FAST RATE ADAPTIVE OUTPUT FEEDBACK CONTROL
OF MULTI-RATE SAMPLED SYSTEMS WITH AN
ADAPTIVE OUTPUT ESTIMATOR

IKURO MIZUMOTO¹, YOTARO FUJIMOTO¹, NAO WATANABE¹ AND ZENTA IWAI²

¹Department of Intelligent Mechanical Systems
Kumamoto University
2-39-1 Kurokami, Kumamoto 860-8555, Japan
ikuro@gpo.kumamoto-u.ac.jp

²Kumamoto Prefectural College of Technology
4455-1 Haramizu, Kikuyou-mach, Kikuchi-gun, Kumamoto 869-1102, Japan
iwai@kumamoto-pct.ac.jp

Received April 2010; revised September 2010

ABSTRACT. This paper deals with the problem of designing an adaptive controller for multi-rate sampled systems with slow output sampling and fast input updating rates. By using an adaptive output estimator based on a reduced first order model of the controlled system, an adaptive output feedback controller with a fast-rate input updating period will be proposed for uncertain multi-rate systems.

Keywords: Multi-rate systems, Adaptive output feedback control, Adaptive output estimator

1. Introduction. There are many systems including several multi-rate sampled chemical and mechanical ones in which higher speed output sampling periods cannot be taken, even though the input actuating period can be taken at relatively high speeds [1,2,16]. Generally, in such systems, feedback controllers are designed according to a single rate based on the slow output sampling rate. In these cases, the control performance within the slow sampling period has not always been ensured. However, if the input can be updated at a faster rate by applying a multi-rate control strategy, one can expect an improvement in the control performance.

Recently, a novel adaptive output estimator that realizes output feedback at fast rates has been proposed [11,12]. In this method, the adaptive output estimator is designed for a reduced simple first order model of the controlled system and so has a relatively simple structure compared with common adaptive state observers. In order to observe the output response within the slow rate sampling periods, several kinds of output estimators and fast rate model identification methods have been proposed for multi-rate systems [3-7]. Most of them, however, are based on a full order model of the considered system. In practical systems, it might be difficult to determine the exact order of the considered systems and for higher order systems, the designed estimator may become complex. In [11,12], by using the proposed fast rate output estimator, a fast rate static output feedback control with a relatively simple controller structure has been proposed based on the system’s almost strictly positive real (ASPR) properties [13,14,17]. However, as has been pointed out, it is difficult to determine an appropriate feedback gain for most practical unknown systems.

In this paper, an adaptive control system design method, which can achieve a fast rate adaptive output feedback using adaptively estimated outputs and adaptively adjusted
feedback gain, is presented for unknown multi-rate sampled systems in which output
signals with a faster sampling period cannot be obtained. We will propose a fast-rate
adaptive output feedback control based on the ASPR property of the system with an
output estimator which can adaptively estimate the outputs at a fast-rate. Although we
may design the feedback gain by a constant high gain as in [11,12], one can automatically
obtain an appropriate feedback gain for uncertain controlled systems by adaptively ad-
justing the feedback gain so as to maintain the control performance with adequate control
input magnitude. The stability of the designed fast rate adaptive output feedback control
system with the estimated fast rate outputs is ensured based on the ASPR-ness of the
controlled system.

2. Problem Statement. We first consider the following linear, time-invariant, \( n_p \)-th
order multirate system in which the input is updated at a fast period of \( T \) and the output
is sampled at a slow period of \( nT \):

\[
x(k+n) = A^n x(k) + B u(k) \\
y(k) = c^T x(k)
\]  \( \text{(1)} \)

with

\[
B = \begin{bmatrix} A^{n-1} b & \cdots & Ab & b \end{bmatrix},
\]

\[
u(k) = [ u(k) \ u(k+1) \ \cdots \ u(k+n-1) ]^T,
\]

where \( A \in \mathbb{R}^{n \times n} \), \( b \in \mathbb{R}^{n} \) and \( c \in \mathbb{R}^{n} \) are system matrices of the following virtual
fast-rate system in which the input is updated at the fast period of \( T \) and the output is
also sampled at the same period of \( T \):

\[
x(k+1) = Ax(k) + bu(k) \\
y(k) = c^T x(k).
\]  \( \text{(2)} \)

It is assumed that this virtual fast-rate system satisfies the following assumptions.

Assumption 2.1. The relative degree of the virtual fast-rate system given in (2) is 1.

Assumption 2.2. The virtual fast-rate system given in (2) is minimum-phase.

Next, consider a bounded reference signal \( r(k) \) which satisfies the following relations:

\[
R(z) [ r(k) ] = 0, \quad R(z) = z^\rho + d_1 z^{\rho-1} + \cdots + d_{\rho}, \quad |r(k)| \leq \bar{r},
\]  \( \text{(3)} \)

where “\( z \)” denotes the forward-shift operator.

The objective of this paper is to design an output feedback controller with the output
\( y(k) \) that can track the reference signal \( r(k) \) at a fast sampling period \( T \).

To this end, we first consider introducing an internal model filter \( \tilde{G}(z) = \frac{\tilde{N}(z)}{\tilde{D}(z)} \) to the
virtual fast-rate system (2) as shown in Figure 1, where,

\[
\tilde{D}(z) = R(z)
\]  \( \text{(4)} \)

and \( \tilde{N}(z) \) is any stable polynomial of order \( \rho \).

\[
\begin{aligned}
\text{Figure 1. The virtual fast-rate system with an internal model filter}
\end{aligned}
\]
Then, the virtual error system with the internal model filter can be represented by the following form [15].

\[
\tilde{x}(k + 1) = \tilde{A}\tilde{x}(k) + \tilde{b}u_f(k) \\
e(k) = \tilde{y}(k) = \tilde{c}^T\tilde{x}(k)
\] (5)

where \(\tilde{x}(k)\) is the state vector of the controlled systems with the internal model filter and \(e(k) = y(k) - \gamma(k)\).

For this virtual error system, we impose the following assumptions.

**Assumption 2.3.** There exists for the virtual fast-rate system with the internal model filter given in (5), a known stable parallel feedforward compensator (PFC):

\[
x_a(k + 1) = A_a x_a(k) + b_a u_f(k) \\
e_a(k) = c_a^T x_a(k) + d_f u_f(k) = e(k) + y_f(k),
\] (7)

such that the resulting augmented system:

\[
x_a(k + 1) = A_a x_a(k) + b_a u_f(k) \\
e_a(k) = c_a^T x_a(k) + d_f u_f(k) = e(k) + y_f(k),
\] (7)

is ASPR [13,14].

Using the lifting technique [9,10], the augmented system given in (7) can be expressed by

\[
x_a(k + n) = A_a^n x_a(k) + B_a u_f(k) \\
e_a(k) = C_a x_a(k) + D_a u_f(k)
\] (8)

with a period \(nT\), where

\[
B_a = \begin{bmatrix} A_a^{n-1}b_a & \cdots & A_a b_a & b_a \end{bmatrix} \\
C_a = \begin{bmatrix} c_a^T \cdots c_a^T A_a \cdots c_a^T A_a^{n-1} \\ \vdots \\ c_a^T A_a^{n-1} \end{bmatrix}, \\
D_a = \begin{bmatrix} d_f & 0 & \cdots & 0 \\ c_a^T b_a & d_f & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ c_a^T A_a^{n-2} b_a & \cdots & c_a^T b_a & d_f \end{bmatrix}, \\
\]

\[
u_f(k) = \begin{bmatrix} u_f(k) \\ u_f(k + 1) \\ \vdots \\ u_f(k + n - 1) \end{bmatrix}, \\
e_a(k) = \begin{bmatrix} e_a(k) \\ e_a(k + 1) \\ \vdots \\ e_a(k + n - 1) \end{bmatrix}.
\]

3. **Adaptive Output Estimator.**

3.1. **Adaptive output estimator design.** In order to make it possible to realize the fast-rate output feedback, we first consider designing an estimator which estimates the output \(y(k + i), i = 1, 2, \cdots, n - 1\) within slow sampling points at a period of \(nT\). By Assumption 2.1, the virtual fast-rate system given in (2) can be represented by
the following canonical form with an appropriate non-singular transformation \( \xi(k) = [y(k) \ y^T(k)]^T = \Phi x(k) \) [8].

\[
\begin{align*}
y(k + 1) &= a_y y(k) + b_y u(k) + c^T_y \eta(k) \\
\eta(k + 1) &= A_y \eta(k) + b_y y(k).
\end{align*}
\]

(9)

Here, we impose the following assumption:

**Assumption 3.1.** Defining

\[
A_{yn} = A^n + \bar{A}_{yn}, \quad \bar{A}_{yn} = \sum_{i=0}^{n-1} b_{ni} c^T_i A^{n-i}_n,
\]

with

\[
b_{ni} = A_{ni-1} b_n + b_{ni-1} a_y, \quad A_{ni} = A_{ni-1} A_n + b_{ni-1} c^T_n,
\]

\[
A_{y0} = I, \quad b_{y0} = 0,
\]

\(A_{yn}\) is a stable matrix.

Since \(A_{yn}\) is a stable matrix from Assumption 3.1, there exists the positive symmetric matrices \(P_\eta = P_\eta^T > 0\) and \(Q_\eta = Q_\eta^T > 0\) such that the following Lyapunov equation is satisfied:

\[
A_{yn}^T P_\eta A_{yn} - P_\eta = -Q_\eta.
\]

(10)

Under this assumption, taking the form in (9) into consideration, \(y(k)\) can be expressed as:

\[
\begin{align*}
y(k) &= a_y \hat{y}(k - 1) + b_y u(k - 1) - a_y \hat{y}(k - 1) + b_y \sum_{i=1}^{n-1} a_{yi} u(k - i - 1) \\
&  + a_{yn} y(k - n) + c^T_{yn} \eta(k - n) \\
&= \theta^*_1 z_1(k) + \theta^*_2 z_2(k) + c^T_{yn} \eta(k - n)
\end{align*}
\]

(11)

where

\[
\begin{align*}
\theta^*_1 &= [a_y, b_y]^T, \quad \theta^*_2 = [a_{y1}, a_{yn}, b_y a_{y1}, \cdots, b_y a_{yn-1}]^T \\
z_1(k) &= \left[\hat{y}(k - 1), u(k - 1)\right]^T, \\
z_2(k) &= \left[-\hat{y}(k - 1), y(k - n), u(k - 2), \cdots, u(k - n)\right]^T.
\end{align*}
\]

(12)

\(\hat{y}(k - 1)\) is the estimated value of \(y(k - 1)\) to be given later, and \(a_{yj}, c_{ynj}\) are defined as follows:

\[
a_{yj} = a_{yj-1} a_y + c^T_{j-1} b_n, \quad a_{y0} = 1 \\
c^T_{ynj} = a_{ynj-1} c^T_n + c^T_{ynj-1} A_n, \quad c_{yn0} = 0
\]

Using this expression of \(y(k)\), under Assumption 3.1, we design the output estimator as follows [12]:

\[
\begin{align*}
\hat{y}(k) &= \theta^*_1 z_1(k) + \beta(k), \\
\hat{y}(k - i) &= \theta^*_1 (k - n) z_1(k - i), \quad (i = 1, \cdots, n - 2), \\
\hat{y}(k - n + 1) &= \theta^*_1 (k - n) \left(\begin{array}{c} y(k - n) \\
u(k - n) \end{array}\right), \\
\beta(k) &= \theta^*_2 z_2(k),
\end{align*}
\]

(14) (15) (16) (17)
where
\[
\theta_1(k) = \begin{bmatrix} \hat{a}_y(k) & \hat{b}_y(k) \end{bmatrix}^T, \\
\theta_2(k) = \begin{bmatrix} a_1(k) & a_n(k) & b_1(k) & \cdots & b_{n-1}(k) \end{bmatrix}^T
\]
\[
\theta_1(k) \text{ and } \theta_2(k) \text{ are estimated values of } \theta_1^* \text{ and } \theta_2^*, \text{ respectively.}
\]

The estimated parameters \( \theta_1(k) \) and \( \theta_2(k) \) are adaptively adjusted by the following parameter adjusting law:
\[
\begin{align*}
\theta_1(k) &= \hat{\sigma}\theta_1(k-n) - \hat{\sigma}\Gamma_1 z_1(k)\epsilon(k) + p(k) \quad (19) \\
\theta_2(k) &= \hat{\sigma}\theta_2(k-n) - \hat{\sigma}\Gamma_2 z_2(k)\epsilon(k) \quad (20)
\end{align*}
\]

with
\[
\Gamma_1 = \text{diag} [\gamma_a, \gamma_b] > 0,
\]
\[
\Gamma_2 = \text{diag} [\gamma_{a_1}, \gamma_{a_2}, \gamma_{b_1}, \gamma_{b_2}, \ldots, \gamma_{b_{n-1}}] > 0, \quad 0 < \hat{\sigma} < 1.
\]
\[
\epsilon(k) = \hat{y}(k) - y(k) \quad \text{and} \quad p(k) = \begin{bmatrix} p_a(k) & p_b(k) \end{bmatrix}^T
\]

are a parameter projection given by
\[
p_a(k) = \begin{cases} 0 & \text{if } a_y \leq f_a(k) \leq \bar{a}_y \\ \hat{\sigma}\gamma_a\hat{y}(k-1)\epsilon(k) & \text{otherwise} \end{cases} \quad (21)
\]
\[
p_b(k) = \begin{cases} 0 & \text{if } b_y \leq f_b(k) \leq \bar{b}_y \\ \hat{\sigma}\gamma_b u(k-1)\epsilon(k) & \text{otherwise} \end{cases} \quad (22)
\]
\[
f_a(k) = \hat{\sigma}\hat{a}_y(k-n) - \hat{\sigma}\gamma_a\hat{y}(k-1)\epsilon(k) \quad (23)
\]
\[
f_b(k) = \hat{\sigma}\hat{b}_y(k-n) - \hat{\sigma}\gamma_b u(k-1)\epsilon(k) \quad (24)
\]

where \( \bar{a}_y, \bar{b}_y \) and \( a_y, b_y \) are known upper and lower bounds of \( a_y, b_y \) respectively. Further, the estimate error \( \epsilon(k) \) used for calculating the adaptive adjusting law (19) and (20) can be generated using the available signals as follows:
\[
\epsilon(k) = (1 + \hat{\sigma}z_1^T(k)\Gamma_1 z_1(k) + \hat{\sigma}z_2^T(k)\Gamma_2 z_2(k))^{-1}
\]
\[
\times (\hat{\sigma}\theta_1^T(k-2)z_1(k) + \hat{\sigma}\theta_2^T(k-2)z_2(k) - y(k))
\]

For the designed adaptive output estimator, we have the following theorem concerning the boundedness of all the signals in the adaptive output estimator [12].

**Theorem 3.1.** For bounded inputs and outputs, all the signals in the adaptive output estimator given in (14) to (17) are uniformly bounded under Assumptions 2.1, 2.2 and 3.1.

4. **Control System Design.**

4.1. **Fast-rate adaptive output feedback controller.** Under Assumption 2.3, it has been clarified that a fast-rate output feedback controller using the adaptive output estimator can be designed as follows with an ideal feedback gain \( K^* = k^*I \)
\[
\mathbf{u}_f(k) = -K^* \begin{bmatrix} e_a(k) \\ \hat{e}_a(k+1) \\ \vdots \\ \hat{e}_a(k+n-1) \end{bmatrix} = -K^* \hat{\mathbf{e}}_a(k), \quad (25)
\]

where
\[
\hat{e}_a(k+i) = \hat{e}(k+i) + y_f(k+i) \\
\hat{e}(k+i) = \hat{y}(k+i) - r(k+i)
\]
\[
i = 1, 2, \ldots, n-1
\]
Unfortunately, since the controlled system is unknown, one can not design an ideal feedback gain, and since the output of the augmented error system \( e_a(k) \) or \( \hat{e}_a(k) \) consists of the control input \( u_f(k) \), the causality problem arises. To solve these problems, we consider an equivalent control input and adaptively adjusting the obtained equivalent feedback gain. The equivalent control input is obtained as:

\[
\mathbf{u}_f(k) = -\frac{k^*}{1 + d_xk^*} \begin{bmatrix}
    \hat{e}(k) \\
    \hat{e}(k + 1) \\
    \vdots \\
    \hat{e}(k + n - 1)
\end{bmatrix} = -\tilde{K}^*\hat{e}(k),
\]

where \( \tilde{K}^* = \frac{k^*}{1 + d_xk^*}I \) and

\[
\hat{e}(k) = y(k) - r(k) + \mathbf{c}_f^T \mathbf{x}_f(k) = c(k) + \mathbf{c}_f^T \mathbf{x}_f(k),
\]

and consider adjusting the equivalent feedback gain \( \tilde{K}^* \).

Consequently, the adaptive output feedback control is designed as

\[
\mathbf{u}_f(k) = -\begin{bmatrix}
    \hat{\theta}_p(k) & 0 & \cdots & 0 \\
    0 & \hat{\theta}_p(k + 1) & \ddots & \vdots \\
    \vdots & \ddots & \ddots & 0 \\
    0 & \cdots & 0 & \hat{\theta}_p(k + n - 1)
\end{bmatrix} \begin{bmatrix}
    \hat{e}(k) \\
    \hat{e}(k + 1) \\
    \vdots \\
    \hat{e}(k + n - 1)
\end{bmatrix}
\]

\[
= -\tilde{\Theta}_p(k)\hat{e}(k)
\]

with the following adaptive adjusting laws:

\[
\hat{\theta}_p(k) = \hat{\sigma}_p\hat{\theta}_p(k - 1) + \hat{\sigma}_p\gamma \hat{e}_a(k)e_a(k) + p(k)
\]

\[
\hat{\theta}_p(k + i) = \hat{\sigma}_p\hat{\theta}_p(k + i - 1) + \hat{\sigma}_p\gamma \hat{\hat{e}}_a(k + i)\hat{e}_a(k + i) + p(k + i)
\]

\[
\sigma_p \geq 0, \quad \hat{\sigma}_p = \frac{1}{1 + \sigma_p}, \quad \gamma > 0, \quad i = 1, \ldots, n - 1
\]

where \( p(k) \) is a parameter projection. Since \( \hat{\sigma}_p < \frac{1}{\sigma_f} \) for \( \sigma_f < \infty \), \( p(k) \) is designed for any lower bound \( \hat{\theta}_{\min} \) as:

\[
p(k) = \begin{cases}
0 & \text{if } \hat{\theta}_{\min} \leq f_p(k) < \frac{1}{\sigma_f} \\
-\hat{\sigma}_p\gamma \hat{e}_a(k)e_a(k) & \text{otherwise}
\end{cases}
\]

\[
p(k + i) = \begin{cases}
0 & \text{if } \hat{\theta}_{\min} \leq f_p(k + i) < \frac{1}{\sigma_f} \\
-\hat{\sigma}_p\gamma \hat{\hat{e}}_a(k + i)\hat{e}_a(k + i) & \text{otherwise}
\end{cases}
\]

\[
(i = 1, 2, \ldots, n - 1)
\]

\[
f_p(k) = \sigma_p\hat{\theta}_p(k - 1) + \sigma_p\gamma \hat{e}_a(k)e_a(k)
\]

\[
f_p(k + i) = \sigma_p\hat{\theta}_p(k + i - 1) + \sigma_p\gamma \hat{\hat{e}}_a(k + i)\hat{e}_a(k + i)
\]

4.2. Stability analysis. Since the control input can be expressed by

\[
\mathbf{u}(k) = -\tilde{\Theta}_p(k)\hat{e}_a(k) = -\tilde{K}^*\hat{e}_a(k) + \mathbf{v}(k)
\]

\[
\mathbf{v}(k) = -\Delta\tilde{\Theta}_p(k)\{\hat{e}_a(k) + \epsilon(k)\} - \tilde{K}^*\epsilon(k)
\]

\[
\epsilon(k) = \hat{\epsilon}_a(k) - \hat{e}_a(k)
\]

\[
\Delta\tilde{\Theta}_p(k) = \tilde{\Theta}_p(k) - \tilde{K}^*
\]
the obtained closed loop system
\[ x_a(k + n) = A_c x_a(k) + B_c \nu(k) \]
\[ e_a(k) = C_c x_a(k) + D_c \nu(k) \] (34)
should be SPR from Assumption 2.3 with an ideal feedback gain \( K^* \) which renders the augmented ASPR system (8) SPR. Thus, for the closed loop system \((A_c, B_c, C_c, D_c)\), there exist a symmetric positive definite matrices \( P, Q \) and appropriate matrices \( L, W \) such that the following K-Y Lemma is satisfied.

\[
\begin{align*}
A_c^T PA_c - P &= -LL^T - Q \\
A_c^T PB_c &= C_c^T - LW \\
B_c^T PB_c &= D_c + D_c^T - WTW
\end{align*}
\] (35)

Now, consider the following positive definite function \( V \):
\[
V(k) = V_a(k) + V_i(k) \tag{36}
\]
\[
V_a(k) = V_1(k) + V_2(k) + V_3(k) \tag{37}
\]
\[
V_1(k) = \rho_1 \epsilon^2(k), \quad V_2(k) = \rho_2 \eta^T(k) P_1 \eta(k)
\]
\[
V_3(k) = \sum_{i=1}^{2} \sigma \Delta \theta_i^T(k) \Gamma_i^{-1} \Delta \theta_i(k)
\]
\[
V_4(k) = V_4(k) + V_5(k), \quad \rho_4 > 0,
\]
\[
V_5(k) = \rho_5 \frac{\rho_5}{(1 + \sigma_p^2) \Delta \tilde{\theta}_p^2(k - 1)}, \quad \rho_5 > 0
\] (38)

Taking the difference \( \Delta V_a(k) = V_a(k) - V_a(k - n) \), \( \Delta V_a(k) \) can be evaluated by [12].
\[
\Delta V_a(k) \leq -\alpha_{\epsilon} |\epsilon(k)|^2 - \rho_1 |\epsilon(k - n)|^2 - \alpha_\eta \|\eta(k - n)\|^2 + \alpha_y |y(k - n)|^2
\]
\[
+ \sum_{i=1}^{n} \alpha_{ui} |u(k - n - i)|^2 - \alpha_{\theta} \|\Delta \theta(k - n)\|^2
\]
\[
- \alpha_{\theta2} \|\Delta \theta(k)\|^2 + \frac{1}{\delta_31} (1 - \rho_3)^2 \sigma^2 \|\Gamma^{1/2} \|\Theta^*\|^2
\] (39)

with appropriate positive constants, \( \alpha_{\epsilon}, \alpha_{\eta}, \alpha_{ui}, \alpha_{\theta}, \alpha_{\theta2} \) and \( \delta_31 \).

Further from (34) and K-Y Lemma (35), the difference \( \Delta V_4(k) = V_4(k) - V_4(k - n) \) can be evaluated as:
\[
\begin{align*}
\Delta V_4(k) &= -\rho_4 x_a^T(k - n) Q x_a(k - n) - \rho_4 \left\{ x_a^T(k - n) L + \nu^T(k - n) W^T \right\} x_a^T(k - n) L \\
& \quad + \nu^T(k - n) W^T \right\} + 2 \rho_3 \tilde{\epsilon}_a^T(k - n) \tilde{\epsilon}_a(k - n)
\end{align*}
\]
\[
= -\rho_4 x_a^T(k - n) Q x_a(k - n) - \rho_4 \left\{ x_a^T(k - n) L + \nu^T(k - n) W^T \right\} x_a^T(k - n) L \\
+ \nu^T(k - n) W^T \right\} + 2 \rho_3 \tilde{\epsilon}_a^T(k - n) \left\{ -\Delta \tilde{\theta}(k - n) \tilde{\epsilon}_a(k - n) - \tilde{K}^* \tilde{\epsilon}_a(k - n) \right\}
\]
\[
\leq -\rho_4 \lambda_{\min}(Q) \|x_a(k - n)\|^2 - \rho_4 \|x_a^T(k - n) L + \nu^T(k - n) W^T \|^2
\]
\[
- 2 \rho_3 \tilde{\epsilon}_a^T(k - n) \Delta \tilde{\theta}(k - n) \tilde{\epsilon}_a(k - n) - 2 \rho_4 \sum_{i=1}^{n-1} \{ \tilde{K}^* \tilde{\epsilon}_a(k - n + i) \tilde{\epsilon}_a(k - n + i) - \tilde{K}^* \tilde{\epsilon}_a(k - n + i) \}
\] (40)

\[
- \Delta \tilde{\theta}(k - n + i) \epsilon_a(k - n + i) \tilde{\epsilon}_a(k - n + i) - \tilde{K}^* \epsilon_a(k - n + i) \tilde{\epsilon}_a(k - n + i)
\]
Here, taking the fact that $\hat{\theta}_p(k) < \frac{1}{d_f}$, it is easy to show that $u(k - n + j)$ can be evaluated.

$$|u(k - n + j)| \leq \alpha_{ux(j)}\|x_a(k - n)\| + \alpha_{ur(j)}\bar{r} \tag{41}$$

with appropriate positive constants $\alpha_{ux(j)}$ and $\alpha_{ur(j)}$. Further, since

$$e_a(k - n) = c_a^T x_a(k - n) + d_f u_f(k - n) \tag{42}$$
$$u_f(k - n) = -\hat{\theta}_p(k - n)\hat{e}_a(k - n) = -\theta_p(k - n)\hat{e}_a(k - n) \tag{43}$$

and from (15), defining

$$\theta_p(k) = \frac{\hat{\theta}_p(k)}{1 - d_f\hat{\theta}_p(k)} \tag{44}$$

we have

$$|e_a(k - n + i)| \leq \frac{1}{1 + d_f\theta_p(k - n + i)}\alpha_{xi}\|x_a(k - n)\| + \frac{1}{d_f}|e(k - n + i)| \tag{45}$$

and

$$|\hat{e}_a(k - n + i)| \leq \frac{1}{1 + d_f\theta_p(k - n + i)}\alpha_{xi}\|x_a(k - n)\| + \frac{1}{1 + d_f\theta_p(k - n + i)}\alpha_{ri}\bar{r} \tag{46}$$

where

$$\alpha_{xi} = \|c_a\| \max \left\{\left\|\prod_{j=1}^{i} \left(A_a - b_a\tilde{\theta}_p(k - n + j - 1)c_a^T\right)\right\|\right\}$$

and

$$\alpha_{xi} = \tilde{a}_g\|\hat{e}\| + \sum_{j=1}^{i} \tilde{a}_y^{i-j}\tilde{b}_y\alpha_{ux(j-1)} + \|c_f\|\|A_f\| + \|c_f\|\|A_f^{i-1}\|\|b_f\|\frac{1}{d_f}\|c_a\|$$
$$+ \sum_{j=2}^{i} \|c_f\|\|A_f^{i-1}\|\|b_f\|\frac{1}{d_f}\alpha_{x(j-1)}$$
$$\alpha_{ri} = \sum_{j=1}^{i} \tilde{a}_y^{i-j}\tilde{b}_y\alpha_{ur(j-1)} + \tilde{a}_g^i + 1 + \sum_{j=2}^{i} \|c_f\|\|A_f^{i-j}\|\|b_f\|\frac{1}{d_f}\alpha_{r(j-1)}$$

Therefore, from (45) and (46), $e_a(k - n + i)$ can be evaluated as:

$$|e_a(k - n + i)| \leq \frac{1}{1 + d_f\theta_p(k - n + i)}(\alpha_{xi} + \alpha_{xi})\|x_a(k - n)\| + \frac{1}{1 + d_f\theta_p(k - n + i)}\alpha_{ri}\bar{r} \tag{47}$$

Consequently, we have

$$\left|\left\{k^*\hat{e}_a(k - n + i)\hat{e}_a(k - n + i) - e_a(k - n + i)\Delta\hat{\theta}_p(k - n + i)\hat{e}_a(k - n + i) - \hat{k}^*e_a(k - n + i)\hat{e}_a(k - n + i)\right\}\right|$$
$$\leq \alpha_{b1i}\|x_a(k - n)\|^2 + \alpha_{b2i}\|x_a(k - n)\| + \alpha_{b3i}\bar{r}^2 \tag{48}$$
where

\[
\alpha_{b1i} = \frac{\bar{k}^*}{(1 + d_f \theta_{p_{\min}})^2} \alpha_{xi}(\tilde{\alpha}_{xi} + \alpha_{xi}) + \frac{1}{d_f} \left\{ \frac{1}{1 + d_f \theta_{p_{\min}}} \alpha_{xi} \leq 4 f_t (\tilde{\alpha}_{xi} + \alpha_{xi}) + \frac{1}{(1 + d_f \theta_{p_{\min}})^2} (\tilde{\alpha}_{xi} + \alpha_{xi})^2 \right\}
\]

\[
\alpha_{b2i} = \frac{\bar{k}^*}{(1 + d_f \theta_{p_{\min}})^2} \alpha_{ri} + \frac{1}{d_f} \left\{ \frac{1}{1 + d_f \theta_{p_{\min}}} \alpha_{xi} \alpha_{ri} + \frac{2}{(1 + d_f \theta_{p_{\min}})^2} \alpha_{ri} (\tilde{\alpha}_{xi} + \alpha_{xi}) \right\}
\]

\[
\alpha_{b3i} = \frac{1}{d_f (1 + d_f \theta_{p_{\min}}^2)} \alpha_{ri}^2 \tilde{r}^2
\]

with

\[
\theta_{p_{\min}} = \min_k \left\{ \frac{\tilde{\theta}_p(k)}{1 - d_f \theta_p(k)} \right\} = \frac{\tilde{\theta}_{\min}}{1 - d_f \theta_{\min}}
\]

Furthermore, it follows for any positive constants \(\delta_{b1i}\) that

\[
-\delta_{b1i} \|x_a(k-n)\|^2 + \alpha_{b2i} \tilde{r} \|x_a(k-n)\| \leq \frac{\alpha_{b2i}^2 \tilde{r}^2}{4 \delta_{b1i}}
\]

(49)

we have

\[
\left| \left\{ \tilde{k}^* \hat{e}_a(k-n+i) \hat{e}_a(k-n+i) \right\} - \tilde{k}^* \hat{e}_a(k-n+i) \hat{e}_a(k-n+i) \right| 
\]

\[
\leq (\delta_{b1i} + \alpha_{b1i}) \|x_a(k-n)\|^2 + \left( \alpha_{b3i} + \frac{\alpha_{b2i}^2}{4 \delta_{b1i}} \right) \tilde{r}^2
\]

(50)

Finally, \(\Delta V_4(k)\) can be evaluated as:

\[
\Delta V_4(k) \leq - \left\{ \rho_4 \lambda_{\min} (Q) - 2 \rho_4 \sum_{i=1}^{n-1} (\delta_{b1i} + \alpha_{b1i}) \right\} \|x_a(k-n)\|^2
\]

\[
+ 2 \rho_4 \sum_{i=1}^{n-1} \left( \alpha_{b3i} + \frac{\alpha_{b2i}^2}{4 \delta_{b1i}} \right) \tilde{r}^2 - \rho_4 \|x_a^T(k-n)L + \tilde{g}(k-n)W^T\|^2
\]

\[
- 2 \rho_4 \tilde{e}_a^T(k-n) \Delta \tilde{\Theta}_p(k-n) \tilde{e}_a(k-n)
\]

(51)

It is noted that sufficiently small \(\tilde{\alpha}_{xi}, \alpha_{xi}\) and \(\alpha_{ri}\) can be obtained by setting appropriate large \(\theta_{p_{\min}}\).

Next, defining \(\Delta V_5(k) = V_5(k) - V_5(k-n)\), \(\Delta V_5(k)\) can be represented by

\[
\Delta V_5(k) = \frac{\rho_4}{(1 + \sigma_p)} \sum_{i=0}^{n-1} \Delta V_{5i}(k)
\]

(52)

\[
\Delta V_{5i}(k) = \Delta \tilde{\theta}_p^2(k-n+i) - \Delta \tilde{\theta}_p^2(k-n+i-1)
\]
In the case where $\theta_{\min} \leq f_{p}(k) < \frac{1}{\delta_{\theta}}$, we have from (30) that

$$\Delta V_{50}(k) \leq - \left\{ (1 + \sigma_p)^2 - 1 - \delta_0 \right\} \Delta \hat{\theta}_p(k - n) + \frac{\sigma_p^2 (1 + \sigma_p)^2}{\delta_\theta} \tilde{k}^2 + 2 (1 + \sigma_p) \gamma \epsilon_a(k - n) \Delta \hat{\theta}_p(k - n) \epsilon_a(k - n)$$

(53)

with any positive constant $\delta_\theta$. Otherwise, $\Delta V_{50}(k)$ can be evaluated by

$$\Delta V_{50}(k) \leq - \left\{ (1 + \sigma_p)^2 - 1 - \delta_0 \right\} \Delta \hat{\theta}_p(k - n) - \sigma_p^2 \tilde{k}^2 + \frac{\sigma_p^2 (1 + \sigma_p)^2}{\delta_\theta} \tilde{k}^2$$

(54)

Thus, $\Delta V_{50}(k)$ can be evaluated as:

$$\Delta V_{50}(k) \leq - \left\{ (1 + \sigma_p)^2 - 1 - \delta_0 \right\} \Delta \hat{\theta}_p(k - n) + \frac{\sigma_p^2 (1 + \sigma_p)^2}{\delta_\theta} \tilde{k}^2 + 2 (1 + \sigma_p) \gamma \epsilon_a(k - n) \Delta \hat{\theta}_p(k - n) \epsilon_a(k - n)$$

(55)

Similarly, with regard to $\Delta V_{5i}(k)$, we have

$$\Delta V_{5i}(k) \leq - \left\{ (1 + \sigma_p)^2 - 1 - \delta_0 \right\} \Delta \hat{\theta}_p(k - n + i) + \frac{\sigma_p^2 (1 + \sigma_p)^2}{\delta_\theta} \tilde{k}^2 + 2 (1 + \sigma_p) \gamma \epsilon_a(k - n + i) \Delta \hat{\theta}_p(k - n + i) \epsilon_a(k - n + i)$$

(56)

Therefore, we obtain

$$\sum_{i=0}^{n-1} \Delta V_{5i}(k) \leq - \left\{ (1 + \sigma_p)^2 - 1 - \delta_0 \right\} \| \Delta \hat{\Theta}_p(k - n) \|^2 + n \frac{\sigma_p^2 (1 + \sigma_p)^2}{\delta_\theta} \tilde{k}^2 + 2 (1 + \sigma_p) \gamma \epsilon_a(k - n) \Delta \hat{\theta}_p(k - n) \epsilon_a(k - n)$$

(57)

Consequently, $\Delta V_{6}(k)$ can be evaluated from (51) and (57) as:

$$\Delta V_{6}(k) = \Delta V_{4}(k) + \Delta V_{5}(k)$$

$$\leq - \left\{ \rho_4 \lambda_{\min}[Q] - 2 \rho_4 \sum_{i=1}^{n-1} (\delta_{b1i} + \alpha_{b1i}) \right\} \| x_a(k - n) \|^2 + 2 \rho_4 \sum_{i=1}^{n-1} \left( \alpha_{b3i} + \frac{\alpha_{b2i}}{4 \delta_{b1i}} \right) \tilde{r}^2$$

$$- \rho_4 \| x_a^T(k - n) L + \nu(k - n) W T \|^2$$

$$- \frac{\rho_4}{1 + \sigma_p} \left\{ (1 + \sigma_p)^2 - 1 - \delta_0 \right\} \| \Delta \hat{\Theta}_p(k - n) \|^2 + n \rho_4 \frac{\sigma_p^2 (1 + \sigma_p)}{\gamma \delta_\theta} \tilde{k}^2$$

(58)

Finally, it follows that

$$|y(k - n)|^2 \leq \| \hat{c} \|^2 \| x_a(k - n) \|^2 + 2 \| \hat{c} \| \| x_a(k - n) \| \tilde{r} + \tilde{r}^2$$

(59)

$$|u(k - n + i)|^2 \leq \alpha^2_{uxi} \| x_a(k - n) \|^2 + 2 \alpha_{ux}(\alpha_{ur(i)} \tilde{r} \| x_a(k - n) \|) + \alpha^2_{ur(i)} \tilde{r}^2$$

(60)

the difference $\Delta V(k)$ can be evaluated as:

$$\Delta V(k) \leq -\alpha_{ux} \| x_a(k - n) \|^2 - \rho_4 \| x_a(k - n) \|\|^2 - \alpha_p \| \eta(k - n) \|^2$$

$$- \alpha_{\theta 1} \| \Delta \hat{\Theta}(k - n) \|^2 - \alpha_{\theta 2} \| \Delta \hat{\Theta}(k) \|^2 - \alpha_{xa} \| x_a(k - n) \|^2$$

$$- \alpha_{\theta p} \| \Delta \hat{\Theta}_p(k - n) \|^2 - \rho_4 \| x_a^T(k - n) L + \nu(k - n) W T \|^2 + R$$

(61)
with any positive constant $\delta_1$, where

$$\alpha_{xa} = \left\{ \rho_4 \lambda_{\min}[Q] - 2 \rho_4 \sum_{i=1}^{n-1} (\delta_{b_{1i}} + \alpha_{b_{1i}}) - \alpha_y \| \hat{e} \|^2 - \sum_{i=1}^{n} \alpha_{ui} \alpha_{auri}^2 - \delta_1 \right\}$$

$$\alpha_p = \frac{\rho_4}{(1 + \sigma_p) \gamma} \left\{ (1 + \sigma_p)^2 - 1 - \delta_0 \right\}$$

$$R = \left\{ \alpha_y \hat{r}^2 + \sum_{i=1}^{n} \alpha_{ui} \alpha_{auri}^2 + 2 \rho_4 \sum_{i=1}^{n-1} \left( \alpha_{b_{3i}} + \frac{\alpha_{b_{2i}}}{4 \delta_{b_{1i}}} \right) + \frac{1}{\delta_1} \left( \alpha_y \| \hat{e} \| + \sum_{i=1}^{n-1} \alpha_{ui} \alpha_{auri} \alpha_{auri} \right)^2 \right\} \hat{r}^2$$

$$+ \frac{(1 - \rho_3)^2}{\delta_{31}} \| \Gamma^{-1} \|^2 \| \theta^* \|^2 + n \rho_4 \sigma_p^2 (1 + \sigma_p) \hat{k}^2$$

Consequently, setting a sufficiently large $\tilde{\theta}_{\min} < \frac{1}{\delta_{19}}$ and considering a sufficiently large $\rho_4$ and sufficiently small $\delta_1$ and $\delta_0$ such that $\alpha_{xa} > 0$ and $\alpha_p > 0$, such a constants certainly exist, we have for $\Delta V(k)$ that

$$\Delta V(k) \leq -\alpha V(k) + R, \quad \alpha > 0$$

and thus, we can conclude that the all the signals in the resulting control system are bounded provided that we set an appropriately large $\tilde{\theta}_{\min} < \frac{1}{\delta_{19}}$ in the controller.

**Theorem 4.1.** Under Assumptions 2.1, 2.2, 2.3 and 3.1, all the signals in the resulting control system with the control input given in (29) to (31) and the adaptive output estimator in (14) to (20) are bounded provided that we set an appropriately large $\tilde{\theta}_{\min} < \frac{1}{\delta_{19}}$ in the controller.

Finally, the overall control system design follows the following steps.

**Step 1:** Design an adaptive output estimator for the system which satisfies Assumptions 2.1, 2.2 and 2.3 as in (14) to (20). If the system does not satisfy assumptions, then design a PFC for adaptive output estimate so as to render the augmented system satisfying the assumptions. This PFC is only used for output estimation (See Section 5 for details.), and the estimated output can be obtained by taking the PFC output from the estimated augmented systems output. The conditions in the assumptions can be checked using approximated system model.

**Step 2:** Design a PFC for the fast rate adaptive output feedback so as to render the augmented fast rate system ASPR.

**Step 3:** Design the adaptive output feedback controller as in (29) to (31).

5. **Validation through Numerical Simulation.** In this section, we confirm the effectiveness of the proposed method through numerical simulations on the liquid level of a SISO three tank process. A simple configuration of the three tank liquid level process is illustrated in Figure 2, in which the output $y_1(t)$ is the liquid level of tank 1 and the control input $u_1(t)$ is the flow rate of the Pump 1 applied to tank 2. It is assumed that a constant flow $u_2(t)$ is supplied to tank 3 from the pump 2. In this simulation, we also assume that the output $y_1(t)$ is sampled at a period of $10T = 100$ [sec] but the input signal $u_1(t)$ can be updated through a zero-order hold at a fast period $T = 10$ [sec].

In the simulation, the system model is given by

$$G_p(z) = \frac{-0.02417z^4 + 0.04422z^3 + 0.04241z^2 + 0.001802z + 5.11 \times 10^{-5}}{z^5 - 2.414z^4 + 1.846z^3 - 0.4091z^2 - 0.02117z - 0.0006621}$$

(63)
and it is supposed that this system is unknown but the approximated fast rate model of the tank system is known as:

\[
G_p^*(z) = \frac{-0.01933z^4 + 0.03538z^3 + 0.03393z^2 + 0.001442z + 4.088 \times 10^{-5}}{z^5 - 2.414z^4 + 1.846z^3 - 0.4091z^2 - 0.02117z - 0.0006621} \tag{64}
\]

This approximated value was given such that

\[
G_p^*(z) = 0.8 \times G_p(z)
\]

Using this approximated model, a PFC which renders the augmented fast rate system ASPR was designed as follows according to the model based PFC design scheme.

\[
G_{PFC1}(z) = G_{ASPR}(z) - G_p^*(z) \tag{65}
\]

with an ideal ASPR model of

\[
G_{ASPR}(z) = \frac{18.52z - 15.99}{z - 0.9636} \tag{66}
\]

Furthermore, since this tank model is not minimum-phase, we introduced additional PFC for output estimation. The PFC was designed as follows in order to render a minimum-phase augmented system with a relative degree of 1.

\[
G_{PFC2}(z) = \frac{0.3765z^2 - 0.4934z + 0.001407}{z^3 - 1.915z^2 + 0.918z - 0.000916} \tag{67}
\]

The overall block diagram of the obtained control system is as shown in Figure 3.

In the simulations, the design parameters in the output estimator and the adaptive controller are set as:

\[
\Gamma_1 = \text{diag}[10, 100000], \quad \Gamma_2 = \text{diag}[0.01, 0.01, \cdots, 0.01], \quad \sigma = 0.01
\]

\[
\gamma_p = 10^{-6}, \quad \sigma_p = 0.5 \times 10^{-3}
\]

and we set in the parameter projection that

\[
0 \leq a_y \leq 500, \quad -500 \leq b_y \leq 500.
\]

The internal model was given by

\[
\bar{G}(z) = \frac{z}{z - 1}
\]

Figure 4 shows the simulation results. The output of the liquid level quickly and accurately reaches the set point and a good control performance with accurate output.
estimation is shown. Figure 5 shows the results for set point changes. We can see that a steady control performance was maintained for each set point.

In order to establish the effectiveness of the proposed control scheme, we tried to control the tank process at a slow rate output feedback control based on ASPR-ness with a PFC. In the slow rate controller, we consider two types of PFCs and these are designed by

\[ G_{PFC_{slow_1}}(z) = G_{ASPR_{slow_1}}(z) - G_p^*(z) \]  

(68)
The first one was designed so as to obtain an approximately equal rising time as the fast rate results and the second one was designed so as to have no overshoot in the slow rate control.

Figure 6 shows a comparison with the results of the proposed control and slow rate control. The control performance for the slow rate control with \( G_{PFC_{slow1}}(z) \) deteriorates with overshooting and oscillation which led a long settling time, the result with \( G_{PFC_{slow2}}(z) \) shows that there is no overshoot but the rising and settling times became longer. The result with the proposed method shows a better control performance.

Figures 7 to 9 show comparisons with the results of the proposed adaptive control method and the control method with fixed feedback gains. As seen in Figure 7, if we set a higher feedback gain, a good result as well as the result of the proposed method will be obtained. However, the magnitude of the control input will become larger according to the largeness of the feedback gain. If we set smaller gains as the results in Figures 8 and 9, although we can reduce the magnitude of the input, the control performance deteriorated compared with the proposed adaptive method. It can be seen that the adaptive method

$$ G_{ASPR_{slow1}}(z) = \frac{18.52z + 2.975}{z - 0.6905} $$

$$ G_{ASPR_{slow2}}(z) = \frac{166.7z - 99.7}{z - 0.03567} $$
can automatically obtain an appropriate feedback gain for uncertain controlled systems so as to maintain the control performance with adequate control input magnitude.

6. **Conclusions.** In this paper, an adaptive output feedback controller with a fast-rate input updating period was proposed for unknown multi-rate sampled systems in which the fast rate output signals cannot be obtained. In the proposed method, one can design a fast-rate adaptive output feedback control system using an adaptive output estimator based on a reduced first order model of the controlled system. The effectiveness of the proposed method was confirmed through numerical simulations for a three tank process.

**REFERENCES**


Figure 7. Comparison with the results of the proposed method and the method with constant feedback gain $k = 0.06$


Figure 8. Comparison with the results of the proposed method and the method with constant feedback gain $k = 0.01$


Figure 9. Comparison with the results of the proposed method and the method with constant feedback gain $k = 0.001$


