STATE FEEDBACK $H_2$ OPTIMAL CONTROLLERS UNDER REGULATION CONSTRAINTS FOR DESCRIPTOR SYSTEMS

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Abstract. This paper is concerned with a non-standard multi-objective state feedback control problem for continuous descriptor systems. In this problem, an output is to be regulated asymptotically with presence of an infinite-energy exo-system, while a desired $H_2$ performance from a finite external disturbance to a tracking error has also to be satisfied. Thanks to the descriptor framework, not only unstable but also nonproper behaviors can be treated. A parametrization of all optimal dynamic and static controllers solving the proposed multi-objective control problem is given. Moreover, an application to the non-standard LQR problem is investigated. A numerical example shows the efficiency of the proposed results.

Keywords: Asymptotic regulation, Controller parametrization, Descriptor systems, $H_2$ optimal control

1. Introduction. Descriptor (or singular) systems have been attracting the attention of many researchers over recent decades due to their capacity to preserve the structure of physical systems and describe non-dynamic constraints and impulsive behaviors [1, 2]. These systems arise in large-scale systems networks [3], circuits [4], boundary control system [5] and power systems. A number of control issues have been successfully extended to descriptor systems and the related results have been reported, for instance, in [1, 2, 6, 7, 8, 9, 10, 11, 12] and the references therein.

On the other hand, the regulation problem (or asymptotic tracking/rejection problem) plays an essential role in control theory, and has been studied by many scholars. The seminal result, known as the Internal Model Principle, was developed in the 1970s [13, 14] to deal with such a problem. Based on this principle, exact asymptotic rejection is achieved by a structured controller containing a copy of the dynamics of the exo-system. Moreover, extensions of this scheme have been considered by integrating other performance objectives, for instance, $H_2$ and $H_{\infty}$ performance. Such multi-objective problems have been extensively investigated in the literature; e.g., see [15, 16, 17, 18, 19] and the references therein. An alternative method for solving these problems consists in reformulating the problems through the use of unstable weighting filters [20, 21]. Moreover, the regulation problem for descriptor systems has also been studied. For example, in [2], the author has provided a solution to this problem in terms of a set of nonlinear matrix equations depending on system parameters and some other parameters. In [22], via solving a generalized Sylvester equation, Lin and Dai have investigated the regulation problem for the special case where the measurement output is identical to the vector to be regulated. However, to the best of the authors’ knowledge, little attention has been paid to the performance control problem under regulation constraints for descriptor systems, except for the results
in [23], where the authors have considered the state feedback $H_2$ control problem with presence of unstable exo-systems.

The main contribution of this paper is the parameterization of all controllers solving the asymptotic regulation problem with guaranteed $H_2$ performance via state feedback for continuous descriptor systems. The plant and the exo-system are both described within the descriptor framework. Hence, it is possible to take into account not only unstable, but nonproper exogenous reference/disturbance models as well. A generalized Sylvester-type equation is proposed to achieve internal stabilization subject to asymptotic regulation constraints. Then, the class of optimal state feedback controllers is explicitly characterized based on the results in [7].

This paper is briefly outlined as follows: Section 2 recalls basic notations of descriptor systems and formulates the problem of the optimal state feedback $H_2$ control with regulation constraints; the main result of the paper is then deduced in Section 3, where a necessary and sufficient condition for the solvability of the proposed problem is given, and the parametrization of the set of all optimal controllers is formulated; finally, a numerical example is included in Section 4 to illustrate the efficiency of the proposed results and the conclusion takes place in the last section.

Notation: The superscripts '⊤' and '*' represent the transpose and complex conjugate transpose, respectively. A continuous descriptor system $G$ associated with the system data $(E, A, B, C, D)$ is represented by $(G): \begin{bmatrix} E & A \\ B & C \\ D \end{bmatrix}$. Moreover, without special stated, all matrices used in this paper are supposed to have appropriate dimensions.

2. Preliminaries and Problem Formulation.

2.1. Preliminaries. In this subsection, we recall some useful notations for continuous descriptor systems, which will be used in the sequel.

Let us consider the following descriptor system:

$$
\begin{cases}
E \dot{x}(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t) + Du(t),
\end{cases}
$$

(1)

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$ are the descriptor variable, measurement and control input vector, respectively. The matrix $E \in \mathbb{R}^{n \times n}$ may be singular, i.e., $\text{rank}(E) = r \leq n$; and $A$, $B$, $C$ and $D$ are known real constant matrices with appropriate dimensions.

The finite eigenvalues of $sE - A$ are called the finite dynamic modes. The infinite eigenvalues of $sE - A$ with the eigenvector $v$ satisfying $Ev = 0$ determine the static modes; while the infinite eigenvalues with the generalized eigenvectors $v_k$ such that $Ev_1 = 0$ and $Ev_k = Av_{k-1}$ $(k \geq 2)$ are the impulsive modes.

The descriptor system (1) is said to be [2].

a): regular if $\text{det}(sE - A)$ is not identically null;
b): impulse-free if $\text{deg}(	ext{det}(sE - A)) = \text{rank}(E)$;
c): stable if all the roots of $\text{det}(sE - A) = 0$ have negative real parts;
d): admissible if it is regular, impulse-free and stable;
e): finite dynamics stabilizable if there exists $F$ such that the pair $(E, A + BF)$ is regular and stable;
f): impulse controllable if there exists $F$ such that the pair $(E, A + BF)$ is regular and impulse-free.

If the descriptor system is regular, then its transfer function can be written as:

$$
G(s) = C(sE - A)^{-1}B + D.
$$

(2)
Definition 2.1. Denote $RH_2$ the set of strictly proper and real rational stable transfer matrices. For $F \in RH_2$, its $H_2$ norm is defined as:

$$
\|F\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}(F^*(j\omega)F(j\omega)) \, d\omega.
$$

Throughout this paper, we define constant matrices $U$, $V \in \mathbb{R}^{n \times (n-r)}$ with full column rank satisfying

$$
E^\top U = 0, \quad EV = 0;
$$

and also $E_L, E_R \in \mathbb{R}^{n \times (n-r)}$ with full column rank satisfying $E_L^\top U = 0$, $E_R^\top V = 0$. Under this definition, $E$ can be decomposed as $E = E_L \Sigma E_R^\top$, with $\Sigma \in \mathbb{R}^{r \times r}$ nonsingular.

Then, the transform matrices $M$, $N$ transforming the descriptor system into the singular value decomposition (SVD) form [24] can be constructed as:

$$
M = \left[ \Sigma_1^{-1}(E_L^\top E_L)^{-1} E_L^\top \right], \quad N = \left[ E_R (E_R^\top E_R)^{-1} V \right].
$$

2.2. Optimal $H_2$ control under regulation constraints. Consider a continuous descriptor plant written by:

$$
(G) : \begin{bmatrix}
E \dot{x} \\
e \\
z \\
y
\end{bmatrix} = \begin{bmatrix}
A & B_w & B_d & B \\
C_e & D_{ew} & D_{ed} & D_{eu} \\
C_z & D_{zw} & D_{zd} & D_{zu} \\
i_n & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
x \\
w \\
d \\
u
\end{bmatrix},
$$

where $e \in \mathbb{R}^{q_e}$, $z \in \mathbb{R}^{q_z}$, $y \in \mathbb{R}^p$, $w \in \mathbb{R}^{l_w}$, $d \in \mathbb{R}^{l_d}$ and $u \in \mathbb{R}^m$ are the tracking error, controlled output, measurement, exogenous disturbance, external disturbance and control input vector, respectively. $A$, $B_w$, $B_d$, $B$, $C_e$, $C_z$, $C$, $D_{ew}$, $D_{ed}$, $D_{eu}$, $D_{zw}$, $D_{zd}$ and $D_{zu}$ are known real constant matrices with appropriate dimensions.

The following assumption is made in the sequel:

(A.1): $(E, A, B)$ is finite dynamics stabilizable and impulse controllable.

The exogenous disturbance $w$ is generated by an exo-system $G_w$ of the form:

$$
(G_w) : E_w \dot{w} = A_w w,
$$

where the matrix $E_w$ may be singular, i.e., $\text{rank}(E_w) = r_w \leq l_w$.

Remark 2.1. For simplifying presentation of the current paper, we assume that $G_w$ only holds unstable and/or impulsive modes. This assumption causes no loss of generality since the output regulation of the stable and static modes of the exo-system is a mere consequence of internal stabilization.

Denote the new descriptor variable as $\chi^\top = [x^\top \ w^\top]$. Then, the descriptor plant $G$ can be rewritten as:

$$
(G) : \begin{bmatrix}
E & 0 \\
0 & E_w
\end{bmatrix}, \quad \begin{bmatrix}
A & B_w & B_d & B \\
0 & A_w & 0 & 0 \\
C_e & D_{ew} & D_{ed} & D_{eu} \\
C_z & D_{zw} & D_{zd} & D_{zu} \\
i_n & 0 & 0 & 0
\end{bmatrix} := \begin{bmatrix}
G_{ed}(s) & G_{eu}(s) \\
G_{zd}(s) & G_{zu}(s) \\
G_{yd}(s) & G_{yu}(s)
\end{bmatrix}.
$$

Let us consider a state feedback controller of the form

$$
(\Sigma_c) : u = F \chi.
$$
Problem 2.1 (Optimal $H_2$ Control Problem with Asymptotic Regulation Constraints). Given the descriptor plant $G$ shown in (8). The optimal $H_2$ control problem with asymptotic regulation constraints consists in finding a state feedback controller $\Sigma_e$ in (9) such that the following conditions hold:

C.1: (Internal Stability) In the absence of the disturbances $w$ and $d$, the closed-loop system formed by $G$ and $\Sigma_e$ is internally stable (admissible);

C.2: (Asymptotic Regulation) $\lim_{t \to \infty} e(t) = 0$ for any $d \in L_2$, and for all $x(0) \in \mathbb{R}^n$ and $w(0) \in \mathbb{R}^{l_w}$;

C.3: (Optimal $H_2$ Performance) The $H_2$ norm of the closed-loop system $T_{zd} = G_{zd} + G_{zu}(1 - G_{yu}\Sigma_e)^{-1}G_{yd}$ is minimized.

3. Main Result. In this section, the solution to the optimal $H_2$ control problem with asymptotic regulation constraints is given.

3.1. Internal stabilization with asymptotic regulation.

Lemma 3.1. Consider the descriptor plant (8). Assume that Assumption (A.1) holds. There exists a state feedback controller in (9) such that the conditions C.1 and C.2 of Problem 2.1 hold if and only if there exists $T \in \mathbb{R}^{n \times l_w}$, $R \in \mathbb{R}^{(n-r) \times l_w}$, $S \in \mathbb{R}^{n \times (l_w-r_w)}$ and $F_a \in \mathbb{R}^{m \times l_w}$ satisfying the following generalized Sylvester-type equation:

$$
BF_a = A(TE_w - VR) - B_w - (ET + SU_w^\top)A_w,
$$

$$
D_{eu}F_a = C_e(TE_w - VR) - D_{ew},
$$

(10)

where $V$ is defined in (4) and $U_w \in \mathbb{R}^{l_w \times (l_w - r_w)}$ is any full column rank matrix satisfying $E_w^\top U_w = 0$. Moreover, the gain $F$ is given as:

$$
F = [F_1 \ F_a + F_1(TE_w - VR)],
$$

(11)

with any $F_1$ such that the pair $(E, A + BF_1)$ is admissible.

Proof: (Sufficiency) Suppose that there exists $T$, $R$, $S$ and $F_a$ satisfying (10) and the state feedback gain $F$ is formed as (11). According to Assumption (A.1), we can always find $F_1$ such that $(E, A + BF_1)$ is admissible. The closed-loop can be written as:

$$
\begin{bmatrix}
E & 0 \\
0 & E_w
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{w}
\end{bmatrix}
= \begin{bmatrix}
A + BF_1 & B_w + B(F_a + F_1(TE_w - VR)) \\
0 & A_w
\end{bmatrix}
\begin{bmatrix}
x \\
w
\end{bmatrix}
+ \begin{bmatrix}
B_d \\
0
\end{bmatrix}d,
$$

$$
e = [C_e + D_{eu}F_1 \ D_{eu} + D_{eu}(F_a + F_1(TE_w - VR))] \begin{bmatrix}
x \\
w
\end{bmatrix}
+ D_{ed}d.
$$

(12)

Let us introduce two non-singular matrices $\mathcal{M}$ and $\mathcal{N}$:

$$
\mathcal{M} = \begin{bmatrix}
I_n & ET + SU_w^\top \\
0 & I_w
\end{bmatrix},
\mathcal{N} = \begin{bmatrix}
I_n & -TE_w + VR \\
0 & I_w
\end{bmatrix}.
$$

Applying the coordinate transformation with $\mathcal{M}$, $\mathcal{N}$ and using the condition (10) yield an alternative representation of the above closed-loop:

$$
\begin{bmatrix}
E & 0 \\
0 & E_w
\end{bmatrix}
\begin{bmatrix}
\dot{\zeta} \\
\dot{w}
\end{bmatrix}
= \begin{bmatrix}
A + BF_1 & 0 \\
0 & A_w
\end{bmatrix}
\begin{bmatrix}
\zeta \\
w
\end{bmatrix}
+ \begin{bmatrix}
B_d \\
0
\end{bmatrix}d,
$$

$$
e = (C_e + D_{eu}F_1)\zeta + D_{ed}d.
$$

(13)

with $\zeta = x - (TE_w - VR)w$.

Since $(E, A + BF_1)$ is admissible and $d \in L_2$, the closed-loop is internally stable. In addition, $\lim_{t \to \infty} e(t) = \lim_{t \to \infty} ((C_e + D_{eu}F_1)\zeta(t) + D_{ed}d(t)) = (C_e + D_{eu}F_1)\lim_{t \to \infty} \zeta(t) + D_{ed}\lim_{t \to \infty} d(t) = 0$. Hence, the conditions C.1 and C.2 are both satisfied.
(Necessity) Suppose that there exists \( F = [F_1 \ F_2] \) with \( F_1 \) satisfying that \((E, A + BF_1)\) is admissible, and the resulting closed-loop:

\[
\begin{bmatrix}
E & 0 \\
0 & E_w
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{w}
\end{bmatrix}
= \begin{bmatrix}
A + BF_1 & B_w + BF_2 \\
0 & A_w
\end{bmatrix}
\begin{bmatrix}
x \\
w
\end{bmatrix}
+ \begin{bmatrix}
B_d \\
0
\end{bmatrix} d,
\]

\[e = \begin{bmatrix} C_e + D_{eu} F_1 & D_{ew} + D_{eu} F_2 \end{bmatrix}
\begin{bmatrix} x \\
w
\end{bmatrix}
+ D_{ed} d,
\]

meets the conditions C.1 and C.2. Applying the coordinate transformation with \( \mathcal{M}, \mathcal{N} \) in (12) leads to

\[
\begin{bmatrix}
E & 0 \\
0 & E_w
\end{bmatrix}
\begin{bmatrix}
\dot{\zeta} \\
\dot{w}
\end{bmatrix}
= \begin{bmatrix}
A + BF_1 & \mathfrak{B}_w \\
0 & A_w
\end{bmatrix}
\begin{bmatrix}
\zeta \\
w
\end{bmatrix}
+ \begin{bmatrix}
B_d \\
0
\end{bmatrix} d,
\]

\[
e = (C_e + D_{eu} F_1) \zeta + \mathfrak{D}_w w + D_{ed} d,
\]

where \( \mathfrak{B}_w = BF_a + B_w + (ET + SU^\top_w) A_w - A(TE_w - VR), \mathfrak{D}_w = D_{eu} F_a + D_{ew} - C_e (TE_w - VR) \) and \( F_a = F_2 - F_1 (TE_w - VR) \).

Since the closed-loop is internally stable, the exo-system should not appear in the closed-loop. Hence, we have \( \mathfrak{B}_w = 0 \). Moreover, the condition C.2 implies \( \mathfrak{D}_w = 0 \), otherwise \( \lim_{t \to \infty} \varepsilon(t) = \mathfrak{D}_w \lim_{t \to \infty} w(t) \neq 0 \), which is contradictory to the asymptotic regulation constraint. Hence, (10) holds. This completes the proof.

**Remark 3.1.** When \( E = I_n, \ E_w = I_{iw} \), the generalized Sylvester-type Equation (10) reduces to the regulator equation reported in [18, 20, 19, 21] for state-space systems. Furthermore, if we take \( E_w = I_{iw} \), the underlying generalized Sylvester-type equation reduces to the generalized Sylvester equation reported in [22].

### 3.2. Optimal \( H_2 \) control with asymptotic regulation.

Based on the solution proposed previously to asymptotic regulation objective, Problem 2.1 is handled in this subsection. The parametrization of the class of optimal controllers is given.

Using the matrices \( U, V \) defined in (4), let us make the following assumptions:

\( (\text{A.2}): (E, A) \) is regular;

\( (\text{A.3}): D_{zu} \) has full column rank;

\( (\text{A.4}): \begin{bmatrix} A - j \omega E & B \\ C_z & D_{zu} \end{bmatrix} \) has no invariant zeros on the imaginary axis including infinity;

\( (\text{A.5}): \ker \begin{bmatrix} U^\top AV & U^\top B \\ C_e V & D_{zu} \end{bmatrix}^\top \subseteq \ker \begin{bmatrix} U^\top B_d \\ D_{zd} \end{bmatrix}^\top; \)

\( (\text{A.6}): \ker U^\top B_d \subseteq \ker D_{zd}. \)

**Remark 3.2.** Assumptions (A.1)-(A.4) ensure the solvability of the generalized algebraic Riccati equation (GARE) (15). These assumptions coincide with the classical assumptions in the \( H_2 \) problem for conventional state-space systems (see [25]). Note that (A.3) can be made without loss of generality under the descriptor framework. If it does not hold, an equivalent realization satisfying this assumption can always be obtained. Moreover, the two kernel conditions (A.5) and (A.6) are necessary and sufficient conditions for strict properness of the underlying closed-loop. Referring the reader to [6, 7] for a comprehensive discussion of Assumptions (A.1)-(A.6).

Now, we are in a position to give the solution to the multi-objective control problem defined in Section 2.

**Theorem 3.1.** (Main Result) Consider the descriptor plant (8), and the matrices \( U, V \) and \( M \) defined in (4) and (5), respectively. Assume that Assumptions (A.1)-(A.6) hold.
There exists a state feedback controller in (9) such that Problem 2.1 is solved if and only if there exists \( T \in \mathbb{R}^{n \times l_w}, R \in \mathbb{R}^{(n-r) \times l_w}, S \in \mathbb{R}^{n \times (l_w-r_w)} \) and \( F_a \in \mathbb{R}^{n \times l_w} \) satisfying (10). Moreover, the class of optimal controllers solving Problem 2.1 is parameterized by:

\[
F(s) = \left[ \mathcal{F}(s) \quad F_a + \mathcal{F}(s)(TE_w - VR) \right]
\]

with

\[
\mathcal{F}(s) := F_c + (I + (\Psi + W(s)\Pi)B)^{-1}(\Psi + W(s)\Pi)(sE - A - BF_c),
\]

where

\begin{enumerate}
  \item[i]: \( F_c := -(D_{zu}^T D_{zu})^{-1}(D_{zu}^T C_z + B^T X), \) in which \( X \) is an admissible solution to the following GARE
  \[
  \begin{cases}
  E^T X = X^T E \geq 0, \\
  A^T X + X^T A + C_z^T C_z - (C_z^T D_{zu} + X^T B)(D_{zu}^T D_{zu})^{-1}(D_{zu}^T C_z + B^T X) = 0;
  \end{cases}
  \]
  \item[ii]: \( \Pi := I - B_d B_d^T, \) in which \( B_d^T \) is the pseudo-inverse of \( B_d; \)
  \item[iii]: \( \Psi := \begin{bmatrix} 0 & \Theta \end{bmatrix} (U^T (A + BF_c)V)^{-1} \) \( M, \) in which \( \Theta \) is the solution to the following equation:
  \[
  \begin{align*}
  &\left( D_{zu} - (C_z + D_{zu} F_c) V (U^T (A + BF_c)V)^{-1} U^T B \right) \\
  = &\left( D_{zd} - (C_z + D_{zu} F_c) V (U^T (A + BF_c)V)^{-1} U^T B_d \right);
  \end{align*}
  \]
  \item[iv]: \( W(s) \in RH_\infty \) such that \( \det (I + (\Psi + W(s)\Pi)B) \neq 0. \)
\end{enumerate}

Furthermore, the minimal value of \( H_2 \) norm of the closed-loop system is \( \|T_{zd}\|_2 = \|G_{F_c,\Psi}\|_2, \) where

\[
\begin{bmatrix} A + BF_c \\ C_z + D_{zu} F_c \end{bmatrix} = \begin{bmatrix} B_d + B \Psi B_d \\ D_{zd} + D_{zu} \Psi B_d \end{bmatrix}.
\]

**Proof:** The asymptotic regulation objective is achieved if and only if there exists \( T, R, S \) and \( F_a \) such that (10) holds. It observes, from the preceding discussion, that when the additional optimal \( H_2 \) performance is considered, Problem 2.1 can be equivalently stated as: for the following plant \( \hat{G} \)

\[
\hat{G} := \begin{bmatrix}
E & 0 & \begin{bmatrix} A & 0 \\
0 & A_w \end{bmatrix} & \begin{bmatrix} B_d \\ 0 \end{bmatrix} & \begin{bmatrix} B \\ 0 \end{bmatrix}
\end{bmatrix}
\]

with \( \mathcal{D}_z = D_{zu} F_a + D_{uw} - C_z(T E_w - VR) \), find a state feedback control law such that Problem 2.1 is solved. It is easy to see that if the control law internally stabilizes the system \( \begin{bmatrix} E, \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix} \end{bmatrix} \), then the asymptotic regulation objective is achieved. Hence, the additional \( H_2 \) performance objective can be regarded as the optimal state feedback \( H_2 \) control problem for the plant \( \hat{G} \) of the form:

\[
\hat{G} := \begin{bmatrix}
E, \begin{bmatrix} A & B \\ C_z & D_{zd} \end{bmatrix} & B \\
I_n & 0 & 0 
\end{bmatrix}.
\]

Since Assumptions (A.1)-(A.4) hold, the GARE (15) admits an admissible solution \( X \). Using the results in [7] gives the class of optimal \( H_2 \) controllers parameterized by (14)
for $\hat{G}$. Moreover, according to the controller structure in (11), the class of optimal $H_2$ controllers solving Problem 2.1 is formed as (13). The proof is completed.

It is well-known that the LQR problem can be viewed as a special case of the state feedback $H_2$ problem. We introduce next an LQR control problem under regulation constraints for descriptor systems.

**Problem 3.1 (LQR Problem under Regulation Constraints).** Consider the following descriptor system:

\[
\begin{cases}
E\dot{x} = Ax + Bw + Bu,

e = C_w x + D_{eu} w + D_{eu} u,
\end{cases}
\]

(18)

where $w$ is generated by the exo-system $G_w$ in (7). The LQR problem with asymptotic regulation constraints consists in finding a state feedback control $u$ such that the conditions C.1 and C.2 of Problem 2.1 hold, and the cost functional $J$:

\[
J = \int_0^\infty (C_z x(t) + D_{zu} u(t))^\top (C_z x(t) + D_{zu} u(t)) dt
\]

is minimized as well.

This LQR problem can be alternatively stated as Problem 2.1 for the plant $(G_{LQ}) :

\[
(G_{LQ}) : \begin{bmatrix}
E & 0 \\
0 & E_w
\end{bmatrix}, \begin{bmatrix}
A & B_w \\
0 & A_w
\end{bmatrix} \begin{bmatrix}
E & 0 \\
B & 0
\end{bmatrix}
\]

(20)

**Corollary 3.1.** Consider the descriptor system (18). Assume that Assumptions (A.1)-(A.4) hold. Problem 3.1 is solvable if and only if Problem 2.1 associated with the descriptor system $G_{LQ}$ (20) is solvable. Moreover, the class of all controllers solving Problem 3.1 is parameterized as follows:

\[
F(s) = [F(s) F_a + F(s)(TE_w - VR)]
\]

(21)

with

\[
F(s) := F_c - (I + (\Gamma + W(s)U) U^\top B)^{-1} (\Gamma + W(s)U) U^\top (A + BF_c),
\]

(22)

where

i): $T, R, S, F_a$ satisfy (10), and $U, V$ are defined in (4);

ii): $F_c := - (D_{zu} D_{wu})^{-1} (D_{wu} C_z + B^\top X)$ with $X$ an admissible solution to the GARE (15);

iii): $\Gamma \in \mathbb{R}^{m \times (n-r)}$ and $W(s) \in RH_{\infty}$ such that $\det(I + (\Gamma + W(s)U) U^\top B) \neq 0$.

Furthermore, the class of all static controllers solving Problem 3.1 is given with parameter $\Gamma \in \mathbb{R}^{m \times (n-r)}$ as:

\[
F = [F_a + F(TE_w - VR)],
\]

(23)

and

\[
F := F_c - (I + \Gamma U^\top B)^{-1} \Gamma U^\top (A + BF_c),
\]

(24)

in which $\det(I + \Gamma U^\top B) \neq 0$. 
4. Numerical Example. Consider the following descriptor system

\[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix} \dot{x} = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix} x + \begin{bmatrix}
10 & 0 \\
0 & 10 \\
\end{bmatrix} w + \begin{bmatrix}
0 \\
1 \\
\end{bmatrix} u,
\]

where \( w \) is a sinusoidal disturbance such that \( \dot{w} = \begin{bmatrix}
0 \\
5 \\
-5 \\
0 \\
\end{bmatrix} w \). We are interested in finding all static feedback controllers for the LQR problem under regulation constraints with regard to the cost functional: \( J = \int_0^\infty \| x_1(t) \|^2 + \| x_2(t) \|^2 + \| u(t) \|^2 dt \).

Note that the reported results in [7] fail to provide a solution owing to the existence of the uncontrollable sinusoidal disturbance \( w \). According to Lemma 3.1, the following data are obtained:

\[
T = \begin{bmatrix}
0 & 1 \\
2.5 & 0 \\
\end{bmatrix}, \quad R = \begin{bmatrix}
-2.5 & 0 \\
0 & -9 \\
\end{bmatrix}, \quad F_a = \begin{bmatrix}
0 & -9 \\
\end{bmatrix}.
\]

The GARE (15) admits two stabilizing solutions

\[
X_1 = \begin{bmatrix}
\sqrt{2} & 0 \\
1 + \sqrt{2} & 1 \\
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
\sqrt{2} & 0 \\
1 - \sqrt{2} & -1 \\
\end{bmatrix}.
\]

Hence, \( F_c = [-1 \mp \sqrt{2} \mp 1] \). According to Corollary 3.1, the class of all static controllers is parameterized as:

\[
F = [-1 \mp \sqrt{2} \pm \psi \sqrt{2} \mp 1 \pm \psi \mp 5 \pm 5 \psi \mp 10 \mp \sqrt{2} \pm \psi \sqrt{2}],
\]

with \( \psi = \frac{\Gamma}{1 + \Gamma} \) and \( \Gamma \neq -1 \). Moreover, the optimal closed-loop system is given by:

\[
(G_{F_c, \psi}) : \begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
\mp \sqrt{2} & \mp 1 \\
1 & 0 \\
\mp 1 & \mp \sqrt{2} \\
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix} = \frac{1}{s + \sqrt{2}} \begin{bmatrix}
1 & 0 \\
-\sqrt{2} & 0 \\
-1 & 0 \\
\end{bmatrix}.
\]

And thus the optimal cost is \( J_{\text{opt}} = \| G_{F_c, \psi} \|_2 = 1.1892, \forall \Gamma \neq -1 \).
5. Conclusion. This paper deals with a non-standard control problem for continuous descriptor systems where a regulation requirement together with an $H_2$ performance has to be satisfied. Thanks to the descriptor framework, both unstable and nonproper exosystems are allowed. It is shown that the asymptotic regulation constraint is achieved if and only if a generalized Sylvester-type equation admits a solution. Parametrizations of all optimal dynamic and static state feedback $H_2$ controllers are given in this case. In addition, the LQR problem subject to regulation constraints is also addressed. The present results will underpin future work on the output feedback $H_2$ and $H_\infty$ control problems under regulation constraints.

REFERENCES

