FEEDBACK STABILIZATION FOR NONLINEAR AFFINE STOCHASTIC SYSTEMS

WEIHAI ZHANG\textsuperscript{1}, BOR-SEN CHEN\textsuperscript{2} AND ZHIGUO YAN\textsuperscript{3}

\textsuperscript{1}Shandong Key Laboratory of Robotics and Intelligent Technology
College of Information and Electrical Engineering
Shandong University of Science and Technology
Economic and Technical Development Zone, Qingdao 266510, P. R. China
w_hzhang@163.com

\textsuperscript{2}Department of Electrical Engineering
National Tsing Hua University
No. 101, Sec. 2, Kuang-Fu Road, Hsin Chu 30013, Taiwan
bschen@ee.nthu.edu.tw

\textsuperscript{3}School of Electrical Engineering and Automation
Tianjin University
No. 92, Weijin Road, Tianjin 300072, P. R. China
yanzg500@sina.com

Received May 2010; revised September 2010

Abstract. This paper studies the state feedback stabilization of nonlinear affine stochastic systems. Sufficient conditions for the local and global asymptotic stabilization in probability are presented by means of Hamilton-Jacobi inequalities. As corollaries, some previous results are improved. For a class of nonlinear stochastic perturbed systems, the output feedback stabilization problem is also considered. One example is given to show the validity of our main results.

Keywords: Asymptotic stability in probability, Hamilton-Jacobi inequality, Feedback stabilization, Linear matrix inequality

1. Introduction. Stochastic stability for Itô systems has been studied for a few decades; see [1-5] for the discussion of stochastic Lyapunov stability and [6] for the work of Lyapunov exponent of linear stochastic systems. For a stochastic control system, stochastic stabilization is a problem that should be first considered in the system analysis and design. It should be noted that the study on stochastic stabilization has received a great deal of attention and has become a popular research issue in recent years; see [7-15]. How to stabilize an unstable stochastic system via state or output feedback is without doubt a very valuable topic in practice. [8, 9, 14] discussed the mean square stabilization of linear stochastic control systems. In recent years, the stabilization of nonlinear stochastic systems has been studied extensively. By means of a positive symmetric solution of a class of generalized algebraic Riccati equations (GAREs), one can construct a quadratic Lyapunov function to present some sufficient conditions for local stabilization of bilinear or general nonlinear stochastic systems with state-dependent noise [7, 11]. By combining the stochastic Lyapunov technique with stochastic version of LaSalle’s invariance principle [3], the global stabilization of affine stochastic systems was also discussed based on the assumption that the unforced systems are stable in probability [12], and the $H_\infty$ stabilization was studied in [13]. For the case when the system state is not completely available, [15] first initiated the study on output feedback stabilization of a class of nonlinear stochastic
systems using the backstepping method. [16-19] were about the stabilization and optimal control of stochastic hybrid systems. There are some recent references studying predictive control, output regulation, tracking performance and global stability of nonlinear systems and neural networks; we refer the reader to [20-23].

This paper mainly focuses on the feedback stabilization of nonlinear stochastic systems. Based on Hamilton-Jacobi inequalities (HJIs), for a class of nonlinear affine stochastic systems, a sufficient condition for local asymptotic state feedback stabilization is given; see Theorem 3.1. As corollaries, some results of [7, 11] on stochastic bilinear systems are improved and extended to slightly more general models. There is no any difficulty that extends Theorem 3.1 to nonlinear stochastic time-varying systems; see Theorem 3.2. Theorems 3.1 and 3.2 are very general in theoretical aspect, which transform the stabilization problem of nonlinear affine stochastic systems into solving HJIs.

When the system state is not completely measurable, we also consider the dynamic output feedback stabilization for a class of nonlinear perturbed stochastic systems. Two results concerning the local and global asymptotic stabilization are obtained via linear matrix inequalities (LMIs), by which an output feedback stabilizing controller can be easily designed. To test the validity of our main results, an example is given to illustrate how to design a state feedback stabilizing controller in the last of this paper.

For convenience, we use the matrix notations such as $A^t, A \geq 0, A > 0$ and $I$ to represent the conventional senses as in [8]. $C^2(U)$: the class of all real-valued functions $V(x)$ defined on $x \in U$ which are continuously twice differentiable in $x \in U$. $C_0^{1,2}([0, \infty) \times U)$: the class of functions $V(t, x)$ defined on $[0, \infty) \times U$ which are continuously twice differentiable in $x \in U$, and once continuously differentiable in $t \in [0, \infty)$. $|x|$: the norm of a vector or a matrix $x$.

2. Preliminaries. We first consider the following nonlinear affine stochastic system governed by Itô-type stochastic differential equation.

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{dx(t)}{dt} = (f(x(t)) + g(x(t))u(t))dt + (h(x(t)) + l(x(t))u(t))dw(t) \\
x(0) = x_0 \in \mathbb{R}^n
\end{array} \right.
\]

(1)

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ ($1 \leq m \leq n$) are respectively called the system state and control input. Without loss of generality, we suppose $w(t)$ to be a standard one-dimensional Wiener process defined on the filtered probability space $(\Omega, F, \{F_t\}_{t \geq 0}, P)$, where $F_t$ is the natural filter, i.e., $F_t := \sigma\{w(s) : 0 \leq s \leq t\}$. $u(\cdot)$ is said to be an admissible control if $u(\cdot)$ is an $F_t$-adapted stochastic process. $f(x), g(x), h(x)$ and $l(x)$ are vector or matrix-valued measurable functions with $f(0) = h(0) = 0$. As it is well known in stochastic differential equation theory [1, 5], in order to guarantee the existence and uniqueness of strong solutions of an unforced stochastic Itô systems ($u \equiv 0$) on any finite interval $[0, T]$, $f(x)$ and $h(x)$ are assumed to satisfy some definite conditions such as Lipschitz and linear growth conditions. Similarly, for stochastic control system (1), we also assume that $f(x), g(x), h(x), l(x)$ and $u$ satisfy definite conditions such that system (1) has a unique strong solution [24]. Under above assumptions, $x \equiv 0$ is an equilibrium point of system (1).

Now, we recall some essential definitions on stochastic stability cited from [1, 5].

**Definition 2.1.** Consider a stochastic unforced affine system

\[
\frac{dx(t)}{dt} = f(x(t))dt + h(x(t))dw(t), \quad f(0) = h(0) = 0. \tag{2}
\]

1) $x \equiv 0$ of (2) is said to be stable in probability if for any $\epsilon > 0$,

\[
\lim_{x_0 \to 0} P \left( \sup_{t \geq 0} |x(t)| > \epsilon \right) = 0. \tag{3}
\]
2) $x \equiv 0$ of (2) is said to be locally asymptotically stable in probability if (3) holds and
\[
\lim_{t \to \infty} P \left( \lim_{t \to \infty} |x(t)| = 0 \right) = 1.
\]

3) $x \equiv 0$ of (2) is said to be globally asymptotically stable in probability if (3) holds and
\[
P \left( \lim_{t \to \infty} |x(t)| = 0 \right) = 1.
\]

**Definition 2.2.** If there exists an admissible control $u^*$ which is $F_t$-adapted, such that $x \equiv 0$ of the closed-loop system
\[
dx(t) = (f(x(t)) + g(x(t))u^*(t)) \, dt + (h(x(t)) + l(x(t))u^*(t)) \, dw(t)
\]
is stable in the sense of Definition 2.1, then system (1) is called stabilizable in the corresponding sense with the feedback stabilizing law $u^*$.

For instance, if $x \equiv 0$ of (6) is globally asymptotically stable in probability, then we say that (1) is globally asymptotically stabilizable in probability.

3. **Feedback Stabilization.** In this section, we will discuss the state feedback stabilization of nonlinear affine stochastic systems and the output feedback stabilization of a class of nonlinear perturbed systems.

3.1. **State feedback stabilization.** Different from the previous methods such as the backstepping design for triangular systems, in what follows, we will first present a very general sufficient criterion for the stabilization of affine stochastic systems based on HJI, which may yield very useful corollaries.

**Theorem 3.1.** If there exists a positive definite Lyapunov function $V(x) \in C^2(U)$ satisfying the following HJI.
\[
\Pi(V(x)) := \frac{\partial V}{\partial x} f(x) + \frac{1}{2} h'(x) \frac{\partial^2 V}{\partial x^2} h(x) - \frac{1}{4} \left( \frac{\partial V}{\partial x} g(x) + h'(x) \frac{\partial^2 V}{\partial x^2} l(x) \right)
\]
\[
\times \left( I + \frac{1}{2} l'(x) \frac{\partial^2 V}{\partial x^2} l(x) \right)^{-1} \left( g'(x) \frac{\partial V}{\partial x} + l'(x) \frac{\partial^2 V}{\partial x^2} h(x) \right) < 0,
\]
\[
V(0) = 0, \quad \det \left( I + \frac{1}{2} l'(x) \frac{\partial^2 V}{\partial x^2} l(x) \right) \neq 0, \quad \forall x \in U
\]

then system (1) is locally asymptotically stabilizable in probability with the feedback stabilizing law
\[
u^*(x) = \frac{-1}{2} \left( I + \frac{1}{2} l'(x) \frac{\partial^2 V}{\partial x^2} l(x) \right)^{-1} \left( g'(x) \frac{\partial V}{\partial x} + l'(x) \frac{\partial^2 V}{\partial x^2} h(x) \right),
\]
where $U$ is some field in $R^n$ including $x = 0$.

Additionally, if $V(\cdot)$ is proper (i.e., for each $a > 0$, $V^{-1}[0,a]$ is compact), $V(x) \in C^2(R^n)$ and (7) holds for any $x \in R^n$, then system (1) is globally asymptotically stabilizable in probability with the same feedback stabilizing law (8).

**Proof:** We first note that (7) is equivalent to
\[
\frac{\partial V}{\partial x} (f + gk) + \frac{1}{2} (h + lk) \frac{\partial^2 V}{\partial x^2} (h + lk) < -k'k
\]
where
\[
k = \frac{-1}{2} \left( I + \frac{1}{2} l'(x) \frac{\partial^2 V}{\partial x^2} l(x) \right)^{-1} \left( g'(x) \frac{\partial V}{\partial x} + l'(x) \frac{\partial^2 V}{\partial x^2} h(x) \right).
\]
Second, if we take \( u(t) = u^*(x(t)) = k(x(t)) \) in (1), then by Itô’s formula, it follows:

\[
L_{u^*} V(x) = \frac{\partial V}{\partial x} (f + gk) + \frac{1}{2} (h + lk)^T \frac{\partial^2 V}{\partial x^2} (h + lk),
\]

where \( L_u \) is the infinitesimal generator of (1). By (9), \( L_{u^*} V(x) < -k'k \leq 0 \). Because \( V(0) = 0 \) and \( V(x) \in C^2(U) \), we have \( \lim_{|x| \to 0} V(x) = 0 \). By Corollary 1 of [1] (page 168), the first part of this theorem is proved.

If \( V(x) \) is proper, then \( \lim_{|x| \to \infty} V(x) = \infty \), a direct application of Theorem 4.4 of [1] (page 170) yields the second assertion. \( \square \)

Remark 3.1. Theorem 3.1 only presents a sufficient condition for the local and global asymptotic stabilization of system (1), respectively. So, the solution \( V(x) \) to HJI (7) does not always exist even if system (1) is stabilizable.

Corollary 3.1. Consider the following nonlinear stochastic system

\[
dx = (Ax + g(x)u)dt + (Cx + Du)dw, \quad g(0) = B \in \mathbb{R}^{n \times m}
\]

with \( g(\cdot) \) being continuous at \( x = 0 \). If one of the following conditions holds, then system (10) is locally asymptotically stabilizable in probability,

(i) The following generalized algebraic Riccati inequality (GARI)

\[
PA + A'P + C'PC - (PB + C'PD)(I + D'PD)^{-1}(B'P + D'PC) < 0
\]

has a solution \( P > 0 \). In this case, a feedback stabilizing law is given by \( u^*(x) = -(I + D'PD)^{-1}(B'P + D'PC)x \).

(ii) The following LMI

\[
\begin{bmatrix}
P_1 A' + AP_1 + Y'B' + BY & CP_1 + DY \\
P_1 C' + Y'D' & -P_1
\end{bmatrix}
\]

has solutions \( P_1 > 0 \) and \( Y \in \mathbb{R}^{m \times m} \). In this case, a feedback stabilizing law is given by \( u^*(x) = YP_1^{-1}x \).

Proof: By taking \( V(x) = x'Px, (7) \) becomes

\[
x'[PA + A'P + C'PC - (Pg(x) + C'PD)(I + D'PD)^{-1}(g'(x)P + D'PC)]x < 0.
\]

If (11) has a solution \( P > 0 \), by the continuity of \( g(x) \) at \( x = 0 \), there exists a neighborhood \( U_0 \) of the origin, such that for all \( x \in U_0 \), (13) holds. By Theorem 3.1, (i) is proved.

To show (ii), we note that (12) is equivalent to

\[
P_1(A + BK)' + (A + BK)P_1 + (C + DK)P_1(C + DK)' < 0
\]

with \( K = YP_1^{-1} \) according to Schur’s complement [9]. (14) holds if there exists a \( P_2 \) such that [8]

\[
(A + BK)'P_2 + P_2(A + BK) + (C + DK)'P_2(C + DK) < 0.
\]

As said in (i), (15) implies that the following

\[
L_{u=Kx}(x'P_2 x) = x'[PA + A'P + C'PC - (Pg(x) + C'PD)(I + D'PD)^{-1}(g'(x)P + D'PC)]x < 0
\]

holds for all \( x \in U_0 \). Again, using Theorem 3.1, the proof of (ii) is complete. \( \square \)

If we take \( g(x) = Bx + b, u \in \mathbb{R}, D = 0 \), then system (10) comes down to

\[
dx = (Ax + u(Bx + b))dt + Cx dw.
\]

**Theorem A** [11] If \((A, b)\) is stabilizable, and

\[
\inf_{k \in \mathbb{R}^{1 \times m}} \left| \int_0^\infty \exp(t(A + bk)')C'C\exp(t(A + bk))dt \right| < 1 \tag{18}
\]

then system (17) is locally asymptotically stabilizable in probability.

We should point out that, although Theorem A and Corollary 3.1 present a sufficient condition for the local asymptotic stabilization of (17), respectively, Corollary 3.1 is an improved version of Theorem A, which can be seen from the following fact: the preconditions of Theorem A imply that of Corollary 3.1, but the converse is not true; see Remark 4 of [8] and the following counterexample.

**Example 3.1.** In (17), we take \(A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} \sqrt{1.09} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \).

It is easy to test that (11) has at least one solution \(P > 0\), so system (10) is locally asymptotically stabilizable in probability by Corollary 3.1. However, (18) does not hold.

In fact, \(\inf_{k \in \mathbb{R}^{1 \times m}} \left| \int_0^\infty \exp(t(A + bk)')C'C\exp(t(A + bk))dt \right| = 1.09 \begin{bmatrix} \frac{11}{12} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} > 1.\)

So from Theorem A, we cannot derive that system (17) is locally asymptotically stabilizable in probability.

A slightly more general model than (10) is as follows:

\[
dx = (Ax + g(x)u + F_0(x, u))dt + (Cx + Du + F_1(x, u))dw, \quad g(0) = B \in \mathbb{R}^{n \times m} \tag{19}
\]

where \(g(\cdot)\) is continuous at \(x = 0\), and \(F_i(\cdot, \cdot)\) \((i = 1, 2)\) satisfy

\[
\lim_{x \to 0} \frac{|F_0(x, u(x))|}{|x|} = \lim_{x \to 0} \frac{|F_1(x, u(x))|}{|x|} = 0, \quad F_0(0, u) \equiv 0, \quad F_1(0, u) \equiv 0, \quad \forall u \in \mathbb{R}. \tag{20}
\]

The constraint condition (20) comes from [11], which means that for any \(u \in \mathbb{R}, |F_i(x, u)| = o(|x|)\). For instance, \(F_i(x, u) = u \sin^2 x\) satisfies (20) for \(i = 0, 1\).

A special version of (19) as:

\[
dx = (Ax + u(Bx + b) + F_0(x, u))dt + (Cx + F_1(x, u))dw, \quad u \in \mathbb{R}^l \tag{21}
\]

was also discussed in [11].

**Proposition 3.1.** Under the assumptions of (20) and \(g(x)\) being continuous at \(x = 0\), if one of the following conditions holds, then system (19) is locally asymptotically stabilizable in probability.

(i) The GARI

\[
P(A + \tfrac{3}{4}I) + (A + \tfrac{3}{4}I)'P + (1 + \gamma)C'PC \tag{22}
\]

\[-(PB + (1 + \gamma)C'PD)(I + (1 + \gamma)D'PD)^{-1}(B'P + (1 + \gamma)D'PC) < 0\]

has a solution \(P > 0\) for some \(\gamma > 0\). In this case, a feedback stabilizing law is given by \(u = -(I + (1 + \gamma)D'PD)^{-1}(B'P + (1 + \gamma)D'PC)x\).

(ii) The following LMI

\[
\left[ \begin{array}{cc}
P_1(A + \tfrac{3}{4}I)' + (A + \tfrac{3}{4}I)P_1 + \gamma'\gamma'(C_1P_1 + DY)

\sqrt{1 + \gamma'}(P_1C' + Y'D') - P_1 \tag{23}
\end{array} \right] < 0
\]

has solutions \(P_1 > 0\) and \(Y \in \mathbb{R}^{m \times n}\). In this case, a feedback stabilizing law is given by \(u = YP_1^{-1}x\).
Proof: Take $V(x) = x'PxF$, where $P$ is a solution of GARI (22). Let $L_0$ be the infinitesimal generator of the closed-loop system.

$$dx = (Ax + g(x)kx + F_0(x, kx))dt + (Cx + Dkx + F_1(x, kx))dw$$

with $k = -(I + (1 + \gamma)D'PD)^{-1}(B'P + (1 + \gamma)D'PC)$, then

$$L_0V(x) = x'[P(A + g(x)k) + (A + g(x)k)'P + (C + Dk)'P(C + Dk)]x$$

$$+x'PF_0(x, kx) + \dot{F_0}(x, kx)Px + x'(C + Dk)'PF_1(x, kx)$$

$$+F_1(x, kx)P(C + Dk)x + F_1'(x, kx)PF_1(x, kx).$$

Using the following well known inequality

$$X'Y + Y'X \leq \gamma X'X + \gamma^{-1}Y'Y, \quad \gamma > 0$$

we have

$$x'PF_0(x, kx) + \dot{F_0}(x, kx)Px \leq \gamma x'Px + \gamma^{-1}F_0'(x, kx)PF_0(x, kx),$$

$$x'(C + Dk)'PF_1(x, kx) + F_1'(x, kx)P(C + Dk)x \leq \gamma x'(C + Dk)'P(C + Dk)x$$

$$+\gamma^{-1}F_1'(x, kx)PF_1(x, kx).$$

So, (25) leads to

$$L_0V(x) \leq x'\left[ P \left( A + g(x)k + \frac{\gamma}{2}I \right) + \left( A + g(x)k + \frac{\gamma}{2}I \right)'P \right.$$  

$$+ (1 + \gamma)(C + Dk)'P(C + Dk) \left. \right] x$$

$$+ (1 + \gamma^{-1})F_0'(x, kx)PF_0(x, kx) + \gamma^{-1}F_0'(x, kx)PF_0(x, kx).$$

(27)

If (22) holds, then

$$P \left( A + Bk + \frac{\gamma}{2}I \right) + \left( A + Bk + \frac{\gamma}{2}I \right)'P + (1 + \gamma)(C + Dk)'P(C + Dk) < -k'k \leq 0.$$

By the continuity of $g(x)$ at $x = 0$, for sufficiently small $\delta > 0$, there exists a neighborhood $U_\delta$ of the origin, such that

$$P(A + g(x)k + \frac{\gamma}{2}I) + \left( A + g(x)k + \frac{\gamma}{2}I \right)'P + (1 + \gamma)(C + Dk)'P(C + Dk) < -\delta I, \quad \forall x \in U_\delta.$$

Additionally, (20) yields that there exists another small neighborhood $U'_\delta$ of the origin, such that $\|F_i(x, kx)\|^2 \leq \frac{\delta}{\max(P_i)}\|x\|^2$, $i = 0, 1, \forall x \in U'_\delta$. Take $U := U_\delta \cap U'_\delta$, then from (27),

$$L_0V(x) \leq -\delta\|x\|^2 + \frac{\delta}{2}\|x\|^2 = -\frac{\delta}{2}\|x\|^2 < 0$$

for all nonzero $x \in U$. So, when (22) holds, (19) is locally asymptotically stabilizable in probability by the result of [1]. The assertion (ii) can be proved similarly to the proof of Corollary 3.1 (ii). □

Because in (19), we allow the control input to enter into the diffusion term, the model (19) is more general than (23) of [11] and (4.1) of [7]. As in Remark 3.2, it is easy to show that Proposition 3.1 improves the first assertion of Theorem 5.1 [11] and Proposition 1 of [7] when our model (19) comes down to (23) of [11] and (4.1) of [7].

By the same discussion as in Theorem 3.1, we are in a position to extend Theorem 3.1 to the nonlinear stochastic time-varying system

$$\begin{align*}
\left\{ \begin{array}{l}
\dot{x}(t) = (f(x(t), t) + g(x(t), t)u(t))dt + (h(x(t), t) + l(x(t), t)u(t))dw(t) \\
x(0) = x_0 \in \mathbb{R}^n.
\end{array} \right.
\end{align*}$$

(28)
Theorem 3.2. Assume there exists a positive definite Lyapunov function \( V(t, x) \in C_0^{1, 2} ([0, \infty) \times U) \), which has an infinitesimal upper limit, i.e., \( \lim_{|x| \to 0} \sup_{t>0} V(t, x) = 0 \). If \( V(t, x) \) solves the following HJI.

\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) + \frac{1}{2} h'(t, x) \frac{\partial^2 V}{\partial x^2} h(t, x) - \frac{1}{4} \left( \frac{\partial^2 V}{\partial x^2} g(t, x) + h'(t, x) \frac{\partial^2 V}{\partial x^2} l(t, x) \right) \\
\left( I + \frac{1}{2} h'(t, x) \frac{\partial^2 V}{\partial x^2} l(t, x) \right)^{-1} \left( g'(t, x) \frac{\partial V}{\partial x} + l'(t, x) \frac{\partial^2 V}{\partial x^2} h(t, x) \right) < 0
\]

\[V(t, 0) = 0, \quad \det \left( I + \frac{1}{2} h'(t, x) \frac{\partial^2 V}{\partial x^2} l(t, x) \right) \neq 0, \quad \forall (t, x) \in [0, \infty) \times U\]  

(29)

then system (28) is locally asymptotically stabilizable in probability with the feedback stabilizing law

\[u^*(t, x) = -\frac{1}{2} \left( I + \frac{1}{2} h'(t, x) \frac{\partial^2 V}{\partial x^2} l(t, x) \right)^{-1} \left( g'(t, x) \frac{\partial V}{\partial x} + l'(t, x) \frac{\partial^2 V}{\partial x^2} h(t, x) \right).\]  

(30)

Furthermore, if \( U = \mathbb{R}^n \), and \( V(t, x) \) satisfies \( \lim_{|x| \to \infty} \inf_{t>0} V(t, x) = \infty \) then (28) is globally asymptotically stabilizable in probability with the same control law (30).

Remark 3.2. The models (1) and (28) are very general than those appeared in the previous references such as [7, 11, 12, 15], so Theorems 3.1 and 3.2 present sufficient conditions for the stabilization of a large class of nonlinear stochastic systems. At present stage, it is not very convenient in using Theorems 3.1 and 3.2 to test the stabilization of a general nonlinear stochastic system, because it is not easy to solve HJIs (7) and (29), which deserves further study.

Remark 3.3. Theorems 3.1 and 3.2 only present sufficient conditions for nonlinear stochastic stabilization, in order to obtain less conservative results in this aspect, it is necessary to study the converse problems of Theorems 3.1 and 3.2, which is without doubt a difficult issue at present stage.

3.2. Output feedback stabilization. In Subsection 3.1, we have discussed the local and global asymptotic stabilization of system (1) and (28) with complete state information, respectively. However, in practice, it is not easy, in some cases, to measure the total state, because the system state is often partially observed. Therefore, it is worthwhile to consider the problem of output feedback stabilization. The related discussion on nonlinear stochastic hybrid systems and Itô systems can be found in [15, 16], respectively. Up to now, there are few papers on output feedback (observer-based) stabilization of general nonlinear stochastic differential systems. The initial work was done by [15] for a class of nonlinear stochastic systems, where a backstepping-based controller design was given. In this subsection, we will employ the LMI-based technique to study the output feedback stabilization of nonlinear perturbed stochastic system.

\[
\begin{cases}
\frac{dx}{dt} = (Ax + Bu + F_0(x, u))dt + (Cx + Du + F_1(x, u))dw \\
x(0) = x_0 \in \mathbb{R}^n, \quad F_i(0, u) = 0, \quad i = 0, 1
\end{cases}
\]

(31)

In the sequel, we assume the output equation is

\[dy(t) = A_1 x(t)dt + G_1 x(t)dv(t)\]

(32)

where \( y(t) \in \mathbb{R}^{n_y}, \ n_y \leq n, \ A_1 \) and \( G_1 \) are constant matrices of appropriate dimensions, \( v(t) \) is a one-dimensional standard Wiener process that is uncorrelated with \( w(t) \). An output feedback controller design is proposed as:

\[d\hat{x} = A_f \hat{x}dt + B_f dy, \quad u = K \hat{x}, \quad \hat{x}(0) = 0.\]

(33)
In this subsection, we will give a sufficient condition for the local/global output feedback stabilization. When discussing the local output feedback stabilization, we assume $F_i$ ($i = 0, 1$) satisfy the condition (20), while when discussing the global output feedback stabilization, we assume $F_i$ ($i = 0, 1$) satisfy the subsequent condition (47).

Set $\xi' = [x' \ x']$, then the augmented system is as follows:

$$d\xi = \tilde{A}\xi dt + \tilde{D}_0\xi dv + \tilde{D}_1\xi dw + \tilde{F}_0 dt + \tilde{F}_1 dw$$

(34)

where

$$\tilde{A} = \begin{bmatrix} A & BK \\ B_fA_1 & A_f \end{bmatrix}, \quad \tilde{D}_0 = \begin{bmatrix} 0 & 0 \\ B_fG_1 & 0 \end{bmatrix}, \quad \tilde{D}_1 = \begin{bmatrix} C & DK \\ 0 & 0 \end{bmatrix},$$

$$\tilde{F}_0 = \begin{bmatrix} F_0(x, K\hat{x}) \\ 0 \end{bmatrix}, \quad \tilde{F}_1 = \begin{bmatrix} F_1(x, K\hat{x}) \\ 0 \end{bmatrix}.$$ 

(35)

(36)

Our design purpose is to specify $A_f, B_f$, and $K$ in (33), such that (34) is locally/globally asymptotically stable in probability. For any $P > 0$, if we take a Lyapunov function $V(\xi) = \xi'P\xi$, then for system (34),

$$LV(\xi) = \xi' (P\tilde{A} + \tilde{A}'P + \tilde{D}_0'PD_0 + \tilde{D}_1'P\tilde{D}_1)\xi$$

$$+ \left( \tilde{F}_0'P\xi + \xi'P\tilde{F}_0 + \tilde{F}_1'P\tilde{F}_1 + 2\tilde{F}_1'P\tilde{D}_1\xi \right)$$

(37)

with $L$ being the infinitesimal generator of (34). By the same discussion as in Proposition 3.1, it is easy to show that under the condition (20), a sufficient condition for (34) to be locally asymptotically stable in probability is

$$P\tilde{A} + \tilde{A}'P + \tilde{D}_0'PD_0 + \tilde{D}_1'P\tilde{D}_1 < 0.$$ 

(38)

By Schur’s complement, (38) is equivalent to

$$\begin{bmatrix} P\tilde{A} + \tilde{A}'P & \tilde{D}_0'P & \tilde{D}_1'P \\ P\tilde{D}_0 & -P & 0 \\ PD_1 & 0 & -P \end{bmatrix} < 0.$$ 

(39)

Set $P = diag(P_{11}, P_{22}) > 0$, and substitute (35) into (39), it yields

$$\begin{bmatrix} P_{11}A + A'P_{11} & P_{11}BK + A'_fP_{22} + G'fP_{22} & C'fP_{22} & C'P_{11} \\ P_{22}B_fA_1 + K'B'P_{11} & P_{22}A_f + A'_fP_{22} & 0 & K'D'P_{11} \\ P_{22}B_fG_1 & 0 & -P_{22} & 0 \\ P_{11}C & P_{11}DK & 0 & -P_{11} \end{bmatrix} := T_1 + T_2 < 0$$

(40)

where

$$T_1 = \begin{bmatrix} P_{11}A + A'P_{11} & A'_fP_{22} & G'fP_{22} & C'P_{11} \\ P_{22}B_fA_1 & P_{22}A_f + A'_fP_{22} & 0 & 0 \\ P_{22}B_fG_1 & 0 & -P_{22} & 0 \\ P_{11}C & 0 & 0 & -P_{11} \end{bmatrix}$$

and

$$T_2 = \begin{bmatrix} 0 & P_{11}BK & 0 & 0 \\ K'B'P_{11} & 0 & 0 & K'D'P_{11} \\ 0 & 0 & 0 & 0 \\ 0 & P_{11}DK & 0 & 0 \end{bmatrix}.$$ 

If $T_1$ and $T_2$ satisfy

$$T_1 < -diag \left( P_{11}^2, P_{11}^2, I, P_{11}^2 \right),$$

$$T_2 < diag \left( P_{11}^2, P_{11}^2, I, P_{11}^2 \right)$$

(41)

(42)
then (40) holds. Let $P_{22}A_f = Z$ and $P_{22}B_f = Z_1$ in (41), then (41) can be transformed into an LMI as:

$$
\begin{bmatrix}
P_{11}A + A'P_{11} & A'_1Z'_1 & G'_1P_{11} & C'P_{11} & 0 & 0 & P_{11} \\
Z_1A_1 & Z + Z' & 0 & 0 & 0 & 0 & P_{11} \\
Z_1G_1 & 0 & -P_{22} + I & 0 & 0 & 0 & 0 \\
P_{11}C & 0 & 0 & -P_{11} & P_{11} & 0 & 0 \\
0 & 0 & 0 & P_{11} & -I & 0 & 0 \\
0 & P_{11} & 0 & 0 & 0 & -I & 0 \\
P_{11} & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0. \tag{43}
$$

Pre- and post-multiplying (42) by $\text{diag}(P_{11}^{-1}, P_{11}^{-1}, I, P_{11}^{-1})$ and taking $Z_2 = K P_{11}^{-1}$, we obtain an LMI on the variable $Z_2$ as:

$$
\begin{bmatrix}
-I & BZ_2 & 0 \\
Z_2'B' & -I & Z_2'D' \\
0 & DZ_2 & -I
\end{bmatrix} < 0. \tag{44}
$$

So, we have the following result:

**Theorem 3.3.** Consider the state Equation (31) with the measurement (32). Assume (20) holds. If LMIs (43) and (44) have solutions $P_{11} > 0$, $P_{22} > 0$, $Z$, $Z_1$, and $Z_2$ of suitable dimensions, then (34) is locally asymptotically stable in probability by the following dynamic output feedback:

$$
d\hat{x} = P_{22}^{-1}Z \dot{x} dt + P_{22}^{-1}Z_1 dy, \quad \hat{x}(0) = 0, \quad u^*(\hat{x}) = K \hat{x}(t) = Z_2 P_{11} \hat{x}.
$$

To address the global output feedback stabilization, we apply (26) with $\gamma = 1$ to (37), it follows that

$$
LV(\xi) \leq \xi' \left( P \ddot{A} + \dot{A}'P + \dot{D}'_0P\dot{D}_0 + 2\dot{D}'_1P\dot{D}_1 + P \right) \xi + \dot{F}_0'P\dot{F}_0 + 2\dot{F}_1'P\dot{F}_1. \tag{45}
$$

Set $P = \text{diag}(P_{11}, P_{22})$, then we have

$$
\dot{F}_0'P\dot{F}_0 = F_0'P_{11}F_0, \quad \dot{F}_1'P\dot{F}_1 = F_1'P_{11}F_1. \tag{46}
$$

If there exists $\alpha > 0$, $\lambda > 0$, such that $P_{11} \leq \alpha I$, and

$$
|F_i(x, K \hat{x})| \leq \lambda|\xi|, \quad i = 0, 1 \tag{47}
$$

for some $K$ and any $x, \hat{x}$, then (46) yields

$$
\dot{F}_i'P\dot{F}_i \leq \lambda^2 \alpha|\xi|^2, \quad i = 0, 1. \tag{48}
$$

Substituting (48) into (45), we have

$$
LV(\xi) \leq \xi' \left( P \ddot{A} + \dot{A}'P + \dot{D}'_0P\dot{D}_0 + 2\dot{D}'_1P\dot{D}_1 + P + 3\lambda^2 \alpha I \right) \xi. \tag{49}
$$

By [1], a sufficient condition for (31) to be globally asymptotically stabilizable in probability is

$$
P \ddot{A} + \dot{A}'P + \dot{D}'_0P\dot{D}_0 + 2\dot{D}'_1P\dot{D}_1 + P + 3\lambda^2 \alpha I < 0
$$

which is equivalent to

$$
\begin{bmatrix}
P_{11}A + A'P_{11} + P_{11} + 3\lambda^2 \alpha I & P_{11}B_K + A'_1B'_1P_{22} & G'_1B'_1P_{22} & \sqrt{2}C'P_{11} \\
P_{22}B_fA_1 + K'B'_1P_{11} & P_{22}A_f + A'_fP_{22} + P_{22} + 3\lambda^2 \alpha I & 0 & \sqrt{2}K'D'_1P_{11} \\
\sqrt{2}P_{11}C & 0 & -P_{22} & 0 \\
0 & \sqrt{2}P_{11}DK & 0 & -P_{11}
\end{bmatrix} < 0.
$$

Repeating the same procedure as in Theorem 3.1, it is not difficult to obtain the following theorem.
Theorem 3.4. Consider stochastic system (31) with the measurement (32), and assume condition (47) holds. If the following LMIs

\[
\begin{bmatrix}
\{P_{11}A + A'P_{11} + P_{11} + 3\lambda^2\alpha I\} & A_1'Z_1' & G_1'Z_1' & \sqrt{2}C'P_{11} & 0 & 0 & P_{11} \\
Z_1A_1 & \{Z + Z' + P_{22}\} & 0 & 0 & 0 & P_{11} & 0 \\
Z_1G_1 & 0 & -P_{22} + I & 0 & 0 & 0 & 0 \\
\sqrt{2}P_{11}C & 0 & 0 & -P_{11} & P_{11} & 0 & 0 \\
0 & 0 & 0 & P_{11} & 0 & 0 & -I \\
P_{11} & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0 \quad (50)
\]

have positive real number solution \(\alpha > 0\), matrix-valued solutions \(P_{11} > 0\), \(P_{22} > 0\), \(Z\), \(Z_1\) and \(Z_2\) of suitable dimensions, then (34) is globally asymptotically stable in probability via the following dynamic output feedback

\[
d^\Delta x = P_{22}^{-1}Z\hat{x}dt + P_{22}^{-1}Z_1dy, \quad \hat{x}(0) = 0, \quad u^*(\hat{x}) = K\hat{x} = Z_2P_{11}\hat{x}.
\]

Remark 3.4. As we have said at the beginning of Section 2, we assume \(w\) to be one-dimensional only for simplicity, all the results given in this paper can be generalized to multiple noises case without any difficulty, e.g., the model (31) can take a more general form as:

\[
dx = (Ax + Bu + F_0(x, u))dt + \sum_{i=1}^{l} (C_ix + D_iu + F_i(x, u))dw_i
\]

where \(\{w_i, i = 1, \cdots , l\}\) are independent standard Wiener processes.

Below, we present an example to illustrate the validity of Theorem 3.1.

Example 3.2. Consider the following nonlinear stochastic system with state and control-dependent noise

\[
dx = \left(\begin{array}{c}
\frac{1}{2}x_1^3 - 2x_1 + \frac{2}{3}x_1^2x_2 \\
x_2^3 - 2x_2 + x_1x_2^2
\end{array}\right) dt + \left(\begin{array}{c}
2x_1 \\
2x_2
\end{array}\right) u + \left(\begin{array}{c}
x_2 \\
x_1x_2
\end{array}\right) dw.
\]

(53)

Based on Theorem 3.1, we need to solve the following HJI.

\[
\Pi(V(x)) = \left[\begin{array}{cc}
\frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2}
\end{array}\right] \left[\begin{array}{c}
\frac{1}{2}x_1^3 - 2x_1 + \frac{2}{3}x_1^2x_2 \\
x_2^3 - 2x_2 + x_1x_2^2
\end{array}\right] + \frac{1}{2} \left[\begin{array}{cc}
x_2^2 & x_1x_2
\end{array}\right] \left[\begin{array}{cc}
\frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1x_2} \\
\frac{\partial^2 V}{\partial x_2x_1} & \frac{\partial^2 V}{\partial x_2^2}
\end{array}\right] \left[\begin{array}{c}
x_2^2 \\
x_1x_2
\end{array}\right]
\]

(54)
\[-\frac{1}{4} \left( \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \\ \end{bmatrix} \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} x_2^2 & x_1 x_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_2^2} \\ \frac{\partial^2 V}{\partial x_1 \partial x_2} & \frac{\partial^2 V}{\partial x_2 \partial x_1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right] \\
\cdot \left( I + \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 V}{\partial x_2^2} \\ \frac{\partial^2 V}{\partial x_1 \partial x_2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} \\
\cdot \left( \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_2^2} \\ \frac{\partial^2 V}{\partial x_1 \partial x_2} & \frac{\partial^2 V}{\partial x_2 \partial x_1} \end{bmatrix} \begin{bmatrix} x_2^3 \\ x_1 x_2 \end{bmatrix} \right) \right] < 0.

(55)

It is easy to test that $V(x) = x_1^2 + x_2^2$ solves HJI (55), because

$$
\Pi(V(x)|_{V(x)=x_1^2+x_2^2} = x_1^4 - 4x_1^2 + \frac{4}{3}x_1^2 x_2 + 2x_2^3 - 4x_2^2 + 2x_1 x_2^3 + (x_1^2 + x_1^2 x_2^2) \\
- \left( \frac{4}{3}x_1^4 + 3x_2^4 + \frac{13}{3}x_1^2 x_2^2 + 2x_1 x_2^3 + \frac{4}{3}x_1^3 x_2 \right) \\
= -\frac{1}{3}x_1^4 - 4x_1^2 - 4x_2^2 - \frac{10}{3}x_1^2 x_2^2 < 0.
$$

So, by Theorem 3.1, system (53) is globally asymptotically stabilizable in probability, and the corresponding state feedback stabilizing law is as the form of

$$
u^*(x) = -\frac{1}{2} \left( I + \frac{1}{2} l' \frac{\partial^2 V}{\partial x^2} l \right)^{-1} \left( g' \frac{\partial V}{\partial x} + l' \frac{\partial^2 V}{\partial x^2} h \right) \\
= -\frac{1}{2} (1 + 2)^{-1} \left( \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_2^3 \\ x_1 x_2 \end{bmatrix} \right) \\
= -\frac{1}{6} \left( 4x_1^2 + 6x_2^2 + 2x_1 x_2 \right).
$$

By taking $(x_1(0), x_2(0)) = (-0.05, 0.08)$, the simulation results are shown in Figures 1 and 2.

**Figure 1.** The state trajectories of $x_1$ and $x_2$

**Figure 2.** The trajectory of $u^*$
4. **Conclusions.** By means of HJIs (7) and (29), the general results of nonlinear stochastic state feedback stabilization have been derived. As corollaries, some previous results have been extended to more general models. Finally, for a class of special nonlinear systems, when the state is not completely observable, a dynamic output feedback controller design is respectively presented for the local and global asymptotic stabilization via LMIs.

**Acknowledgment.** This work is supported by the National Natural Science Foundation of China (No. 60874032), the Key Project of Natural Science Foundation of Shandong Province (No. ZR2009GZ001), the Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20103718110006) and the NSC of Taiwan under Contract NSC98-2221-E-007-113-MY3.

**REFERENCES**


