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NUMERICAL SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND BY GENERAL LEGENDRE WAVELETS

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Abstract. *A general Legendre wavelets method is applied to Fredholm integral equation of the second kind. Using the general operational matrix of integration we approximate the solution of Fredholm integral equation of the second kind by general Legendre wavelets. Numerical examples are illustrated.*

Keywords: General Legendre wavelets, Fredholm integral equation, Numerical solution

1. **Introduction.** It is known that some boundary value problems can be transformed into Fredholm integral equations. For example, the boundary value problem

$$\frac{d^2x}{dt^2} + \lambda x = \varphi(t), \quad x(0) = x(1) = 0$$

can be transformed into the Fredholm integral equation of the second kind

$$x(t) = \lambda \int_0^1 K(t,s)x(s)\mathrm{d}s + u(t), \quad t \in [0,1],$$

where

$$K(t,s) = \begin{cases} s(1-t), & 0 \le s \le t, \\ t(1-s), & t < s \le 1, \end{cases} \quad u(t) = -\int_0^1 t(1-s)\varphi(s)\mathrm{d}s.$$

It is seen that Fredholm integral equations may be applied to boundary value problems in practice. It is known that the analytic solutions of Fredholm integral equations are obtained difficultly. There are different methods to approximate the solutions of Fredholm integral equations. Wavelets theory has been applied in a wide range of engineering disciplines [1]. A Legendre wavelet method for solving nonlinear optimal control problems with inequality constraints was presented in [2]. The authors in [3] used this Legendre wavelet method to solve the nonlinear Volterra-Fredholm integral equations. By piecewise constant two-dimensional block-pulse functions and their operational matrix of integration, two-dimensional first kind integral equations were reduced to a lower triangular system in [4]. Orthogonal functions or polynomials, such as Walsh [5], block-pulse functions [6], Chebyshev [7], Laguerre [8], Legendre [9], Fourier [10] and hybrid functions [11, 12, 13] are developed for solving various systems. Now we use the general Legendre wavelets to solve Fredholm integral equation of the second kind. The advantage of the general Legendre wavelets gives a kind of fast algorithm for easy implementation. Using the general operational matrix and expression of an integral function Fredholm integral equation of the second kind is transformed into a system of algebraic equations by the general Legendre wavelets. Approximate solutions of Fredholm integral equation of the second kind are derived. The numerical examples demonstrate that the algorithm is valid.

2. Preliminaries.

2.1. **Definitions.** The general Legendre wavelets $\psi_{km}(t)$, $k=1,2,\dots,N$, $m=0,1,\dots,M-1$, on the interval $[a,b)$ are defined as

$$\psi_{km}(t)=\begin{cases}\left[(2m+1)d_k^{-1}\right]^{\frac{1}{2}}L_m(d_k^{-1}(2t-t_{k-1}-t_k)), & t_{k-1}\le t<t_k,\\ 0, \quad \text{otherwise},\end{cases}\tag{1}$$

where $t_0=a$, $t_N=b$ and $d_k=t_k-t_{k-1}$, $k=1,2,\dots,N$. And $L_m(t)$, $m=0,1,\dots,$ are Legendre polynomials on the interval $[-1,1]$ given by the following recursive formula:

$$\begin{cases}L_0(t)=1, \quad L_1(t)=t,\\ (m+1)L_{m+1}(t)=(2m+1)tL_m(t)-mL_{m-1}(t).\end{cases}\tag{2}$$

2.2. **Function approximation.** A vector function $f(t)$ on the interval $[a,b)$ is expressed as

$$f(t)\simeq\sum_{k=1}^{N}\sum_{m=0}^{M-1}f_{km}\psi_{km}(t),\tag{3}$$

where

$$f_{km}=\left[(2m+1)d_k\right]^{\frac{1}{2}}2^{-1}\int_{-1}^{1}f\left(2^{-1}\left(d_kt+t_{k-1}+t_k\right)\right)L_m(t)\mathrm{d}t.$$

Rewrite $f(t)$ as

$$f(t)\simeq\sum_{k=1}^{N}F_k\varPsi_k(t)=F\varPsi(t),\tag{4}$$

where

$$F_k=[f_{k0},\dots,f_{k,M-1}],\quad F=[F_1,\dots,F_N],$$
$$\varPsi_k(t)=[\psi_{k0}(t),\dots,\psi_{k,M-1}(t)]^{\mathrm{T}},\quad \varPsi(t)=[\varPsi_1^{\mathrm{T}}(t),\dots,\varPsi_N^{\mathrm{T}}(t)]^{\mathrm{T}}.$$

For corresponding F_k and F, we denote

$$\hat{F}_k=\left[f_{k0}^{\mathrm{T}},\dots,f_{k,M-1}^{\mathrm{T}}\right]^{\mathrm{T}},\quad \hat{F}=\left[\hat{F}_1^{\mathrm{T}},\dots,\hat{F}_N^{\mathrm{T}}\right]^{\mathrm{T}},$$

where T is the transpose.

2.3. **Expression of an integral function.** Let a matrix function $M(t)$ be appropriate to a vector function $f(t)$. We express $M(t)$ and $f(t)$, respectively, as

$$M(t)\simeq\sum_{k=1}^{N}\sum_{m=0}^{M-1}M_{km}\psi_{km}(t),\quad f(t)\simeq\sum_{k=1}^{N}\sum_{m=0}^{M-1}f_{km}\psi_{km}(t).$$

Then,

$$M(t)f(t)\simeq\sum_{k=1}^{N}\sum_{i=0}^{M-1}\sum_{j=0}^{M-1}M_{ki}f_{kj}\psi_{ki}(t)\psi_{kj}(t).$$

From

$$\psi_{ki}(t)\psi_{kj}(t)\simeq\sum_{m=0}^{M-1}d_{km}^{(ij)}\psi_{km}(t),$$

where

$$d_{km}^{(ij)}=[(2i+1)(2j+1)(2m+1)d_k^{-1}]^{\frac{1}{2}}2^{-1}\int_{-1}^{1}L_i(t)L_j(t)L_m(t)\mathrm{d}t,$$

we have

$$M(t)f(t) \simeq \sum_{k=1}^{N}\sum_{m=0}^{M-1} \tilde{M}_{km}\psi_{km}(t) = \sum_{k=1}^{N} \tilde{M}_k \mathit{\Psi}_k(t), \tag{5}$$

where

$$\begin{aligned}
&\tilde{M}_{km} = \hat{M}_{km}\hat{F}_k, \quad \hat{F}_k = \left[f_{k0}^{\mathrm{T}}, \dots, f_{k,M-1}^{\mathrm{T}}\right]^{\mathrm{T}}, \quad \hat{\tilde{M}}_k = \hat{M}_k\hat{F}_k,\\
&\hat{\tilde{M}}_k = \left[\tilde{M}_{k0}^{\mathrm{T}}, \dots, \tilde{M}_{k,M-1}^{\mathrm{T}}\right]^{\mathrm{T}}, \quad \hat{M}_k = \left[\hat{M}_{k0}^{\mathrm{T}}, \dots, \hat{M}_{k,M-1}^{\mathrm{T}}\right]^{\mathrm{T}},\\
&\hat{M}_{km} = \left[\sum_{i=0}^{M-1} d_{km}^{(i0)} M_{ki}, \dots, \sum_{i=0}^{M-1} d_{km}^{(i,M-1)} M_{ki}\right].
\end{aligned}$$

Therefore,

$$K(s,t)f(t) \simeq \sum_{k=1}^{N}\sum_{m=0}^{M-1} \tilde{K}_{km}(s)\psi_{km}(t), \tag{6}$$

where

$$\begin{aligned}
&\tilde{K}_{km}(s) = \hat{K}_{km}(s)\hat{F}_k, \quad \hat{K}_{km}(s) = \left[\sum_{i=0}^{M-1} d_{km}^{(i0)} K_{ki}(s), \dots, \sum_{i=0}^{M-1} d_{km}^{(i,M-1)} K_{ki}(s)\right],\\
&K_{ki}(s) = [(2i+1)d_k]^{\frac{1}{2}} 2^{-1} \int_{-1}^{1} K(s, 2^{-1}(d_k t + t_{k-1} + t_k))L_i(t)\mathrm{d}t.
\end{aligned}$$

Let

$$\begin{aligned}
g(t) &= \int_a^b K(t,s)f(s)\mathrm{d}s \simeq \sum_{k=1}^{N}\sum_{m=0}^{M-1} g_{km}\psi_{km}(t), \qquad (7)\\
\hat{K}_{km}(t) &\simeq \sum_{j=1}^{N}\sum_{l=0}^{M-1} \hat{K}_{km}^{(jl)}\psi_{jl}(t),
\end{aligned}$$

where

$$\begin{aligned}
\hat{K}_{km}^{(jl)} &= \left[\sum_{i=0}^{M-1} d_{km}^{(i0)} K_{ki}^{(jl)}, \dots, \sum_{i=0}^{M-1} d_{km}^{(i,M-1)} K_{ki}^{(jl)}\right],\\
K_{ki}^{(jl)} &= [(2l+1)d_j]^{\frac{1}{2}} 2^{-1} \int_{-1}^{1} K_{ki}(2^{-1}(d_j t + t_{j-1} + t_j))L_l(t)\mathrm{d}t.
\end{aligned}$$

By Equation (6), we have

$$g(t) \simeq \sum_{j=1}^{N}\sum_{l=0}^{M-1}\sum_{k=1}^{N}\sum_{m=0}^{M-1} \hat{K}_{km}^{(jl)}\left(\int_{t_{k-1}}^{t_k} \psi_{km}(t)\mathrm{d}t\right)\hat{F}_k\psi_{jl}(t).$$

So,

$$\hat{G} = \sum_{i=1}^{N}\sum_{j=1}^{N} E_{ij}^{(N)} \otimes \sqrt{d_j} K_j^{(i)} \hat{F}, \tag{8}$$

where $\hat{G}$ has the meaning similar to $\hat{F}$, $E_{ij}^{(N)}$ is the $N \times N$ matrix with 1 at its entry (i,j) and zeros elsewhere, $\otimes$ denotes Kronecker product and

$$K_j^{(i)} = \left[{K_{j0}^{(i0)}}^{\mathrm{T}}, \dots, {K_{j0}^{(i,M-1)}}^{\mathrm{T}}\right]^{\mathrm{T}}.$$

3. **Analysis of Fredholm Integral Equation of the Second Kind.** Now we consider the following Fredholm integral equation of the second kind:

$$A(t)x(t) = \lambda \int_a^b K(t,s)x(s)\mathrm{d}s + B(t)u(t), \quad t \in [a,b], \tag{9}$$

where $x(t)$ is an unknown l-dimensional function and $u(t)$ a given r-dimensional function. $A(t)$, $K(t,s)$ and $B(t)$ are given matrix functions of appropriate dimensions and λ a given constant. Suppose that Equation (9) has the unique solution. We express $A(t)$, $x(t)$, $\int_a^b K(t,s)x(s)\mathrm{d}s$, $B(t)$ and $u(t)$, respectively. By Equations (5) and (6), we have

$$A(t)x(t) \simeq \sum_{k=1}^{N}\sum_{m=0}^{M-1} \tilde{A}_{km}h_{km}(t), \quad B(t)u(t) \simeq \sum_{k=1}^{N}\sum_{m=0}^{M-1} \tilde{B}_{km}h_{km}(t),$$

$$\int_a^b K(t,s)x(s)\mathrm{d}s \simeq \sum_{k=1}^{N}\sum_{m=0}^{M-1} g_{km}h_{km}(t).$$

Substituting the above equations into Equation (9), we have

$$\left[\tilde{A}_1, \ldots, \tilde{A}_N\right] = \lambda\left[G_1, \ldots, G_N\right] + \left[\tilde{B}_1, \ldots, \tilde{B}_N\right].$$

Rewrite the above equation as

$$\left[\hat{\tilde{A}}_1^{\mathrm{T}}, \ldots, \hat{\tilde{A}}_N^{\mathrm{T}}\right]^{\mathrm{T}} = \lambda\left[\hat{G}_1^{\mathrm{T}}, \ldots, \hat{G}_N^{\mathrm{T}}\right]^{\mathrm{T}} + \left[\hat{\tilde{B}}_1^{\mathrm{T}}, \ldots, \hat{\tilde{B}}_N^{\mathrm{T}}\right]^{\mathrm{T}},$$

where

$$\hat{\tilde{A}}_k = \left[\tilde{A}_{k0}^{\mathrm{T}}, \ldots, \tilde{A}_{k,M-1}^{\mathrm{T}}\right]^{\mathrm{T}},$$

$\hat{G}_N$ and $\hat{\tilde{B}}_N$ have the similar meaning as $\hat{\tilde{A}}_N$. So,

$$\left[\left(\hat{A}_1\hat{X}_1\right)^{\mathrm{T}}, \ldots, \left(\hat{A}_N\hat{X}_N\right)^{\mathrm{T}}\right]^{\mathrm{T}} = \lambda\hat{G} + \left[\left(\hat{B}_1\hat{U}_1\right)^{\mathrm{T}}, \ldots, \left(\hat{B}_N\hat{U}_N\right)^{\mathrm{T}}\right]^{\mathrm{T}}.$$

Using Kronecker product $\otimes$, we have

$$\hat{X} = \left[\sum_{k=1}^{N}\left(E_{kk}^{(N)} \otimes \hat{A}_k\right) - \lambda\sum_{i=1}^{N}\sum_{j=1}^{N} E_{ij}^{(N)} \otimes \sqrt{d_j}K_j^{(i)}\right]^{-1} \sum_{k=1}^{N}\left(E_{kk}^{(N)} \otimes \hat{B}_k\right)\hat{U}. \tag{10}$$

By the method in [11] an error estimation follows. Let $\hat{x}_N^M(t)$ be an Legendre wavelets solution of Equation (9). Then, $\hat{x}_N^M(t)$ satisfies

$$A(t)\hat{x}_N^M(t) = \lambda \int_a^b K(t,s)\hat{x}_N^M(s)\mathrm{d}s + B(t)u(t) + r_N^M(t), \quad t \in [a,b], \tag{11}$$

where

$$r_N^M(t) = A(t)\hat{x}_N^M(t) - \lambda \int_a^b K(t,s)\hat{x}_N^M(s)\mathrm{d}s - B(t)u(t).$$

Combining Equation (9) and Equation (11), we have

$$A(t)\left[x(t) - \hat{x}_N^M(t)\right] = \lambda \int_a^b K(t,s)\left[x(s) - \hat{x}_N^M(s)\right]\mathrm{d}s - r_N^M(t), \quad t \in [a,b]. \tag{12}$$

We can solve Equation (12) as an estimation of the error function of this general Legendre wavelets method.

4. **Example.**

Example 4.1. *In Equation (9), taking* $A(t) = 1$, $\lambda = 1$, $N(t) = t + s$, $B(t) = 1$ *and* $u(t) = e^t + (1 - e)t - 1$ *yields the Fredholm integral equation of the second kind* [11].

$$x(t) = \int_0^1 (t+s)x(s)\mathrm{d}s + e^t + (1-e)t - 1,$$

where $x(t)$ *is an 1-dimensional function.*

By Equation (10) and choosing $M = 4$ and $N = 4$, we have the Legendre wavelets solution $\hat{x}_N^M(t)$:

$$\begin{aligned}\hat{x}_N^M(t) =& \frac{985}{1734}\psi_{10}(t) + \frac{153}{3736}\psi_{11}(t) + \frac{45}{34061}\psi_{12}(t) + \frac{9}{322492}\psi_{13}(t)\\ &+ \frac{1283}{1759}\psi_{20}(t) + \frac{412}{7835}\psi_{21}(t) + \frac{29}{17095}\psi_{22}(t) + \frac{3}{83719}\psi_{23}(t)\\ &+ \frac{1801}{1923}\psi_{30}(t) + \frac{153}{2266}\psi_{31}(t) + \frac{78}{35809}\psi_{32}(t) + \frac{23}{499870}\psi_{33}(t)\\ &+ \frac{469}{390}\psi_{40}(t) + \frac{189}{2180}\psi_{41}(t) + \frac{76}{27173}\psi_{42}(t) + \frac{20}{338521}\psi_{43}(t).\end{aligned}$$

We give the absolute errors in Table 1.

TABLE 1. Absolute errors $\left|x(t) - \hat{x}_N^M(t)\right|$ (Example 4.1)

t	Legendre wavelets ($M = 4, N = 4$)	Legendre wavelets ($M = 8, N = 4$)	Hybrid Taylor [11] ($M = 4, N = 4$)	Hybrid Taylor [11] ($M = 8, N = 4$)
0.2	0.1081×10^{-5}	0.0710×10^{-13}	0.0021	0.4918×10^{-8}
0.4	0.0799×10^{-5}	0.0133×10^{-13}	0.0027	0.6380×10^{-8}
0.6	0.0989×10^{-5}	0.0244×10^{-13}	0.0034	0.7793×10^{-8}
0.8	0.2265×10^{-5}	0.1554×10^{-13}	0.0040	0.9199×10^{-8}
1.0	0.5660×10^{-5}	0.7149×10^{-13}	0.0043	0.9780×10^{-8}

Example 4.2. *Consider the Fredholm integral equation of the second kind described by*

$$\begin{aligned}\begin{bmatrix} -2t^2e^{-t} & 1 \\ e^{-t} & -e^t \end{bmatrix} x(t) =& \int_1^2 \begin{bmatrix} t^2+s^2 & -t^2s \\ 1 & s \end{bmatrix} x(s)\mathrm{d}t\\ &+ \begin{bmatrix} e^{-t} + (2e^{-1} + e - e^2 - 3e^{-2} - 2)t^2 + e - 2e^2 \\ 3e^{-2} - e^2 + e - 2e^{-1} \end{bmatrix},\end{aligned}$$

where $x(t)$ *is a 2-dimensional vector function.*

By Equation (10), we have the general Legendre wavelets solution $\hat{x}_N^M(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$ for given natural numbers M and N. Table 2 shows the absolute errors.

Example 4.3. *Consider the Fredholm integral equation of the second kind described by*

$$\begin{aligned}\begin{bmatrix} -2t^2e^{t-1} & te^{\frac{2-t}{2}} & 2e^{\frac{1-2t}{2}} \\ -2e^{t-1} & 0 & 2t^2e^{\frac{1-2t}{2}} \\ -3e^{t-1} & -2e^{\frac{1-t}{2}} & t^2e^{1-t} \end{bmatrix} x(t) =& \int_{\frac{1}{2}}^{\frac{5}{4}} \begin{bmatrix} t^2s & 1 & ts^2 \\ 2 & t^2 & 0 \\ 3 & s & t^2+s^2 \end{bmatrix} x(s)\mathrm{d}s\\ &+ \begin{bmatrix} \left(\frac{9}{4}e^{-\frac{5}{4}} - \frac{3}{2}e^{-\frac{1}{2}} - 2e^{-1}\right)t^2 + \left(e + \frac{5}{4}e^{\frac{1}{2}} - \frac{17}{16}e^{\frac{5}{4}}\right)t + 2\left(e^{\frac{1}{2}} - e^{\frac{5}{8}} + e^{\frac{1}{4}}\right) \\ 2\left(e^{\frac{1}{2}} - e^{\frac{5}{8}} + e^{\frac{1}{4}}\right)t^2 + 2\left(e^{-\frac{5}{4}} - e^{-\frac{1}{2}} - e^{-1}\right) \\ \left(e - e^{\frac{5}{4}} + e^{\frac{1}{2}}\right)t^2 + 3\left(e^{-\frac{5}{4}} - e^{-\frac{1}{2}} - e^{\frac{1}{4}} - e^{-1}\right) + \frac{3}{2}e^{\frac{5}{8}} - \frac{17}{16}e^{\frac{5}{4}} - \frac{3}{4}e^{\frac{1}{2}} \end{bmatrix}\end{aligned}$$

where $x(t)$ is a 3-dimensional vector function.

By Equation (10), we have the general Legendre wavelets solution $\hat{x}_N^M(t) = [x_1(t), x_2(t), x_3(t)]^{\mathrm{T}}$ for given natural numbers M and N. The absolute errors can be seen in Table 2.

Table 2. Absolute errors $||x(t) - \hat{x}_N^M(t)||_2$ of Legendre wavelets (Example 4.2)

t	$(M=5,N=4)$	$(M=6,N=4)$	$(M=7,N=4)$	$(M=8,N=4)$
1.0	4.2193×10^{-8}	2.4467×10^{-9}	7.3681×10^{-12}	3.0274×10^{-13}
1.2	3.8362×10^{-8}	3.7486×10^{-10}	6.4495×10^{-12}	4.7714×10^{-14}
1.4	4.8673×10^{-8}	3.9959×10^{-10}	5.0930×10^{-12}	3.0296×10^{-14}
1.6	4.2650×10^{-8}	7.4648×10^{-11}	4.3933×10^{-12}	3.7340×10^{-14}
1.8	2.1266×10^{-8}	3.1891×10^{-10}	5.8080×10^{-12}	8.0841×10^{-14}
2.0	2.4361×10^{-7}	1.7937×10^{-9}	3.2462×10^{-11}	9.0616×10^{-14}

Table 3. Absolute errors $||x(t) - \hat{x}_N^M(t)||_2$ of Legendre wavelets (Example 4.3)

t	$(M=5,N=3)$	$(M=6,N=4)$	$(M=7,N=5)$	$(M=8,N=6)$
0.50	1.0173×10^{-7}	1.8112×10^{-10}	5.9306×10^{-13}	3.0364×10^{-15}
0.65	4.1389×10^{-7}	2.9341×10^{-11}	8.6958×10^{-13}	3.0947×10^{-15}
0.80	3.7592×10^{-7}	1.6537×10^{-10}	3.2570×10^{-12}	1.5067×10^{-14}
0.95	5.0479×10^{-7}	2.8982×10^{-10}	7.7878×10^{-13}	5.0378×10^{-14}
1.10	1.7636×10^{-7}	6.6399×10^{-11}	1.7907×10^{-13}	6.6354×10^{-15}
1.25	3.6289×10^{-7}	1.0490×10^{-10}	5.9833×10^{-13}	4.3203×10^{-15}

5. **Conclusion.** Using the general Legendre wavelets to solve Fredholm integral equation of the second kind has the advantages that the integration internal can be divided arbitrarily and series numbers can be chosen willfully. Applying the excellent properties of the general Legendre wavelets, the general algorithm for Fredholm integral equation of the second kind is derived. The results in Table 1 imply that the general Legendre wavelets solutions converge to the exact solution more rapidly than the hybrid Taylor solutions. Since it is not necessary to choose N as the power of 2, the present approach is more convenient for application on an arbitrary interval.

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