ROBUST GUARANTEED COST CONTROLLER DESIGN FOR NETWORKED CONTROL SYSTEMS: DISCRETIZED APPROACH

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ABSTRACT. The paper addresses the problem of static output feedback guaranteed cost controller design for Networked Control Systems (NCSs) with time-varying delay and polytopic uncertainties. Based on partitioning scheme of time-varying delay and using integral quadratic constraint (IQC), a new discretized Lyapunov-Krasovskii functional method is obtained to design a PI controller achieving a guaranteed cost such that the NCSs can be stabilized for all admissible uncertainties and time-varying delays. Finally, some numerical examples are given to illustrate the effectiveness of the proposed method.

Keywords: Lyapunov-Krasovskii functional (LKF), Networked control systems (NCSs), Polytopic system, Time-varying delay, Integral quadratic constraint (IQC)

1. Introduction. Feedback control systems wherein the loops are closed through real-time networks are called Networked Control Systems (NCSs) (Ray and Halevi [1]; Nilson [6]; Walsh et al. [5]; Zhang et al. [16]). Advantages of using NCSs in the control area include simplicity, cost-effectiveness, ease of system diagnosis and maintenance, increased system agility and testability. However, the integration of communication real-time networks into feedback control loops inevitable leads to some problems. As a result, it leads to a network-induced delay in networked control closed-loop system. The existence of such kind of delay in a network-based control loop can induce instability or poor performance of control systems (Jiang and Han [17]).

There are two approaches for controller design and study of closed-loop system stability in the time domain: Razumikhin theorem and Lyapunov-Krasovskii functional (LKF) approach. It is well known that the LKF approach often provides less conservative results than Razumikhin theorem (Friedman and Niculescu [3]; Richard [7]; Kharitonov and Melchor-Aguilar [13]). The challenge of all approaches using simple LKF is the conservatism of the respective algorithms. The delay-independent stability condition is very conservative if the delay is known. Although the simple delay-dependent condition using model transformation is intended to improve the situation, it is not necessary less conservative in all situations. Furthermore, the method using implicit model transformation is less conservative than two previous ones; however, it still includes certain conservatism and requires the system to be stable if the delay is set to zero (Gu and Niculescu [8]).

To reduce the conservatism efficiently, two techniques have been developed. The first one is partitioning the delay to $N_d$ parts and using the discretized scheme of the Lyapunov-Krasovskii matrices for these parts. It has been shown that if $N_d \to \infty$, the sufficient stability conditions for time delay systems approach to necessary ones (Gu et al. [9]). The price of an increasing $N_d$ is the increased number of variables to be optimized. Another
technique, developed in a Lyapunov and robust stability frameworks, uses an augmented state vector formulation to construct a new LKF for the original system. Hence, a partitioning delay scheme is developed in order to construct a LKF which depends on a discretizing version of the whole state \( x_i(\theta) \) (Gouaisbaut and Peaucelle [4]).

The guaranteed cost control approach has been extended to the uncertain time-delay systems, for the state feedback case, see (Yu and Chu [10]; Lee and Gyu Lee [2]; Zhang et al. [11]; Zhu et al. [19]) and for output feedback (Chen et al. [15]; Xia et al. [21]; Vesely and Nguyen Quang [14]). In the latter paper (Vesely and Nguyen Quang [14]), the authors considered the design of robust guaranteed cost PID controller for NCSs. However, in the above papers the obtained stability conditions are only sufficient, which can be far from necessary and sufficient ones, and the control state/output algorithm involves conservatism.

Motivated by the above observation, in this paper, a new discretized Lyapunov-Krasovskii functional method is studied to design a robust output feedback PI controller achieving a guaranteed cost such that the NCSs can be stabilized for all admissible polytopic-type uncertainties and time-varying delays with less conservatism than previous works. And thus, for a given \( N_d \) a partitioning scheme of time-varying delay and integral quadratic constraint (IQC) are used from (Ariba and Gouaisbaut [20]) to overcome the conservatism in the output feedback design procedures. Sufficient stability condition for existence of a guaranteed cost robust output feedback controller is established in terms of matrix inequalities.

This paper is organized as follows. Section 2 gives the problem formulation. Section 3 explains main results of the paper. Finally, in Section 4 some numerical examples are presented to show that with increasing \( N_d \) the conservatism of output feedback design procedure decreases which show the effectiveness of the proposed method.

Notation: Throughout this paper, for real matrix \( M \), the notation \( M > 0 \) (respectively \( M > 0 \)) means that matrix \( M \) is symmetric and positive semi-definite (respectively positive definite); \( * \) denotes a block that is transposed and complex conjugate to the respective symmetrically placed one. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. Problem Statement and Preliminaries. Consider the following linear time-delay system

\[
\dot{x}(t) = A(x) x(t) + A_d(x) x(t - \tau(t)) + B(x) u(t) \\
y(t) = C x(t) \\
x(t) = \varphi(t), t \in [-\tau_M; 0]
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the control input, \( y(t) \in \mathbb{R}^l \) is the output (measured output). The matrices \( A(x), A_d(x), B(\xi) \in S \) belong to a polytope \( S \) with \( N \) vertices \( S_1, S_2, \ldots, S_N \) which can be formally defined as:

\[
S := \left\{ A(x), A_d(x) \in \mathbb{R}^{n \times n}, B(\xi) \in \mathbb{R}^{n \times m} : A(\xi) = \sum_{i=1}^{N} \xi_i A_i, \quad \right. \\
\left. A_d(\xi) = \sum_{i=1}^{N} \xi_i A_{di}, B(\xi) = \sum_{i=1}^{N} \xi_i B_i, \sum_{i=1}^{N} \xi_i = 1, \xi_i \geq 0 \right\}
\]

where \( A_i, A_{di}, B_i \) are constant matrices with appropriate dimensions and \( \xi_i \) is time-invariant uncertainty; \( \tau_M \) is the upper bound of time delay and \( \varphi(t) \) is continuously differentiable initial function. Note \( S \) is a convex and bounded domain.

We assume that a real-time communication network is integrated into feedback control loops of system (1), and the network induced delay in NCS \( \tau(t) \) is given by \( 0 < \tau(t) \leq \tau_M \) and the derivative of \( \tau(t) \) is bounded as \( |\dot{\tau}(t)| \leq \mu \leq 1 \).
Suppose now that the time interval \([t - \tau(t), t]\) is partitioned into \(N_d\) parts. The discretization-like method is employed considering the state vector shifted by a fraction \(\frac{\tau(t)}{N_d}\) of the delay. The discretized extended states are constructed with signals (Ariba and Gouaisbaut [20]):

\[
x_i(t) = x(t_i(t)), \quad i = \{0, \ldots, N_d - 1\}
\]

where

\[
t_i(t) = \begin{cases} 
    t - \frac{1}{N_d}\tau(t), & i = 0 \\
    t - \frac{i+1}{N_d}\tau(t) + \sum_{k=0}^{i-1} \frac{i-k}{N_d} \delta_k(t), & i \geq 1 
\end{cases}
\]

\[
\delta_0(t) = \tau(t) - \tau\left(t - \frac{\tau(t)}{N_d}\right) = \int_{t - \frac{\tau(t)}{N_d}}^{t} \dot{x}(s)ds, \quad \delta_k(t) = \delta_{k-1}\left(t - \frac{\tau(t)}{N_d}\right), 1 \leq k \leq i - 1
\]

\[
\delta_k(t) \leq \frac{\tau M \mu}{N_d}
\]

Variables \(x_i(t)\) defined by (3)-(6) can be rewritten as:

\[
x_i(t) = x(t) - \int_{t_i(t)}^{t} \dot{x}(s)ds - \int_{t_i(t)}^{t_{i-1}(t)} \int_{t_i(t)}^{t_{i-1}(t)} \dot{x}(s)ds
\]

The last component \(x_{N_d-1}(t)\) is hardly suitable to describe the delayed instantaneous state \(x(t - \tau(t))\). To cope with the relation between these two signals, we introduce an additional operator \(\nabla[\cdot]\) (Ariba and Gouaisbaut [20]) from \(L_2\) to \(L_2\) as

\[
\nabla : x(t) \rightarrow \int_{t - \tau(t)}^{t} x(s)ds = \int_{t - \tau(t)}^{t - \tau(t) + \triangle(t)} x(s)ds
\]

where

\[
\triangle(t) = t_{N_d-1}(t) - (t - \tau(t)) = \sum_{k=0}^{N_d-2} \frac{N_d - 1 - k}{N_d} \delta_k(t) \leq \frac{\tau M \mu (N_d - 1)}{2N_d}
\]

The \(L_2\)-norm of the operator \(\nabla\) is defined by:

\[
\|\nabla(x)\|_{L_2}^2 = \int_0^\infty \left( \int_{t - \tau(t)}^{t - \tau(t) + \triangle(t)} x(s)ds \right)^2 dt
\]

Then, following the Cauchy-Schwarz inequality, the operator \(\nabla\) is bounded as (Ariba and Gouaisbaut [20]):

\[
\|\nabla(x)\|_{L_2}^2 \leq \frac{\tau M \mu (N_d - 1)}{2N_d} \int_0^\infty \left( \int_0^{\tau M \mu (N_d - 1) / 2N_d} \|x(s + t - \tau(t))\|^2du \right) ds
\]

\[
\leq \left( \frac{\tau M \mu (N_d - 1)}{2N_d} \right)^2 \frac{1}{1 - \tau} \|x\|_{L_2}^2
\]
Consider the following PI control algorithm for system (1),

\[ u(t) = K_P y(t - \tau(t)) + K_I \int_0^t y(t - \tau(t)) dt \]  \hspace{1cm} (12)

Let us now derive the closed loop system model suitable for further developments. Considering \( z(t) = \int y(t - \tau(t)) dt \) and using Newton-Leibniz formulas to obtain

\[ x(t - \tau(t)) = x(t) - \int_{t-\tau(t)}^t \dot{x}(s) ds, \]

the PI control algorithm (12) can be rewritten in the form

\[ u(t) = F C_n X(t) - F_P C_P \int_{t-\tau(t)}^t \dot{X}(s) ds \]

\[ = F C_n X(t) + (-F_P C_P) \left[ \sum_{i=1}^{N_d} \int_{t-t_{N_d-i-1}(t)}^{t-t_{N_d-i-1}(t)} \dot{X}(s) ds + \int_{t-\tau(t)}^t \dot{X}(s) ds \right] \]  \hspace{1cm} (13)

where \( X(t) = [x^T(t) \ z^T(t)]^T, F = [K_P \ K_I], F_P = [K_P \ 0], C_n = [C \ 0 \ 0] \) and

\( C_P = \begin{bmatrix} C & 0 & 0 \end{bmatrix}. \)

Taking \( \dot{z}(t) = C_i x(t - \tau(t)) = C_i x(t) - C_i \int_{t-\tau(t)}^t \dot{x}(s) ds \) where \( C_i \) is output matrix for integral output feedback, the system (1) can be expanded in the following form

\[ \dot{X}(t) = A_n(\xi) X(t) + B_n(\xi) u(t) - A_{dn}(\xi) \int_{t-\tau(t)}^t \dot{X}(s) ds \]  \hspace{1cm} (14)

where

\[ A_n(\xi) = \sum_{i=1}^{N} \xi_i A_{ni}, \quad A_{ni} = \begin{bmatrix} A_i + A_{di} & 0 \\ C_i & 0 \end{bmatrix} \]

\[ B_n(\xi) = \sum_{i=1}^{N} \xi_i B_{ni}, \quad B_{ni} = \begin{bmatrix} B_i \\ 0 \end{bmatrix} \]

\[ A_{dn}(\xi) = \sum_{i=1}^{N} \xi_i A_{dni}, \quad A_{dni} = \begin{bmatrix} A_{di} \\ C_i \\ 0 \end{bmatrix} \]  \hspace{1cm} (15)

Note that the dimension of state vector of the augmented system with integrated PI controller is extended to \( n := n + I. \)

Considering the required stability degree \( \gamma > 0 \) and applying the PI control algorithm (13) to system (14) the resulting closed-loop feedback system is obtained in the form

\[ \dot{X}(t) + A_c(\xi) X(t) + A_{dc}(\xi) \int_{t-\tau(t)}^t \dot{X}(s) ds = 0 \]  \hspace{1cm} (16)

where

\[ A_c(\xi) = \sum_{i=1}^{N} \xi_i A_{ci}, \quad A_{ci} = -(A_{ni} + B_{ni} F C_n + \gamma I) \]

\[ A_{dc}(\xi) = \sum_{i=1}^{N} \xi_i A_{dci}, \quad A_{dci} = A_{dni} + B_{ni} F_P C_P \]  \hspace{1cm} (17)
Having positive definite symmetric matrices $Q, R$ we will consider the cost function

$$ J = \int_0^\infty J(t)dt, \quad J(t) = X^T(t)QX(t) + u^T(t)Ru(t) \quad (18) $$

Consider

$$ \eta^T(t) = \begin{bmatrix} \dot{X}(t)^T & X^T(t) & \int_{t-\tau(t)}^{t} \dot{X}(s)ds & \int_{t-N_d-1(t)}^{t-N_d-2(t)} \dot{X}(s)ds & \int_{t-N_d-2(t)}^{t-N_d-3(t)} \dot{X}(s)ds & \int_{t-N_d-3(t)}^{t-N_d-4(t)} \dot{X}(s)ds \end{bmatrix} $$

by substituting $u(t)$ from (13) to $u^T(t)Ru(t)$, we obtain

$$ J(t) = \eta^T(t) \begin{bmatrix} * & 0 & 0 & 0 & 0 \\
0 & Q + C_n^TF^TRFC_n & -C_n^TFC_n & 0 & 0 \\
* & I_P^TC_n^TFC_n & I_P^TF^TRFC_nC_PI_P & * & * \\
* & * & * & * & * \\
0 & 0 & 0 & 0 & 0 \end{bmatrix} $$

where $I_P = [I \ I \ \ldots \ I]_{(N_d+1)n \times n}$.

2.1. Several definitions and theorems. Associated with the cost function (18), the guaranteed cost controller is defined as follows:

**Definition 2.1.** Consider the uncertain system (1). If there exists a controller (12) and a positive scalar such that for all uncertainties (2), the closed-loop system (16) is asymptotically stable and the closed-loop value of the cost function (18) satisfies $J \leq J_0$, then $J_0$ is said to be the guaranteed cost and the controller (12) is said to be the guaranteed cost controller.

**Definition 2.2.** Uncertain system (16) is robustly stable in the uncertainty box (2) if and only if there exists LKF $V(\xi, t)$ such that for the time derivative of $V(\xi, t)$ with respect to (16) the following inequality holds

$$ \frac{d}{dt}V(\xi, t) \leq 0 \quad (20) $$

According to references, there is no general and systematic way to formally determine $V(\xi, t)$. Such LKF is called the parameter dependent LKF and for a particular structure of $V(\xi, t)$ inequality (20) guarantees the parameter dependent quadratic stability.

Finally we recall the well known results from LQ theory.

**Lemma 2.1.** Consider the continuous-time delay system (14) with control algorithm (12). The control algorithm (12) is the guaranteed cost control for system (14) if and only if there exists LKF $V(\xi, t)$ and IQC $\Pi(\xi, t)$ such that the following condition holds:

$$ \frac{d}{dt}(V(\xi, t) + \Pi(\xi, t)) + J(t) \leq 0 \quad (21) $$

The term $V(\xi, t) + \Pi(\xi, t)$ is the complete LKF (Ariba and Gouaisbaut [20]).

The objective of this paper is to develop a procedure to design a robust PI controller of the form (12) which ensures the closed-loop system stability and guaranteed cost.

3. Main Results. Our main result presented in this section is sufficient robust stability condition for uncertain time delay system (1) with PI controller of form (12), that is static output feedback controller, guaranteeing both robust stability and the upper bound on integral cost function (18). This robust stability condition, summarized in Theorem 3.1 is, to the authors best knowledge, less conservative than the previous results from literature and provides the tool for the respective PI controller design, as illustrated on examples in Section 4.
Theorem 3.1. Consider the uncertain linear time-delay system (1) with network-induced delay \( \tau(t) \) satisfying \( 0 < \tau(t) \leq \tau_M, \| \dot{\tau}(t) \| \leq \mu \leq 1 \) and the cost function (18). Assume that there exists a PI controller of form (12), scalar \( J_0 \), and matrices \( P_i > 0, Q_{0i} > 0, Q_{1i} > 0, Q_{2i} > 0, R_{0i} > 0, R_{1i} > 0 (i = \{1, \ldots, N\}) \), \( N_1, N_2, N_3 \) that satisfy the following matrix inequality

\[
W_i = \begin{bmatrix} w_{11}^i & w_{12}^i & w_{13}^i \\ w_{21}^i & w_{22}^i & w_{23}^i \\ w_{31}^i & w_{32}^i & w_{33}^i \end{bmatrix} + M_{Q0}^T \begin{bmatrix} \mu Q_{0i} & (1-\mu)Q_{0i} \\ 0 & -(1-\mu)Q_{0i} \end{bmatrix} M_{Q0} + \\
+ M_{R0}^T \begin{bmatrix} \tau_M R_{0i} & 0 \\ 0 & -\tau_M R_{0i} \end{bmatrix} M_{R0} + M_{R1}^T \begin{bmatrix} \tau_M N_2 R_{1i} & 0 \\ 0 & -\tau_M N_2 R_{1i} \end{bmatrix} M_{R1} + \\
+ \left\{ M_{Q1a} R_{1i} M_{Q1a} - (1-\frac{\tau_M}{N_2}) M_{Q1a} R_{1i} M_{Q1a} \right\} \leq 0
\]

where

\[
w_{11}^i = N_1 + N_1^T + \left[ \frac{\tau_M \mu (N_1-1)}{2N_2} \right]^2 \frac{1}{1-\mu} Q_{2i} \\
w_{12}^i = N_1 A_{n1} + N_1^T + P_i \\
w_{13}^i = N_1 A_{nc} I_P + N_1^T \\
w_{22}^i = N_2 A_{n2} + A_{n2}^T N_2^T + C_n^T F_T R_{F} C_n + Q \\
w_{23}^i = N_2 A_{nc} I_P + A_{nc}^T N_2^T + C_n^T F_T R_{F} C_P I_P \\
w_{33}^i = N_3 A_{nc} I_P + A_{nc}^T N_3^T - \text{diag}(0_{N_{2m}}, Q_{2i}) + I_P C_P^T F_T R_{F} R_{F} C_P I_P
\]

matrices \( M_{Q0}, M_{R0}, M_{R1} \in R^{2n \times (N_2+3)n} \) are of the form

\[
M_{Q0} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & I & I \end{bmatrix}, M_{R0} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \end{bmatrix} \\
M_{R1} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \end{bmatrix}
\]

and matrices \( M_{Q1a}, M_{Q1b} \in R^{N \times (N_2+3)n} \) are of the form

\[
M_{Q1a} = \begin{bmatrix} 0 & I & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & -I & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & I & -I & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & I & -I & \cdots & -I & 0 & 0 \end{bmatrix}, M_{Q1b} = \begin{bmatrix} 0 & I & -I & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & -I & -I & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & I & -I & -I & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & I & -I & -I & \cdots & -I & 0 & 0 \end{bmatrix}
\]

then the uncertain system (1) with controller (12) is parameter-dependent quadratically asymptotically stable and the cost function (18) satisfies the following bound

\[
J \leq J_0 = \sqrt{\lambda_{MP}^2 + \lambda_{MQ0}^2 + \lambda_{MR0}^2 + \lambda_{MR1}^2 + \lambda_{MQ1}^2} \ast J_M
\]

where

\[
\lambda_{MP} = \max_{i=\{1,\ldots,N\}} (\max(\text{Eigenvalue}(P_i))), \lambda_{MQ0} = \max_{i=\{1,\ldots,N\}} (\max(\text{Eigenvalue}(Q_{0i}))) \\
\lambda_{MR0} = \max_{i=\{1,\ldots,N\}} (\max(\text{Eigenvalue}(R_{0i}))), \lambda_{MR1} = \max_{i=\{1,\ldots,N\}} (\max(\text{Eigenvalue}(R_{1i}))) \\
\lambda_{MQ1} = \max_{i=\{1,\ldots,N\}} (\max(\text{Eigenvalue}(Q_{1i})))
\]
where

\[ J_M = \sqrt{\|x_0\|^4 + \left( \int_{-\tau_M}^{0} \|\varphi(s)\|^2 ds \right)^2 + \left( \int_{-\tau_M}^{0} \theta \int_{\theta}^{0} \|\dot{\varphi}(s)\|^2 ds \right)^2 + \left( \int_{-\tau_N_d}^{0} \|\varphi(s)\|^2 ds \right)^2 + \left( \int_{-\tau_N_d}^{0} \theta \int_{\theta}^{0} \|\dot{\varphi}(s)\|^2 ds \right)^2}. \]

\textbf{Proof:} The proof is based on the Lyapunov-Krasovskii approach and Integral Quadratic Constraint (IQC). Let us define the following Lyapunov-Krasovskii functional candidate:

\[ V(\xi, t) = V_1(\xi, t) + V_2(\xi, t) + V_3(\xi, t) + V_4(\xi, t) + V_5(\xi, t) \quad (25) \]

where

\[ V_1(\xi, t) = X^T(t)P(\xi)X(t) \quad (26) \]

\[ V_2(\xi, t) = \int_{t-\tau(t)}^{t} X^T(s)Q_0(\xi)X(s)ds \quad (27) \]

\[ V_3(\xi, t) = \int_{-\tau_M}^{0} \int_{t+\theta}^{t} X^T(s)R_0(\xi)\dot{X}(s)ds \quad (28) \]

\[ V_4(\xi, t) = \int_{t-\tau(t)}^{t} \left[ \begin{array}{c} X(s) \\ X_0(s) \\ X_1(s) \\ \vdots \\
X_{N_d-2}(s) \end{array} \right]^T Q_1(\xi) \left[ \begin{array}{c} X(s) \\ X_0(s) \\ X_1(s) \\ \vdots \\
X_{N_d-2}(s) \end{array} \right] ds \quad (29) \]

\[ V_5(\xi, t) = \int_{-\tau_M}^{0} \int_{t+\theta}^{0} X^T(s)R_1(\xi)\dot{X}(s)ds \quad (30) \]

The IQC is defined as

\[ \Pi(\xi, t) = \int_{0}^{t} \left( \frac{\tau_M\mu(N_d-1)}{2N_d} \right)^2 \frac{1}{1-\mu} X^T(s)Q_2(\xi)\dot{X}(s) - \nabla[X(s)]^TQ_2(\xi)\nabla[X(s)] \right] ds \quad (31) \]

\[ \nabla[\cdot] \] is an operator defined by (8). Consider \( P(\xi) = \sum_{i=1}^{N} \xi_i P_i, \quad P_i > 0; \quad Q_k(\xi) = \sum_{i=1}^{N} \xi_i Q_{ki}, \quad Q_{ki} > 0, \quad k = \{0, 1, 2\} \) and \( R_j(\xi) = \sum_{i=1}^{N} \xi_i R_{iji}, \quad R_{iji} > 0, \quad j = \{0, 1\} \); then \( V(\xi, t) > 0 \). The IQC \( \Pi(\xi, t) \) is positive definite (see proof in Ariba and Gouaisbaut [20]).

The derivative of each element of (25) along the trajectories of (1) provides

\[ \dot{V}_1(\xi, t) = 2X^T(t)P(\xi)\dot{X}(t) \quad (32) \]

\[ \dot{V}_2(\xi, t) = X^T(t)Q_0(\xi)X(t) - (1-\mu)X^T(t-\tau(t))Q_0(\xi)X(t-\tau(t)) \]

\[ = \left[ \begin{array}{c} X(t) \\ \int_{t-\tau(t)}^{t} \dot{X}(s)ds \end{array} \right]^T \left[ \begin{array}{cc} \mu Q_0(\xi) & (1-\mu)Q_0(\xi) \\ * & -(1-\mu)Q_0(\xi) \end{array} \right] \left[ \begin{array}{c} X(t) \\ \int_{t-\tau(t)}^{t} \dot{X}(s)ds \end{array} \right] \quad (33) \]

\[ = \eta^T(t)M_{Q_0}^T \left[ \begin{array}{cc} \mu Q_0(\xi) & (1-\mu)Q_0(\xi) \\ * & -(1-\mu)Q_0(\xi) \end{array} \right] M_{Q_0}\eta(t) \]
\[
\dot{V}_3(\xi, t) = \int_{-\tau_M}^{0} \dot{X}^T(t)R_0(\xi)\dot{X}(t)d\theta - \int_{-\tau_M}^{0} \dot{X}^T(t+\theta)R_0(\xi)\dot{X}(t+\theta)d\theta \\
= \tau_M\dot{X}^T(t)R_0(\xi)\dot{X}(t) - \int_{t-\tau_M}^{t} \dot{X}^T(s)R_0(\xi)\dot{X}(s)ds
\]

(34)

Making use of the Jensen’s inequality, \( \dot{V}_3(\xi, t) \) can be bounded by

\[
\dot{V}_3(\xi, t) \leq \tau_M\dot{X}^T(t)R_0(\xi)\dot{X}(t) - \frac{1}{\tau_M} \left( \int_{t-\tau(t)}^{t} \dot{X}^T(s)ds \right) R_0(\xi) \left( \int_{t-\tau(t)}^{t} \dot{X}(s)ds \right)
\]

(35)

\[
\dot{V}_4(\xi, t) = \left[ \begin{array}{c}
X(t) \\
X_0(t) \\
\vdots \\
X_{N_d-2}(t)
\end{array} \right]^T Q_1(\xi) \left[ \begin{array}{c}
X(t) \\
X_0(t) \\
\vdots \\
X_{N_d-2}(t)
\end{array} \right] - \left(1 - \frac{\mu}{N_d} \right) \left[ \begin{array}{c}
X_0(t) \\
X_1(t) \\
\vdots \\
X_{N_d-1}(t)
\end{array} \right]^T \left[ \begin{array}{c}
X_0(t) \\
X_1(t) \\
\vdots \\
X_{N_d-1}(t)
\end{array} \right] Q_1(\xi)
\]

(36)

\[
\dot{V}_5(\xi, t) \leq \int_{t-\tau_M}^{t} \dot{X}^T(s)ds \left[ \begin{array}{c}
\frac{\tau_M}{N_d} R_1(\xi) \\
\vdots \\
\frac{\tau_M}{N_d} R_1(\xi)
\end{array} \right] \left[ \begin{array}{c}
X(t) \\
\vdots \\
X(t)
\end{array} \right] - \int_{t-\tau_M}^{t} \dot{X}^T(s)ds \left[ \begin{array}{c}
\frac{\tau_M}{N_d} R_1(\xi) \\
\vdots \\
\frac{\tau_M}{N_d} R_1(\xi)
\end{array} \right] M_{R1} \eta(t)
\]

(37)

For the derivative of the IQC (31) along the trajectories of (1), we obtain

\[
\dot{\Pi}(\xi, t) = \left[ \frac{\tau_M(N_d-1)}{2N_d} \right]^2 \left[ 1 - t_{-1N_d} \right] \dot{X}^T(t)Q_2(\xi)\dot{X}(t)
\]

(38)

where the matrices \( M_{Q_0}, M_{R_0}, M_{R_1}, M_{Q_1a}, M_{Q_1b} \) were defined in (23).

Applying the free-weighting matrices technique (Lin and Cheng [18]), the Equation (16) is represented in the following equivalent form

\[
\alpha(t) = 2\eta^T(t) \left[ \begin{array}{c}
N_1^T \\
N_1^T \\
N_1^T \\
N_1^T
\end{array} \right] \left[ \begin{array}{c}
I \\
A_c(\xi) \\
A_c(\xi) \\
A_c(\xi)
\end{array} \right] A_{de} I_P \eta(t)
\]

(39)
Due to Lemma 2.1, the closed-loop system (16) is robustly asymptotically stable with guaranteed cost for the cost function (18) if

\[ V(\xi, t) + \dot{V}(\xi, t) + J(t) + \alpha(t) \leq \eta^T(t)W(\xi)\eta(t) \iff W(\xi) \leq 0 \quad (40) \]

where \( W(\xi) = \sum_{i=1}^{N} \xi_i W_i \). If \( W_i \leq 0 \) for each \( i = \{1, ..., N\} \), then \( W(\xi) = \sum_{i=1}^{N} \xi_i W_i \leq 0 \). The bound on guaranteed cost can be obtained in the following way. Inequality (40) implies

\[ \dot{V}(\xi, t) + \bar{V}(\xi, t) \leq -J(t) \leq 0 \quad (J(t) \geq 0), \]

respectively. \( J(t) \leq \dot{V}(\xi, t) - \bar{V}(\xi, t) \). With \( X(t) = [\varphi^T(t), 0]^T, \forall t \in [-\tau_M, 0] \), after integrating both sizes of \( J(t) \leq \dot{V}(\xi, t) - \bar{V}(\xi, t) \), we obtain

\[
J \leq V_0 \leq \lambda_{MP} \|x_0\|^2 + \lambda_{MQ_0} \int_{-775}^{0} \|\varphi(s)\|^2 ds + \lambda_{MR_0} \int_{-775}^{0} d\theta \int_{0}^{\theta} \|\varphi(s)\|^2 ds \\
+ \lambda_{MQ_1} \left( N_d \int_{-\frac{771}{\tau_d}}^{0} \|\varphi(s)\|^2 ds \right) + \lambda_{MR_1} \int_{-\frac{771}{\tau_d}}^{0} d\theta \int_{0}^{\theta} \|\varphi(s)\|^2 ds
\]  

It is known that for two arbitrary vectors \( \tilde{X}, \tilde{Y} \), the following inequality holds:

\[ |\tilde{X}^T \tilde{Y}| \leq \|\tilde{X}\| \|\tilde{Y}\| \quad (41) \]

Consider \( \tilde{X}^T = [\lambda_{MP}, \lambda_{MQ_0}, \lambda_{MR_0}, \lambda_{MQ_1}, \lambda_{MR_1}] \) and

\[
\tilde{Y}^T = \begin{bmatrix} \|x_0\|^2 & \int_{-775}^{0} \|\varphi(s)\|^2 ds & \int_{-\tau_d}^{0} d\theta \int_{0}^{\theta} \|\varphi(s)\|^2 ds & N_d \int_{-\frac{771}{\tau_d}}^{0} \|\varphi(s)\|^2 ds & \int_{-\frac{771}{\tau_d}}^{0} d\theta \int_{0}^{\theta} \|\varphi(s)\|^2 ds \end{bmatrix}
\]

Applying the inequality (41), to the above inequality the upper bound of cost function (18) \( J_0 \) is obtained as (24) which end the proof of Theorem 3.1.

Theorem 3.1 states that if there exists a feasible solution of (22) with respect to \( P_1 > 0, Q_{00} > 0, Q_{10} > 0, Q_{20} > 0, R_{00} > 0, R_{10} > 0 (i = \{1, ..., N\}) \), \( N_1, N_2, N_3 \), then there exists a PI controller \( F = [K_P, K_I] \) guaranteeing both robust parameter-dependent quadratic stability and the upper bound on integral cost function (18) - guaranteed cost.

4. Numerical Examples. In this section, we present the results of numerical calculations for two examples to demonstrate the effectiveness of the proposed method. In the first example, we compare results obtained by the proposed method, where PI controller has been designed for partitionings of time-delay \( N_d = \{1, 2, 3, 5\} \) with the previous method in (Vesely and Nguyen Quang [14]) on 1000 randomly generated examples. These examples are generated without checking their static output feedback stabilizability. In the second example, we apply the proposed method in order to design robust output feedback PI controller for the example (Esfahani et al. [12]). These examples are generated without checking their static output feedback stabilizability. Numerical calculations have been realized by using PEN-BMI in both examples.

**Example 4.1.** For each case, 1000 examples have been generated with following parameters: the system of 1st, 2nd and 3rd order, with upper bound on unstable eigenvalues: \( 0 < \max(\text{real}(\text{eig})) \leq 0.1 \), each system is SISO system with one uncertainty (two vertices are calculated), generated time-delay mean value \( \tau_{middle} = 200 [\text{ms}] \), time-delay change rate \( \dot{\tau}(t) \leq 0.5 \). For comparison we include also the results for 2nd order stable system. In Table 1, numbers of successful solutions (out of 1000 generated examples) are summarized.

The parameters of cost function are \( R = r I \), \( r = 1 \); \( Q = q I \), \( q = 0.1 \).
The results show that increasing the number of parts $N_d$, to which the time-delay interval is divided, reduces the conservatism of robust stability condition (21), therefore, the number of successful PI controller designs increases with the increased $N_d$.

Table 1. Numerical results for various partitions $N_d$ of time-delay interval: number of successful PI design (out of 1000 generated examples)

<table>
<thead>
<tr>
<th>$N_d$ (or method)</th>
<th>Order of system</th>
<th>$1^{st}$</th>
<th>$2^{nd}$</th>
<th>$3^{rd}$</th>
<th>$2^{nd}$ (stable, max(real(eig) ≤ -0.1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>749</td>
<td>640</td>
<td>530</td>
<td>907</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>763</td>
<td>647</td>
<td>530</td>
<td>909</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>764</td>
<td>650</td>
<td>535</td>
<td>908</td>
</tr>
</tbody>
</table>

Example 4.2. Consider the uncertain time-delay system

\[
\dot{x}(t) = \begin{bmatrix} 1 + 0.33F(t) & 0.42F(t) \\ 0.53F(t) & -2 + 0.67F(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0.32 \\ 0 & -0.5 \end{bmatrix} x(t - \tau) + \begin{bmatrix} 0.47 \\ 0.75 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} x(t), \\
F^2(t) \leq 1.
\]

In this case, the maximal time delay $\tau_M[s]$ has been determined, for which the PI controller design has been successful. The results of calculation for $r = 0.001$, $q = 0.2$, degree of stability $\gamma = 0.001$, $\tau_0 = 1000$ are summarized in Table 2.

The obtained results clearly show that increasing $N_d$ reduces the conservatism of robust stability condition: this is indicated in Table 2. by increasing time-delay bound $\tau_M$ for which the robust stability and performance bound are guaranteed by the designed robust PI controller.

Since the controller design requires solution of BMI, and the size of the problem significantly increases with increased $N_d$, this limits the choice of increasing $N_d$. Nevertheless, even the choice of $N_d = \{2, 3\}$ brought notable progress.

Table 2. Numerical results for Example 4.2: maximal time delay for which the PI controller has been calculated

<table>
<thead>
<tr>
<th>$N_d$</th>
<th>$\tau_M[s]$</th>
<th>$F = [K_p \ K_I]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.325</td>
<td>[3.4847 0.0081]</td>
</tr>
<tr>
<td>2</td>
<td>0.328</td>
<td>[3.4199 0.063]</td>
</tr>
<tr>
<td>3</td>
<td>0.330</td>
<td>[3.3512 0.0579]</td>
</tr>
</tbody>
</table>

5. Conclusions. The paper addresses the problem of output feedback guaranteed cost controller design for Networked Uncertain Control Systems with time-varying delay and polytopic uncertainties. Based on partitioning scheme of time-varying delay and using integral quadratic constraint, a new discretized Lyapunov-Krasovskii functional method is proposed to design a robust PI controller achieving a guaranteed cost and parameter-dependent quadratic stability such that the NCSs is stable for all admissible uncertainties and bounded time-varying delays. The solution has been obtained using PEN-BMI script.

The obtained numerical results show that when partitioning scheme of time-varying delay and IQC are used the robust controller design procedure conservatism is decreased.
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REFERENCES