - a) \frac{\alpha}{\Lambda}; \alpha] \cup [c - (c - b) \frac{\alpha}{\Lambda}, q - (q - b) \alpha; \alpha])$$

$$\cup \bigcup_{\Lambda \leq \alpha \leq 1} [p + (b - p) \alpha, q - (q - b) \alpha; \alpha]$$

From Kaufmann and Gupta [3], when $a < b, c < d, a, b, c, d \in R$, the interval operations are as follows:

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$k \cdot [a, b] = \begin{cases} [ka, kb], & \text{if } k > 0 \\ [kb, ka], & \text{if } k < 0 \end{cases}$$

Let $\overline{B} = [\overline{B}^L, \overline{B}^U] = ([a, b, c; \lambda), (p, b, q)]$ and $\overline{D} = [\overline{D}^L, \overline{D}^U] = ([u, v, w; \lambda), (s, v, t)] \in F_{IV}(\lambda)$.

**Definition 2.5.** If $\overline{B} = [\overline{B}^L, \overline{B}^U]$ and $\overline{D} = [\overline{D}^L, \overline{D}^U] \in F_{IV}(\lambda)$, $k \in R$. We define

$$\overline{B} \oplus \overline{D} = [\overline{B}^L \oplus \overline{D}^L, \overline{B}^U \oplus \overline{D}^U], \overline{B} \ominus \overline{D} = [\overline{B}^L \ominus \overline{D}^L, \overline{B}^U \ominus \overline{D}^U] \text{ and } k\overline{B} = [k\overline{B}^L, k\overline{B}^U].$$
Then, we have the following results.

\[ \tilde{B} \oplus \tilde{D} = [(a + u, b + v, c + w; \lambda), (p + s, b + v, q + t)] \in F_{IV}(\lambda) \]
\[ \tilde{B} \ominus \tilde{D} = [(a - w, b - v, c - u; \lambda), (p - t, b - v, q - s)] \in F_{IV}(\lambda) \]
\[ k \cdot \tilde{B} = \begin{cases} 
[(ka, kb, kc; \lambda), (kp, kb, kq)] \in F_{IV}(\lambda) & \text{if } k > 0 \\
[(kc, kb, ka; \lambda), (kq, kb, kp)] \in F_{IV}(\lambda) & \text{if } k < 0
\end{cases} \]

We used treatments similar to that in Yao and Wu [4] to consider the signed distance of the level \((\lambda, 1)\) i-v of fuzzy number in \(F_{IV}(\lambda)\).

**Definition 2.6.** If \(a, 0 \in R\), the signed distance from \(a\) to \(0\) is defined as \(d_0(a, 0) = a.[4]\)

Now, we will consider how to define the signed distance of the level \((\lambda, 1)\) i-v fuzzy number in \(F_{IV}(\lambda)\). From (8),

\[ \tilde{B} = \bigcup_{0 \leq a < \lambda} ([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha); a] \cup [\tilde{B}^L_1(\alpha), \tilde{B}^U_1(\alpha); a]) \cup \bigcup_{\lambda \leq a \leq 1} [\tilde{B}^U_1(\alpha), \tilde{B}^U_1(\alpha); a] \] (11)

When \(0 \leq a < \lambda\), by (11) and Figure 2, we have \(d_0(\tilde{B}^U_1(\alpha), 0) = \tilde{B}^U_1(\alpha), d_0(\tilde{B}^L_1(\alpha), 0) = \tilde{B}^L_1(\alpha)\) and \(d_0(\tilde{B}^U_1(\alpha), 0) = \tilde{B}^U_1(\alpha)\). We define the signed distance from \([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha)]\) to \(0\) to be \(d_0([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha)], 0) = \frac{1}{2}[d_0(\tilde{B}^U_1(\alpha), 0) + d_0(\tilde{B}^L_1(\alpha), 0)] = \frac{1}{2}[\tilde{B}^U_1(\alpha) + \tilde{B}^L_1(\alpha)]\). Similarly, the signed distance from \([\tilde{B}^L_1(\alpha), \tilde{B}^U_1(\alpha)]\) to \(0\) is defined as:

\[ d_0([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha)] \cup [\tilde{B}^L_1(\alpha), \tilde{B}^U_1(\alpha)], 0) = \frac{1}{4}[a + c + p + q + (2b - a - c)\frac{\alpha}{\lambda} + (2b - p - q)\alpha] \] (12)

For each \(\alpha \in [0, \lambda]\), the following have a one-to-one correspondence. \([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha)] \leftrightarrow [\tilde{B}^L_1(\alpha), \tilde{B}^L_1(\alpha); a], [\tilde{B}^L_1(\alpha), \tilde{B}^U_1(\alpha)] \leftrightarrow [\tilde{B}^L_1(\alpha), \tilde{B}^L_1(\alpha); a]\) and \(0 \leftrightarrow \hat{0}\).

Therefore, by (12), the signed distance from \([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha); a] \cup [\tilde{B}^L_1(\alpha), \tilde{B}^U_1(\alpha); a]\) to \(\hat{0}\) can be defined as:

\[ d([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha); a] \cup [\tilde{B}^L_1(\alpha), \tilde{B}^U_1(\alpha); a], \hat{0}) = \frac{1}{4}[a + c + p + q + (2b - a - c)\frac{\alpha}{\lambda} + (2b - p - q)\alpha] \] (13)

This formula is a continuous function of \(\alpha\). If \(0 \leq \alpha \leq \lambda\), we use the definite integral to find its mean value.

\[ \frac{1}{\lambda} \int_0^\lambda d([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha); a] \cup [\tilde{B}^L_1(\alpha), \tilde{B}^U_1(\alpha); a], \hat{0}) \, d\alpha = \frac{1}{8}[2b + a + c + 2p + 2q + (2b - p - q)\lambda] \] (13)

Similarly, if \(\lambda \leq \alpha \leq 1\), we can define the signed distance from \([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha); a]\) to \(\hat{0}\) to be

\[ d([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha); a], \hat{0}) = \frac{1}{2}[p + q + (2b - p - q)\alpha] \]

This formula is also a continuous function of \(\alpha\). We also use the definite integral to find its mean value.

\[ \frac{1}{1 - \lambda} \int_\lambda^1 d([\tilde{B}^U_1(\alpha), \tilde{B}^L_1(\alpha); a], \hat{0}) \, d\alpha = \frac{1}{4}[2b + p + q + (2b - p - q)\lambda] \] (14)

From (11)-(14), we get the following definition.
Definition 2.7. If $\tilde{B} = [\tilde{B}_L, \tilde{B}_U] \in FIV(\lambda)$, the signed distance from $\tilde{B}$ to $\tilde{0}$ is defined as

(a) $0 < \lambda < 1$, $d(\tilde{B}, \tilde{0}) = \frac{1}{\lambda} \int [13] + [14] = \frac{1}{16} [6b + a + c + 4p + 4q + 3(2b - p - q)\lambda]$.

(b) $0 < \lambda = 1$, $d(\tilde{B}, \tilde{0}) = \frac{1}{4} \int_0^1 [\tilde{B}_U^L(\alpha) + \tilde{B}_L^L(\alpha) + \tilde{B}_L^U(\alpha) + \tilde{B}_R^U(\alpha)]d\alpha = \frac{1}{8} (4b + a + c + p + q)$.

3. Fuzzy Price and Fuzzy Revenue with Level $(\lambda, 1)$ i-v Fuzzy Numbers. In a monopoly market, let the demand function be

$$P(x) = a - bx + cx^2, \quad 0 \leq x \leq x_{00}$$

(15)

where $x$ is the demand quantity and $P(x)$ is the unit price when the demand quantity is $x$.

$$x_{00} = \frac{b - \sqrt{b^2 - 4ac}}{2c} \quad \text{or} \quad x_{00} = \frac{b}{2c}$$

In this case, the revenue function is

$$R(x) = x \cdot P(x) = ax - bx^2 + cx^3$$

(16)

In monopoly market, the monopolist can find the optimum $P(x_{00})$ such that $R(x_{00})$ is the maximum. In (15), $a$, $b$ and $c$ are fixed numbers which are independent of $x$. In a perfect competitive market, the decision maker can determine $a$, $b$ and $c$ in a planning period $T$. Since in this period, the economic situation will influence the values of $a$, $b$ and $c$, it is not appropriate to fix the values of $a$, $b$ and $c$. The decision maker can choose $a$, $b$ and $c$ in the following interval,

$$[a - \Delta_1, a + \Delta_2], \quad 0 < \Delta_1 < a, \quad 0 < \Delta_2$$

(17)

$$[b - \Delta_5, b + \Delta_6], \quad 0 < \Delta_5 < b, \quad 0 < \Delta_6$$

(18)

$$[c - \Delta_9, c + \Delta_{10}], \quad 0 < \Delta_9 < c, \quad 0 < \Delta_{10}$$

(19)

Since $[a - \Delta_1, a + \Delta_2]$ is an interval and is not a value, the decision maker can choose an appropriate value from the interval $[a - \Delta_1, a + \Delta_2]$ as an estimate of $a$. If the decision maker chooses $a$, then there is no error. The error for this value is zero. We can use confidence level to express this idea. If the error is zero, the confidence level will be the maximum and is set to 1. The farther from $a$, the decision maker chooses, the smaller the confidence level is. When the decision maker chooses $a - \Delta_1$ or $a + \Delta_2$, the confidence level is the minimum, i.e., 0. Corresponding to the interval $[a - \Delta_1, a + \Delta_2]$. Therefore, we set the following level 1 triangular fuzzy number $\tilde{a}^U$:

$$\tilde{a}^U = (a - \Delta_1, a, a + \Delta_2)$$

(20)

Figure 3. Level 1 triangular fuzzy number $\tilde{a}^U$

From Figure 3, the membership grade of $\tilde{a}^U$ is 1 at the point $a$. The farther the point deviates from $a$, the smaller the point membership grade. The membership grade is 0 on
the point \( a - \Delta_1 \) or \( a + \Delta_2 \). Therefore, the membership grade shares the same property with the confidence level. Hence, it is reasonable to choose (20) as the level 1 triangular fuzzy number \( \tilde{a}^U \) when we consider the confidence level as the membership grade in the interval \([a - \Delta_1, a + \Delta_2]\). In a planning period \( T \), it is not appropriate in a real situation the membership grade of \( a \) is always set to 1. We can change the membership grade of \( a \) to lie in the interval \([\lambda, 1]\), \( 0 < \lambda < 1 \). We fuzzify \( a \) in (15) to a level \((\lambda, 1)\) i-v fuzzy number \( \tilde{a} \). Let

\[
\tilde{a}^L = (a - \Delta_3, a, a + \Delta_4; \lambda) \tag{21}
\]

\[
\tilde{a} = [\tilde{a}^L, \tilde{a}^U] \tag{22}
\]

\[
0 < \Delta_3 < \Delta_1 < a, 0 < \Delta_4 < \Delta_2 \tag{23}
\]

**Figure 4.** Level \((\lambda, 1)\) i-v fuzzy numbers \( \tilde{a} \)

Similarly, we fuzzify \( b, c \) to the following level \((\lambda, 1)\) i-v fuzzy number \( \tilde{b} \) and \( \tilde{c} \),

\[
\tilde{b} = [\tilde{b}^L, \tilde{b}^U] \tag{24}
\]

\[
\tilde{b}^L = (b - \Delta_7, b, b + \Delta_8; \lambda), \tilde{b}^U = (b - \Delta_5, b, b + \Delta_6) \tag{25}
\]

\[
0 < \Delta_7 < \Delta_5 < b, 0 < \Delta_8 < \Delta_6 \tag{26}
\]

**Figure 5.** Level \((\lambda, 1)\) i-v fuzzy numbers \( \tilde{b} \)

\[
\tilde{c} = [\tilde{c}^L, \tilde{c}^U] \tag{27}
\]

\[
\tilde{c}^L = (c - \Delta_{11}, c, c + \Delta_{12}; \lambda), \tilde{c}^U = (c - \Delta_9, c, c + \Delta_{10}) \tag{28}
\]

\[
0 < \Delta_{11} < \Delta_9 < c, 0 < \Delta_{12} < \Delta_{10} \tag{29}
\]

We use (22), (24) and (27) to fuzzify (15) and obtain the fuzzy demand function

\[
\tilde{P}(x) = \tilde{a} \odot \tilde{b} x \odot \tilde{c} x^2 \tag{30}
\]

From (10) and (20)-(29), we have

\[
\tilde{P}(x) = [\tilde{a}^L, \tilde{a}^U] \odot [\tilde{b}^L, \tilde{b}^U] x \odot [\tilde{c}^L, \tilde{c}^U] x^2 = [\tilde{P}^L(x), \tilde{P}^U(x)], \tag{31}
\]
where
\[
P_L(x) = (a - \Delta_3 - (b + \Delta_6)x + (c - \Delta_{11})x^2, a - bx + cx^2, \\
   a + \Delta_4 - (b - \Delta_7)x + (c + \Delta_{12})x^2; \lambda)
\]
\[
= (a - bx + cx^2 - \Delta_3 - \Delta_6x - \Delta_{11}x^2, a - bx + cx^2, \\
   a - bx + cx^2 + \Delta_4 + \Delta_7x + \Delta_{12}x^2; \lambda)
\]
\[
P_U(x) = (a - \Delta_1 - (b + \Delta_6)x + (c - \Delta_9)x^2, a - bx + cx^2, \\
   a + \Delta_2 - (b - \Delta_5)x + (c + \Delta_{10})x^2)
\]
\[
= (a - bx + cx^2 - \Delta_1 - \Delta_6x - \Delta_9x^2, a - bx + cx^2, \\
   a - bx + cx^2 + \Delta_2 + \Delta_5x + \Delta_{10}x^2)
\]  \tag{32}

We fuzzify (16) and have the fuzzy revenue function
\[
\tilde{R}(x) = \tilde{a}x \ominus \tilde{b}x^2 \oplus \tilde{c}x^3 
\]  \tag{33}

With the same arguments as (31), we get
\[
\tilde{R}(x) = [\tilde{R}_L(x), \tilde{R}_U(x)], 
\]  \tag{34}

where
\[
\tilde{R}_L(x) = ((a - \Delta_3)x - (b + \Delta_6)x^2 + (c - \Delta_{11})x^3, ax - bx^2 + cx^3, \\
   (a + \Delta_4)x - (b - \Delta_7)x^2 + (c + \Delta_{12})x^3; \lambda)
\]
\[
\tilde{R}_U(x) = ((a - \Delta_1)x - (b + \Delta_6)x^2 + (c - \Delta_9)x^3, ax - bx^2 + cx^3, \\
   (a + \Delta_2)x - (b - \Delta_5)x^2 + (c + \Delta_{10})x^3) 
\]  \tag{35}

If we defuzzify \( \tilde{P}(x) \) in (31) and \( \tilde{R}(x) \) in (34) through signed distance in Definition 2.7, we have
\[
P_s(x) = d(\tilde{P}(x), \tilde{0})
\]
\[
= \frac{1}{16} \{ 6(a - bx + cx^2) + (a - bx + cx^2 - \Delta_3 - \Delta_6x - \Delta_{11}x^2) \\
   + (a - bx + cx^2 + \Delta_4 + \Delta_7x + \Delta_{12}x^2) \\
   + 4(a - bx + cx^2 - \Delta_1 - \Delta_6x - \Delta_9x^2) \\
   + 4(a - bx + cx^2 + \Delta_2 + \Delta_5x + \Delta_{10}x^2) \\
   + 3[2(a - bx + cx^2) - (a - bx + cx^2 - \Delta_1 - \Delta_6x - \Delta_9x^2) \\
   - (a - bx + cx^2 + \Delta_2 + \Delta_5x + \Delta_{10}x^2)] \lambda \}
\]
From (37), we get
\begin{align*}
4922 \ J.-S. \ SU, \ T.-S. \ SHIH \ AND \ H.-M. \ LEE \\
\frac{1}{16} \{ & 16(\Delta_1 - \Delta_3 - 4\Delta_1 + 4\Delta_2 + 3\lambda(\Delta_1 - \Delta_2)) \\
+ & [\Delta_7 - \Delta_8 - 4\Delta_6 + 4\Delta_5 + 3\lambda(\Delta_6 - \Delta_5)]x \\
+ & [\Delta_{12} - \Delta_{11} - 4\Delta_9 + 4\Delta_{10} + 3\lambda(\Delta_9 - \Delta_{10})]x^2 \} \\
= & a - bx + cx^2 + \frac{1}{16} [\Delta_4 - \Delta_3 + (4 - 3\lambda)(\Delta_2 - \Delta_1)] \\
+ & \frac{1}{16} [\Delta_7 - \Delta_8 + (4 - 3\lambda)(\Delta_5 - \Delta_6)]x \\
+ & \frac{1}{16} [\Delta_{12} - \Delta_{11} + (4 - 3\lambda)(\Delta_{10} - \Delta_9)]x^2
\end{align*}

Equation (36) is an estimate of the demand function in the fuzzy sense.
When \( \Delta_j = \Delta_{j+1}, \ j = 1, 3, 5, 7, 9, 11, \) then \( P_*(x) \) reduce to crisp \( a - bx + cx^2 = P(x) \).
Let
\begin{align*}
A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) & = a + \frac{1}{16} [\Delta_4 - \Delta_3 + (4 - 3\lambda)(\Delta_2 - \Delta_1)] \\
B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) & = b - \frac{1}{16} [\Delta_7 - \Delta_8 + (4 - 3\lambda)(\Delta_5 - \Delta_6)] \\
C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}) & = c + \frac{1}{16} [\Delta_{12} - \Delta_{11} + (4 - 3\lambda)(\Delta_{10} - \Delta_9)]
\end{align*}

From (37), we get
\begin{align*}
A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) & = \frac{1}{16} [\Delta_4 + (4 - 3\lambda)\Delta_2 + (a - \Delta_3) + (4 - 3\lambda)(a - \Delta_1) + (11 + 3\lambda)a] \\
& = \frac{1}{16} \Delta_4 + \frac{4 - 3\lambda}{16} \Delta_2 + \frac{1}{16} (a - \Delta_3) + \frac{4 - 3\lambda}{16} (a - \Delta_1) + \frac{11 + 3\lambda}{16} a \\
& > 0
\end{align*}
\begin{align*}
B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) & = \frac{1}{16} [\Delta_8 + (4 - 3\lambda)\Delta_6 + (b - \Delta_7) + (4 - 3\lambda)(b - \Delta_5) + (11 + 3\lambda)b] \\
& = \frac{1}{16} \Delta_8 + \frac{4 - 3\lambda}{16} \Delta_6 + \frac{1}{16} (b - \Delta_7) + \frac{4 - 3\lambda}{16} (b - \Delta_5) + \frac{11 + 3\lambda}{16} b \\
& > 0
\end{align*}
\begin{align*}
C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}) & = \frac{1}{16} [\Delta_{12} + (4 - 3\lambda)\Delta_{10} + (c - \Delta_{11}) + (4 - 3\lambda)(c - \Delta_9) + (11 + 3\lambda)c] \\
& = \frac{1}{16} \Delta_{12} + \frac{4 - 3\lambda}{16} \Delta_{10} + \frac{1}{16} (c - \Delta_{11}) + \frac{4 - 3\lambda}{16} (c - \Delta_9) + \frac{11 + 3\lambda}{16} c \\
& > 0
\end{align*}

Using (36)-(38), we can rewrite
\begin{equation}
P_*(x) = A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) - B(\Delta_5, \Delta_6, \Delta_7, \Delta_8)x + C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})x^2
\end{equation}
From (38), we know that
\( A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) > 0, \ B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) > 0 \) and \( C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}) > 0. \)
Let \( D_*(x) = (B(\Delta_5, \Delta_6, \Delta_7, \Delta_8))^2 - 4A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) \cdot C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}). \)
If \( D_* \geq 0 \), set \( x_{ss} = \frac{B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) - \sqrt{D_*}}{2C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})} \)

If \( D_* < 0 \), set \( x_{ss} = \frac{B(\Delta_5, \Delta_6, \Delta_7, \Delta_8)}{2C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})} \)

\[
0 \leq x \leq x_{ss}
\]

Similarly, if we defuzzify \( \tilde{R}(x) \) in (34) through signed distance in Definition 2.7, we have

\[
R_* \equiv d(\tilde{R}(x), \tilde{0}) = A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) x - B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) x^2 + C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}) x^3
\]

\[
0 \leq x \leq x_{ss}
\]

Equation (41) is the revenue function in the fuzzy sense.

When \( \Delta_j = \Delta_{j+1}, j = 1, 3, 5, 7, 9, 11 \), by (37) and (41), we find

\[
A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) = a,
B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) = b,
C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}) = c.
\]

It follows that \( R_*(x) \) reduces to crisp \( ax - bx^2 + cx^3 = R(x) \).

Using (41) by differentiation, we get

\[
\frac{d}{dx} R_*(x) = A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) - 2B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) x + 3C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}) x^2.
\]

Set

\[
3C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}) x^2 - 2B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) x + A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) = 0
\]

\[
(42)
\]

The discriminant value of (42) is

\[
D_0 = (B(\Delta_5, \Delta_6, \Delta_7, \Delta_8))^2 - 3A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) \cdot C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}).
\]

If \( D_0 > 0 \), then (42) can be written as \((x - \alpha)(x - \beta) = 0\), where

\[
\alpha = \frac{B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) - \sqrt{D_0}}{3C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})} \quad \text{and} \quad \beta = \frac{B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) + \sqrt{D_0}}{3C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})}.
\]

Therefore,

\[
\frac{d}{dx} R_*(x) = 3C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})(x - \alpha)(x - \beta).
\]

Since \( C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}) > 0 \), we have the result as shown in Figure 7.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sign of ( \frac{d}{dx} R_*(x) )</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>change of ( R_*(x) )</td>
<td>↗</td>
<td>↘</td>
<td>↗</td>
</tr>
</tbody>
</table>

**Figure 7.** Test change of \( R_*(x) \)

If \( D_0 = 0 \), then (42) can be written as \((x - \gamma)^2 = 0\), where

\[
\gamma = \frac{B(\Delta_5, \Delta_6, \Delta_7, \Delta_8)}{3C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})} > 0 \quad \text{and} \quad x_{ss} - \gamma = \frac{B(\Delta_5, \Delta_6, \Delta_7, \Delta_8)}{6C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})} > 0.
\]

Therefore,

\[
\frac{d}{dx} R_*(x) = 3C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})(x - \gamma)^2 > 0.
\]
It follows that \( R_s(x) \) is increasing on \( 0 \leq x \leq x_\ast \). The result is that \( R_s(x_\ast) \) is the maximum.

If \( D_0 < 0 \), by the fact that \( C(\Delta_0, \Delta_{10}, \Delta_{11}, \Delta_{12}) > 0 \), then \( \frac{d}{dx} R_s(x) > 0 \). The result is the same as that \( D_0 = 0 \), i.e., \( R_s(x_\ast) \) is the maximum. From Figure 7 and the reasons above, we get Proposition 3.1.

**Proposition 3.1.** If we use the level \((\lambda, 1)\) i-v fuzzy numbers \( \tilde{a}, \tilde{b} \) and \( \tilde{c} \) in (22), (24) and (27) to fuzzify the demand function (15) and revenue function (16), we get

1. **Level \((\lambda, 1)\) i-v fuzzy demand function** \( \tilde{P}(x) = \tilde{a} \odot \tilde{b} x + \tilde{c} x^2 = [\tilde{P}^L(x), \tilde{P}^U(x)] \) by (30)-(32);
2. **Level \((\lambda, 1)\) i-v fuzzy revenue function** \( \tilde{R}(x) = \tilde{a} x \odot \tilde{b} x^2 + \tilde{c} x^3 = [\tilde{R}^L(x), \tilde{R}^U(x)] \) by (33)-(35);
3. When \( \Delta_j, j = 1, 2, \ldots, 12 \) satisfies
   \[
   0 < \Delta_3 < \Delta_1 < a, \quad 0 < \Delta_4 < \Delta_2 \\
   0 < \Delta_7 < \Delta_5 < b, \quad 0 < \Delta_8 < \Delta_6 \\
   0 < \Delta_{11} < \Delta_9 < c, \quad 0 < \Delta_{12} < \Delta_{10}.
   \]

3.1 If \( D_0 > 0 \), from Figure 7, we have

3.1.1 If \( x_\ast < x < \beta, \ R_s(x_\ast) \) is maximum and we let \( x_{0D} = x_\ast \).

3.1.2 If \( 0 < \alpha < x_\ast < \beta, \ R_s(\alpha) \) is maximum and we let \( x_{0D} = \alpha \).

3.1.3 If \( 0 < \alpha < \beta < x_\ast, \ \max(R_s(\alpha), R_s(x_\ast)) \equiv R_s(x_{0D}) \) is maximum.

3.1.4 If \( \alpha < 0 < \beta < x_\ast, \ \max(R_s(0), R_s(x_\ast)) \equiv R_s(x_{0D}) \) is maximum.

3.1.5 If \( \alpha < 0 < x_\ast < \beta, \ R_s(0) \) is the maximum and we let \( x_{0D} = 0 \).

3.1.6 If \( \alpha < \beta < 0 < x_\ast, \ R_s(x_\ast) \) is the maximum and we let \( x_{0D} = x_\ast \).

3.1.2 If \( D_0 \leq 0 \), \( R_s(x_\ast) \) is the maximum and we let \( x_{0D} = x_\ast \).

4. **Numerical Examples.** Example 1: Let \( P(x) = 72 - 17x + x^2 \), \( 0 \leq x \leq 8, a = 72, b = 17, c = 1 \).

4.1 Crisp case:

The revenue function is \( R(x) = x \cdot P(x) = 72x - 17x^2 + x^3 \). When \( x^* = 2.8187 \), we have \( R^*(x^*) = 90.27498 \) is maximum value.

4.2 Fuzzy case:

In the following Table 1, \( \Delta_j, j = 1, 2, \ldots, 12 \) satisfies (23), (26) and (29), i.e.,

\[
\begin{align*}
0 &< \Delta_3 < \Delta_1 < a = 72, \quad 0 < \Delta_4 < \Delta_2 \quad (23) \\
0 &< \Delta_7 < \Delta_5 < b = 17, \quad 0 < \Delta_8 < \Delta_6 \quad (26) \\
0 &< \Delta_{11} < \Delta_9 < c = 1, \quad 0 < \Delta_{12} < \Delta_{10} \quad (29)
\end{align*}
\]

<table>
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<tr>
<th>case</th>
<th>( \Delta_1 )</th>
<th>( \Delta_2 )</th>
<th>( \Delta_3 )</th>
<th>( \Delta_4 )</th>
<th>( \Delta_5 )</th>
<th>( \Delta_6 )</th>
<th>( \Delta_7 )</th>
<th>( \Delta_8 )</th>
<th>( \Delta_9 )</th>
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<td>1</td>
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<td>0.9</td>
<td>0.8</td>
<td>0.6</td>
<td>0.5</td>
</tr>
</tbody>
</table>

**Table 1.** \( \Delta_j, j = 1, 2, \ldots, 12 \)
For fuzzy cases, we use the following formulas.

\[
A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) = 72 + \frac{1}{16} [\Delta_4 - \Delta_3 + (4 - 3\lambda) (\Delta_2 - \Delta_1)] (> 0)
\]

\[
B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) = 17 - \frac{1}{16} [\Delta_7 - \Delta_8 + (4 - 3\lambda) (\Delta_5 - \Delta_6)] (> 0)
\]

\[
C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}) = 1 + \frac{1}{16} [\Delta_{12} - \Delta_{11} + (4 - 3\lambda) (\Delta_{10} - \Delta_9)] (> 0)
\]

\[
R_s(x) = A(\Delta_1, \Delta_2, \Delta_3, \Delta_4)x - B(\Delta_5, \Delta_6, \Delta_7, \Delta_8)x^2 + C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})x^3
\]

\[
x^{**} = \frac{B(\Delta_5, \Delta_6, \Delta_7, \Delta_8)}{2C(\Delta_1, \Delta_2, \Delta_3, \Delta_4)}
\]

\[
D_0 = (B(\Delta_5, \Delta_6, \Delta_7, \Delta_8))^2 - 3A(\Delta_1, \Delta_2, \Delta_3, \Delta_4) \cdot C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})
\]

\[
\alpha = \frac{B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) - \sqrt{D_0}}{3C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})} \quad \text{(if } D_0 > 0),
\]

\[
\beta = \frac{B(\Delta_5, \Delta_6, \Delta_7, \Delta_8) + \sqrt{D_0}}{3C(\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12})} \quad \text{(if } D_0 > 0)
\]

We compute the fuzzy case data value in Table 2. Applying Proposition 3.1, we get the optimal solutions \(x_{0D}\) and the maximum revenue \(R_s(x_{0D})\) summarized in Table 3.

**Table 2. Fuzzy cases evaluate values**

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>Case A ((\Delta_1, \Delta_2, \Delta_3, \Delta_4))</th>
<th>Case B ((\Delta_5, \Delta_6, \Delta_7, \Delta_8))</th>
<th>Case C ((\Delta_9, \Delta_{10}, \Delta_{11}, \Delta_{12}))</th>
<th>(D_0)</th>
<th>(x^{**})</th>
<th>(\alpha)</th>
<th>(\beta)</th>
<th>criterion</th>
</tr>
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<td>16.881</td>
<td>1.030</td>
<td>63.057</td>
<td>8.1948</td>
<td>2.8933</td>
<td>8.0340</td>
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<tr>
<td>2</td>
<td>71.697</td>
<td>17.063</td>
<td>1.018</td>
<td>72.140</td>
<td>8.3794</td>
<td>2.8095</td>
<td>8.3670</td>
<td>(0 &lt; \alpha &lt; \beta &lt; x^{**})</td>
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<tr>
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<td>62.795</td>
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<td>2.8920</td>
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<td>68.515</td>
<td>8.3955</td>
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<td>(0 &lt; \alpha &lt; \beta &lt; x^{**})</td>
</tr>
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<td>4</td>
<td>71.650</td>
<td>16.800</td>
<td>1.020</td>
<td>62.991</td>
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<td>(0 &lt; \alpha &lt; \beta &lt; x^{**})</td>
</tr>
<tr>
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<td>17.031</td>
<td>0.884</td>
<td>99.685</td>
<td>9.629</td>
<td>2.6561</td>
<td>10.1825</td>
<td>(0 &lt; x^{**} &lt; \beta &lt; \alpha)</td>
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<tr>
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<td>1.019</td>
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<td>(0 &lt; \alpha &lt; \beta &lt; x^{**})</td>
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<tr>
<td>7</td>
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<td>17.353</td>
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<tr>
<td>8</td>
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<td>16.943</td>
<td>0.974</td>
<td>75.451</td>
<td>8.694</td>
<td>2.8245</td>
<td>8.7676</td>
<td>(0 &lt; \alpha &lt; x^{**} &lt; \beta)</td>
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</table>
5. Conclusion. We made some comments about this paper in the following:

A] Triangular fuzzy numbers are a special case of the level (λ, 1) i-v fuzzy number. If \( \Delta_3 = \Delta_4 = 0, \Delta_7 = \Delta_8 = 0, \Delta_{11} = \Delta_{12} = 0 \) and \( \lambda > 0 \) in (21), (25), (28), \( \hat{a} \) in (22), \( \hat{b} \) in (24) and \( \hat{c} \) in (36) will reduce to triangular fuzzy numbers \( \bar{a}^U = (a - \Delta_1, a + \Delta_2), \bar{b}^U = (b - \Delta_5, b + \Delta_6) \) and \( \bar{c}^U = (c - \Delta_9, c + \Delta_{10}) \), respectively.

From (36), we get the demand function \( P_\lambda = a - bx + cx^2 + \frac{1}{4}(\Delta_2 - \Delta_1) + \frac{1}{4}(\Delta_5 - \Delta_6)x + \frac{1}{4}(\Delta_{10} - \Delta_9)x^2 \) in the fuzzy sense.

Using (23), (26) and (29), we have \( 0 < \Delta_1 < a, 0 < \Delta_2, 0 < \Delta_5 < b, 0 < \Delta_6, 0 < \Delta_9 < c, 0 < \Delta_{10} \).

Let
\[
A_\lambda(\Delta_1, \Delta_2) = a + \frac{1}{4}(\Delta_2 - \Delta_1) > 0
\]
\[
B_\lambda(\Delta_5, \Delta_6) = b - \frac{1}{4}(\Delta_5 - \Delta_6) > 0
\]
\[
C_\lambda(\Delta_9, \Delta_{10}) = c + \frac{1}{4}(\Delta_{10} - \Delta_9) = \frac{1}{4}\Delta_{10} + \frac{1}{4}(c - \Delta_9) + \frac{3}{4}c > 0
\]
We then obtain \( P_\lambda = A_\lambda(\Delta_1, \Delta_2) - B_\lambda(\Delta_5, \Delta_6)x + C_\lambda(\Delta_9, \Delta_{10})x^2 \).

Using (41), we get the fuzzy revenue function \( R_\lambda(x) = A_\lambda(\Delta_1, \Delta_2)x - B_\lambda(\Delta_5, \Delta_6)x^2 + C_\lambda(\Delta_9, \Delta_{10})x^3 \).

B] In Table 3, cases 1, 3, 4, 6 and 8 occur when \( \lambda \) is smaller, the revenue is bigger and the fuzzy case revenue values are bigger than the crisp case \( R^*(x^*) = 90.27498 \).

C] In Table 3, cases 2, 5 and 7 occur when \( \lambda \) is smaller, the revenue is smaller and fuzzy case revenue values are smaller than the crisp case \( R^*(x^*) = 90.27498 \).

D] The proposed method in this study is more flexible and useful than the ones they have presented before. In this study we use level (\( \lambda, 1 \)) interval-valued fuzzy numbers to consider the fuzzy price and the fuzzy revenue. Due to the deriving membership function is a very tedious and difficult task, and for a perfect competitive market the demand function will vary according to the economic situation. Therefore, this paper is better than studied by others in reality applications.

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REFERENCES