International Journal of Innovative Computing, Information and Control
Volume **8**, Number **4**, April **2012**

ICIC International ©2012 ISSN 1349-4198
pp. 2613–2624

ON POLE PLACEMENT IN LMI REGION FOR DESCRIPTOR LINEAR SYSTEMS

Jianjun Bai[1], Hongye Su[1], Jing Wang[2] and Bingxin Shi[3]

[1]State Key Laboratory of Industrial Control Technology
Institute of Cyber-System and Control
Zhejiang University
No. 38, Zheda Road, Yuquan Campus, Hangzhou 310027, P. R. China
baijianjun83@gmail.com

[2]Laiwu Vocational and Technical College
Laiwu 271100, P. R. China

[3]Jidong Oilfield Oil-Gas Gathering and Transportation Company
Tangshan 063200, P. R. China

Received January 2011; revised May 2011

Abstract. *The problem of robust $\mathcal{D}$ admissibility analysis and admissibilization is studied for uncertain descriptor linear systems. Two classes of uncertainties are considered in this paper, i.e., the norm bounded uncertainty and the polytopic uncertainty. With the introduction of some free matrices, a new necessary and sufficient condition for the considered descriptor linear system to be $\mathcal{D}$ admissible is presented, based on which, sufficient conditions for uncertain descriptor linear systems to be robustly $\mathcal{D}$ admissible are proposed. The state feedback control problems are also addressed and explicit expressions of the desired state feedback controller are given in terms of strict linear matrix inequalities (LMIs), which are easily checked. Numerical examples illustrate the efficiency of the obtained approach.*

Keywords: Descriptor linear systems, Robust $\mathcal{D}$ admissibilization, Linear matrix inequality (LMI)

1. **Introduction.** Descriptor systems have been of interest in the literature since they have many important applications in the practice, such as circuit systems, power systems, aerospace engineering and chemical processing [1]. It is known that the control of the descriptor systems is much more complicated because controllers must be designed such that the closed-loop systems are not only stable, but also regular and impulse-free for continuous systems or causal for discrete-time systems, which are not required to be considered in state-space systems [1, 2, 3]. Many classical concepts and results for state-space systems such as stability, controllability and observability have been successfully extended to descriptor systems, see [1] and the references therein. Many efficient methods such as Lyapunov theorem [4, 5, 6], positive realness [7, 8] and dissipative theorem [9] are also developed for the analysis and synthesis problem of the descriptor systems. The stabilization problem was considered in [10, 11, 12], and H_∞ control problem was solved in terms of non-strict LMIs [13]. Strict LMI conditions for the descriptor system to be robust stable were proposed in [2, 3, 14]. The case that the derivative matrix E is with uncertainties was considered in [14, 15, 16].

On the other hand, pole placement is a well-known method to reach some desired transient performances. In practice, regional pole placement is always considered instead of the exact pole placement for the existence of parameter uncertainties. For the state-space case, there are many results [17, 18, 19, 22, 24]. In [17], the linear matrix inequality (LMI) region was first introduced, which is symmetric to the real axis and possesses the property of convexity, and the robust pole placement in LMI region was considered in [18]. A more general LMI region, which also referred to as EMI (Ellipsoid Matrix Region) [20], was introduced in [19] and some less conservative results were proposed using parameter dependent Lyapunov function method [21, 22] or introducing some free matrices [23, 24]. The problem of placing the poles in the clustering regions was addressed in [20]. For the descriptor case, there are also some results [25, 26, 27, 28, 29, 30]. Under the assumption that the nominal descriptor system was $\mathcal{D}$ stable, the robust $\mathcal{D}$ stability problem with one parameter family of perturbations was considered in [25] based on linear fractional transformation and guardian map theory. Circle region was considered in [26], and based on the spectral radius theory, parameters with structured uncertainties were considered. In [27], a wonderful result was acquired, in which the LMI region was considered and a necessary and sufficient criterion for the $\mathcal{D}$ admissibility test was proposed in terms of strict LMIs. Strict LMI conditions were also obtained in [28, 29] which could be solved efficiently [31, 32]. Recently, the $\mathcal{D}$ dissipativity problem was also addressed in [30], and many problems could be solved in this framework.

So far, most of the results are focused on the analysis of $\mathcal{D}$ admissibility, and few results are addressed about the controller design. In this paper, the problem of robust $\mathcal{D}$ admissibilization is considered for uncertain descriptor systems and we only consider the case that there is no uncertainty with the derivative matrix E. First, a necessary and sufficient condition with some free matrices is proposed, and then the $\mathcal{D}$ admissibilization problem is investigated. And explicit expressions of the desired state feedback controller are given for both kind of uncertainties. Numerical examples illustrate the efficiency of the proposed approach.

Notation: Throughout this paper, R^n denotes the n-dimensional real Euclidean space, C denotes the complex plane, I_k is the $k \times k$ identity matrix, the superscripts 'T' and '-1' stand for the matrix transpose and inverse respectively, $\bar{z}$ denotes the conjugate of z, '$*$' denotes the symmetric element in a symmetric matrix, C^- denotes the left-hand side of complex plane and $D_{\text{int}}(0,1)$ denotes the unitary disk centered at the origin. $W > 0$ ($W \geq 0$) means that W is real, symmetric and positive definite (positive semidefinite), $\otimes$ denotes the Kronecker product, $\delta[\cdot]$ denotes the differential operator for continuous systems (i.e., $\delta[x(t)] = \dot{x}(t)$) and the shift operator for discrete-time systems (i.e., $\delta[x(t)] = x(t+1)$), $\lambda(E, A)$ denotes the set of finite eigenvalues of the (E, A) pair, i.e., $\lambda(E, A) = \{s | \det(sE - A) = 0\}$, $\text{Sym}(\cdot)$ denotes the matrix plus its transpose, i.e., $\text{Sym}(A) = A + A^T$. If not explicitly stated, the matrices are assumed to have compatible dimensions.

2. **Problem Statement and Preliminaries.**

Consider the following descriptor systems

$$E\delta[x(t)] = Ax(t) \tag{1}$$

where $x \in R^n$ is the state vector, $u(t) \in R^m$ is the control input vector. The matrix $E \in R^{n \times n}$ may be singular and we assume $r = \text{rank}\, E \leq n$ without loss of generality. A, B are known real constant matrices with appropriate dimensions.

The following definitions and lemmas are essential for the development of our main results.

Definition 2.1 (Dai [1])**.**

1. *The system* (1) *is said to be regular, if* $\det(sE - A)$ *is not identically zero.*
2. *The system* (1) *is said to be impulse-free (causal), if* $\deg(\det(sE - A)) = \text{rank}\, E$.

3. The system (1) *is said to be stable, if* $\lambda(E, A) \subset C^-$ *for continuous descriptor systems or* $\lambda(E, A) \subset D_{int}(0, 1)$ *for discrete-time descriptor systems.*

4. The system (1) *is said to be admissible, if it is regular, impulse-free (causal), and stable.*

If the descriptor linear system (1) is regular, then there exist two nonsingular matrices M and N such that

$$\hat{E} = MEN = \begin{bmatrix} I_r & 0 \\ 0 & J \end{bmatrix}, \quad \hat{A} = MAN = \begin{bmatrix} A_r & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

The pair $(\hat{E}, \hat{A})$ is called the *Kronecker-Weierstrass form* of (E, A). The matrix J is a nilpotent matrix, and r is the number of finite eigenvalues of (E, A). Since M and N are nonsingular, we have

$$\lambda(E, A) = \lambda(\hat{E}, \hat{A}) = \lambda(I_r, A_r),$$

i.e., the stability of any regular descriptor system (E, A) can be completely determined by that of the state-space system (I_r, A_r). Obviously, the pair (E, A) is impulse-free if and only if $J = 0$.

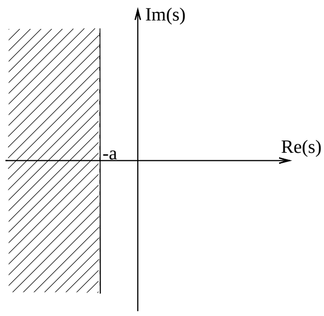

FIGURE 1. $\mathcal{D}$ region for continuous system

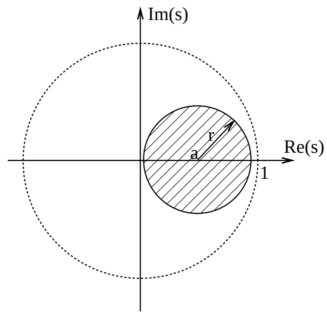

FIGURE 2. $\mathcal{D}$ region for discrete time system

Definition 2.2 (Kuo [27]).
The system (1) *is called* $\mathcal{D}$ *admissible if it is regular, impulse-free (causal), and* $\lambda(E, A) \in \mathcal{D}$.

Under the assumption that the descriptor variable x is measurable, the problem of determining a feedback gain matrix K so that the control signal

$$u(t) = Kx(t)$$

will make the closed-loop descriptor system

$$E\delta[x(t)] = (A + BK)x(t)$$

admissible is called the $\mathcal{D}$ *admissibilization* problem of the considered system. Whenever such a K exists, pair (E, A, B) is called $\mathcal{D}$ *admissibilizable.*

In this paper, the LMI region is given as [19]

$$\mathcal{D} = \left\{ z \in C : R_1 + R_2 z + R_2^T \bar{z} + R_3 z \bar{z} < 0 \right\}$$

where $R_1 = R_1^T \in R^{d \times d}$ and $0 \le R_3 = R_3^T \in R^{d \times d}$. For notational simplicity, write

$$R = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}$$

and d is called the order of this $\mathcal{D}$ region.

Two typical regions are shown in Figure 1 and Figure 2, which could be formulated by the following choices of R, respectively.

$$R_C = \begin{bmatrix} 2a & 1 \\ 1 & 0 \end{bmatrix}, \quad R_D = \begin{bmatrix} a^2 - r^2 & -a \\ -a & 1 \end{bmatrix}$$

In Figure 1, when $a = 0$, it becomes the left-hand side of complex plane C^-. In Figure 2, when $a = 0$ and $r = 1$, it becomes the unitary disk centered at the origin $D_{\text{int}}(0, 1)$.

A necessary and sufficient condition for the $\mathcal{D}$ admissibility of descriptor system (1) is proposed as follows.

Lemma 2.1 (Kuo [27])**.** *The system* (1) *is* $\mathcal{D}$ *admissible if and only if there exist* $n \times n$ *matrices* $P > 0$ *and* Q *satisfying*

$$M(P, Q, E, A) < 0 \tag{2a}$$

$$E^T Q E \geq 0 \tag{2b}$$

with $M(P, Q, E, A) = R_1 \otimes (E^T P E) + R_2 \otimes (E^T P A) + R_2^T \otimes (A^T P E) + R_3 \otimes (A^T P A) + I_d \otimes (A^T Q A)$.

Lemma 2.2 (Peterson [36])**.** *Given matrices* $\Omega = \Omega^T$, M_2, N_2 *with appropriate dimensions,*

$$\Omega + M_2 F(\sigma) N_2 + N_2^T F^T(\sigma) M_2^T < 0$$

for all $F(\sigma)$ *satisfying* $F^T(\sigma) F(\sigma) < I$, *if and only if there exists a scalar* ε *such that*

$$\Omega + \varepsilon^{-1} M_2 M_2^T + \varepsilon N_2^T N_2 < 0$$

3. **Main Results.** In this section, we give a solution to the pole placement of the descriptor system. First, we present a necessary and sufficient condition for the descriptor system (1) to be $\mathcal{D}$ admissible.

Theorem 3.1. *The descriptor linear system* (1) *is* $\mathcal{D}$ *admissible, if and only if there exist matrices* $P > 0$, $Q_1 > 0$, $Y < 0$ *and matrices* U, M, N, S, F, G *such that*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & \Xi_{22} & \Xi_{23} \\ * & * & -I_d \otimes (G + G^T) \end{bmatrix} < 0 \tag{3}$$

where $U_1 \in R^{(n-r)\times n}$ *is with full row rank and satisfies* $U_1 E = 0$, *and*

$\Xi_{11} = U(I_d \otimes A) + (I_d \otimes A^T)U^T + R_1 \otimes (E^T P E)$,
$\Xi_{12} = R_2 \otimes (E^T P) + (I_d \otimes A^T)M + I_d \otimes (SA) - U$,
$\Xi_{13} = I_d \otimes (A^T N - S)$,
$\Xi_{22} = R_3 \otimes P + I_d \otimes (Q_1 + U_1^T Y U_1 + F^T A + A^T F) - M - M^T$,
$\Xi_{23} = I_d \otimes (A^T G - F^T - N)$.

Proof:

Sufficiency. Let $Q = Q_1 + U_1^T Y U_1$. Since Q_1 is positive-definite and $U_1 E = 0$, (2b) is satisfied. Then pre- and post-multiplying (3) by $[I_{dn} \quad I_d \otimes A^T \quad I_d \otimes (A^T)^2]$ and its transpose gives (2a).

Necessity. Suppose that there exist matrix $P > 0$ and matrix Q satisfying (2a) and (2b). From (2b), we can choose $Q = Q_1 + U_1^T Y U_1$, without loss of generality (see [33]), with $Q_1 > 0$, $Y < 0$, $U_1 \in R^{(n-r)\times n}$ is with full row rank and satisfies $U_1 E = 0$. Then, from (2a), due to the strict property of the inequality, there must exist a sufficiently small scalar $\beta > 0$ such that

$$M(P, Q_1 + U_1^T Y U_1, E, A) + \beta I_d \otimes \left((A^T)^2 A^2\right) < 0,$$

which can be equivalently rewritten as

$$R_1 \otimes (E^T PE) + R_2 \otimes (E^T PA) + R_2^T \otimes (A^T PE) - \gamma I_d \otimes (A^T A) + (I_d \otimes A^T)(R_3 \otimes P + I_d \otimes (Q_1 + U_1^T Y U_1) + \gamma I_{dn})(I \otimes A) + \beta I_d \otimes ((A^T)^2 A^2) < 0$$

Choosing a sufficiently large γ such that $R_3 \otimes P + I_d \otimes (Q_1 + U_1^T Y U_1) + \gamma I_{dn} > 0$, and using the Schur complement argument, this implies

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} \\ * & \Upsilon_{22} & 0 \\ * & * & -\beta I_{dn} \end{bmatrix} < 0$$

with

$\Upsilon_{11} = R_1 \otimes (E^T PE) + R_2 \otimes (E^T PA) + R_2^T \otimes (A^T PE) - \gamma I_d \otimes (A^T A)$,
$\Upsilon_{12} = (I_d \otimes A^T)(R_3 \otimes P + I_d \otimes (Q_1 + U_1^T Y U_1) + \gamma I_{dn})$,
$\Upsilon_{13} = \beta I_d \otimes (A^T)^2$,
$\Upsilon_{22} = -(R_3 \otimes P + I_d \otimes (Q_1 + U_1^T Y U_1) + \gamma I_{dn})$.

Taking

$$U = R_2 \otimes (E^T P) - \frac{\gamma}{2}(I_d \otimes A^T), \qquad N = \beta A^T, \quad S = 0$$
$$M = R_3 \otimes P + I_d \otimes (Q_1 + U_1^T Y U_1) + \frac{\gamma}{2} I_{dn}, \quad F = -\frac{\beta}{2} A, \quad G = \frac{\beta}{2} I_n$$

it means that there exist $P > 0$, $Q_1 > 0$, $Y < 0$ and matrices U, S, M, N, F, G such that

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & \Xi_{22} + \beta I_d \otimes (A^T A) & \Xi_{23} \\ * & * & I_d \otimes (-G - G^T) \end{bmatrix} < 0$$

This completes the proof.

Remark 3.1. *For the state-space case, i.e., $E = I$, $U_1 = 0$ and the condition reduces to* [24, Lemma 3]. *Thus, Theorem 3.1 can be regarded as an extension of the result of* [24] *to descriptor linear systems.*

Noting that the matrix R is real, the region $\mathcal{D}$ is symmetric with respect to the real axis and the solution of $\det(sE - A) = 0$ is the same as that of $\det(sE^T - A^T) = 0$, thus, system (1) is equivalent to

$$E^T \delta[y(t)] = A^T y(t) \tag{4}$$

as far as only the $\mathcal{D}$ admissible problem is concerned. Replacing E with E^T, A with A^T in (3), the following result is obtained.

Theorem 3.2. *The descriptor linear system* (1) *is $\mathcal{D}$ admissible, if and only if there exist matrices $P > 0$, $Q_1 > 0$, $Y < 0$ and matrices U, M, S, N, F, G such that*

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ * & \Phi_{22} & \Phi_{23} \\ * & * & -I_d \otimes (G + G^T) \end{bmatrix} < 0 \tag{5}$$

where $U_1 \in R^{(n-r)\times n}$ is with full row rank and satisfies $U_1 E^T = 0$, and
$\Phi_{11} = U(I_d \otimes A^T) + (I_d \otimes A)U^T + R_1 \otimes (EPE^T)$,
$\Phi_{12} = R_2 \otimes (EP) + (I_d \otimes A)M + I_d \otimes (SA^T) - U$,
$\Phi_{13} = I_d \otimes (AN - S)$,
$\Phi_{22} = R_3 \otimes P + I_d \otimes (Q_1 + U_1^T Y U_1 + F^T A^T + AF) - M - M^T$,
$\Phi_{23} = I_d \otimes (AG - F^T - N)$.

Based on Theorem 3.2, sufficient conditions for the robust $\mathcal{D}$ admissibility for two classes of uncertain descriptor systems are given as follows.

3.1. **Norm bounded uncertainties.** Consider a class of uncertain descriptor linear system described by

$$E\delta[x(t)] = (A + \Delta A)x(t) + (B + \Delta B)u(t) \tag{6}$$

where ΔA, ΔB are unknown matrices representing parameter uncertainties, and are assumed to be of the form of

$$\Delta A = M_a F(\sigma) N_a, \quad \Delta B = M_b F(\sigma) N_b, \tag{7}$$

where M_a, N_a, M_b, N_b are real constant matrices with appropriate dimensions, and $F(\sigma)$ is an unknown, real constant matrix with Lebesgue-measurable elements satisfying

$$F^T(\sigma)F(\sigma) \le I \tag{8}$$

Theorem 3.3. *The unforced descriptor system* (6) *is robustly $\mathcal{D}$ admissible, if there exist matrices $P > 0$, $Q_1 > 0$, $Y < 0$, matrices U, M, S, N, F, G and positive scalars ε_1, ε_2 such that*

$$\begin{bmatrix} \Phi_{11} + \varepsilon_1 I_d \otimes (M_a M_a^T) & \Phi_{12} & \Phi_{13} & U(I_d \otimes N_a^T) & I_d \otimes (S N_a^T) \\ * & \Phi_{22} + \varepsilon_2 I_d \otimes (M_a M_a^T) & \Phi_{23} & M^T(I_d \otimes N_a^T) & I_d \otimes (F^T N_a^T) \\ * & * & -I_d \otimes (G - G^T) & I_d \otimes (N^T N_a^T) & I_d \otimes (G^T N_a^T) \\ * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0 \tag{9}$$

Proof: Substituting $(A + \Delta A)$ instead of A in (5) and considering (7), we obtain

$$\begin{aligned} \Phi + &\mathrm{Sym}\left(\begin{bmatrix} I_{dn} \\ 0 \\ 0 \end{bmatrix} (I_d \otimes M_a F(\sigma) N_a) \begin{bmatrix} U^T & M & N \end{bmatrix} \right) \\ &+ \mathrm{Sym}\left(\begin{bmatrix} 0 \\ I_{dn} \\ 0 \end{bmatrix} (I_d \otimes M_a F(\sigma) N_a) \begin{bmatrix} S^T & F & G \end{bmatrix} \right) < 0 \end{aligned}$$

Applying Lemma 2.2 and Schur complement, (9) can be obtained, which completes the proof.

Remark 3.2. *Theorem 3.3 becomes also necessary if the order of the region $d = 1$.*

Now, we are in the position to design a controller for the descriptor linear system (6) such that the closed-loop system is robustly $\mathcal{D}$ admissible.

Theorem 3.4. *The descriptor linear system* (6) *is robustly $\mathcal{D}$ admissibilizable by state feedback, if there exist matrices $P > 0$, $Q_1 > 0$, $Y < 0$, matrices H, V and positive scalars*

ε_1, ε_2, ε_3, ε_4 such that

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & (R_2-\beta_5 I_d)\otimes(H^T N_a^T) & \beta_1 I_d\otimes(H^T N_a^T) & (R_2-\beta_5 I_d)\otimes(V^T N_b^T) & \beta_1 I_d\otimes(V^T N_b^T) \\ * & \Psi_{22} & \Psi_{23} & (R_3-\beta_6)\otimes(H^T N_a^T) & \beta_3 I_d\otimes(H^T N_a^T) & (R_3-\beta_6)\otimes(V^T N_b^T) & \beta_3 I_d\otimes(V^T N_b^T) \\ * & * & \Psi_{33} & \beta_2 I_d\otimes(H^T N_a^T) & \beta_4 I_d\otimes(H^T N_a^T) & \beta_2 I_d\otimes(V^T N_b^T) & \beta_4 I_d\otimes(V^T N_b^T) \\ * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & * & -\varepsilon_4 I \end{bmatrix} < 0 \tag{10}$$

and a suitable state feedback gain is given by

$$K = VH^{-1} \tag{11}$$

where $\beta_i, i = 1, \ldots, 6$ are tuning parameters, and

$$\begin{aligned}
\Psi_{11} =& R_1\otimes(EPE^T)+\varepsilon_1 I_d\otimes(M_a M_a^T)+\varepsilon_3 I_d\otimes(M_b M_b^T)+Sym((R_2^T-\beta_5 I_d)\\
&\otimes(AH+BV))\\
\Psi_{12} =& R_2\otimes(EP)+(R_3-\beta_6 I_d)\otimes(AH+BV)+\beta_1 I_d\otimes(AH+BV)^T\\
&-(R_2-\beta_5 I_d)\otimes H^T\\
\Psi_{13} =& \beta_2 I_d\otimes(AH+BV)-\beta_1 I_d\otimes H^T\\
\Psi_{22} =& R_3\otimes P+I_d\otimes(Q_1+U_1^T Y U_1)+\varepsilon_2 I_d\otimes(M_a M_a^T)+\varepsilon_4 I_d\otimes(M_b M_b^T)\\
&+Sym(\beta_3 I_d\otimes(AH+BV)-(R_3-\beta_6 I_d)\otimes H)\\
\Psi_{23} =& \beta_4 I_d\otimes(AH)+\beta_4 I_d\otimes(BV)-\beta_3 I_d\otimes H^T-\beta_2 I_d\otimes H\\
\Psi_{33} =& -\beta_4 I_d\otimes H-\beta_4 I_d\otimes H^T
\end{aligned}$$

Proof: Substituting $A+(B+\Delta B)K$ instead of A in (9), and using the Schur complement lemma again, we obtain that the closed-loop descriptor system is robust $\mathcal{D}$ admissible if there exist $P>0$, $Q_1>0$, $Y<0$, matrices U, S, M, N, F, G and positive scalars ε_1,

ε_2, ε_3, ε_4 such that

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & U(I_d \otimes N_a^T) & I_d \otimes (SN_a^T) & U(I_d \otimes K^T N_b^T) & I_d \otimes (SK^T N_b^T) \\ * & \Pi_{22} & \Pi_{23} & M^T(I_d \otimes N_a^T) & I_d \otimes (F^T N_a^T) & M^T(I_d \otimes K^T N_b^T) & I_d \otimes (F^T K^T N_b^T) \\ * & * & -I_d \otimes (G+G^T) & I_d \otimes (N^T N_a^T) & I_d \otimes (G^T N_a^T) & I_d \otimes (N^T K^T N_b^T) & I_d \otimes (G^T K^T N_b^T) \\ * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & 0 & -\varepsilon_3 I & 0 \\ * & * & * & * & 0 & 0 & -\varepsilon_4 I \end{bmatrix} < 0 \quad (12)$$

where

$\Pi_{11} = U(I_d \otimes (A+BK)^T) + (I_d \otimes (A+BK))U^T + R_1 \otimes (EPE^T) + \varepsilon_1 I_d \otimes (M_a M_a^T)$
$\quad +\varepsilon_3 I_d \otimes (M_b M_b^T)$,

$\Pi_{12} = R_2 \otimes (EP) + (I_d \otimes (A+BK))M + I_d \otimes (S(A+BK)^T) - U$,

$\Pi_{13} = I_d \otimes ((A+BK)N - S)$,

$\Pi_{22} = R_3 \otimes P + I_d \otimes (Q_1 + U_1^T Y U_1) + I_d \otimes (F^T(A+BK)^T) + (A+BK))F)$
$\quad -M - M^T + \varepsilon_2 I_d \otimes (M_a M_a^T) + \varepsilon_4 I_d \otimes (M_b M_b^T)$,

$\Pi_{23} = I_d \otimes (A+BK)G - F^T - N)$.

From (10), it is obvious that H is nonsingular. Then taking $K = VH^{-1}$, $U = (R_2 - \beta_5 I_d) \otimes H^T$, $M = (R_3 - \beta_6 I_d) \otimes H$, $S = \beta_1 H^T$, $N = \beta_2 H$, $F = \beta_3 H$ and $G = \beta_4 H$, we can obtain (10) from (12). This completes the proof.

3.2. **Polytopic uncertainties.** Consider the descriptor system with polytopic uncertainties,

$$E\delta[x(t)] = A(\alpha)x(t) + B(\alpha)u(t) \quad (13)$$

where matrices $A(\alpha)$, $B(\alpha)$ are in the following convex sets,

$$\mathcal{A} = \left\{ A(\alpha) : A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i, \ \ \sum_{i=1}^{N} \alpha_i = 1; \ \ \alpha_i \geq 0, \ \ i = 1, 2, \cdots, N \right\} \quad (14a)$$

$$\mathcal{B} = \left\{ B(\alpha) : B(\alpha) = \sum_{i=1}^{N} \alpha_i B_i, \ \ \sum_{i=1}^{N} \alpha_i = 1; \ \ \alpha_i \geq 0, \ \ i = 1, 2, \cdots, N \right\} \quad (14b)$$

and N is the number of the vertices of the polytope.

Theorem 3.5. *The unforced descriptor system* (13) *is robustly $\mathcal{D}$ admissible, if there exist $P_i > 0$, $Q_{1i} > 0$, $Y_i < 0$, $i = 1, 2, \cdots, N$, and matrices U, M, S, N, F, G such that*

$$\Psi_i = \begin{bmatrix} \Psi_{i11} & \Psi_{i12} & \Psi_{i13} \\ * & \Psi_{i22} & \Psi_{i23} \\ * & * & -I_d \otimes (G+G^T) \end{bmatrix} < 0, \quad i = 1, 2, \cdots, N \quad (15)$$

where $U_1 \in R^{(n-r)\times n}$ is with full row rank and satisfies $U_1 E^T = 0$, and

$\Psi_{i11} = U(I_d \otimes A_i^T) + (I_d \otimes A_i)U^T + R_1 \otimes (EP_i E^T)$,

$\Psi_{i12} = R_2 \otimes (EP_i) + (I_d \otimes A_i)M + I_d \otimes (SA_i^T) - U$,

$\Psi_{i13} = I_d \otimes (A_i N - S)$,

$\Psi_{i22} = R_3 \otimes P_i + I_d \otimes (Q_{1i} + U_1^T Y_i U_1 + F^T A_i^T + A_i F) - M - M^T,$
$\Psi_{i23} = I_d \otimes (A_i G - F^T - N).$

Proof: It is similar as the proof in [19].

Now, we consider the controller design problem for the polytopic uncertain descriptor system.

Theorem 3.6. *The descriptor linear system* (13) *is robustly $\mathcal{D}$ admissibilizable by state feedback, if there exist $P_i > 0$, $Q_{1i} > 0$, $Y_i < 0$, $i = 1, 2, \cdots, N$, and matrices H, V such that*

$$\Pi_i = \begin{bmatrix} \Pi_{i11} & \Pi_{i12} & \Pi_{i13} \\ * & \Pi_{i22} & \Pi_{i23} \\ * & * & -\beta_4 I_d \otimes (H + H^T) \end{bmatrix} < 0, \quad i = 1, 2, \cdots, N \tag{16}$$

where β_i, $i = 1, \ldots, 6$ are tuning parameters, and

$$\begin{aligned}
\Pi_{i11} =& R_1 \otimes (E P_i E^T) + Sym\big((R_2^T - \beta_5 I_d) \otimes (A_i H + B_i V)\big) \\
\Pi_{i12} =& R_2 \otimes (E P_i) + (R_3 - \beta_6 I_d) \otimes (A_i H + B_i V) + \beta_1 I_d \otimes (A_i H + B_i V)^T \\
& - (R_2 - \beta_5 I_d) \otimes H^T \\
\Pi_{i13} =& \beta_2 I_d \otimes (A_i H + B_i V) - \beta_1 I_d \otimes H^T \\
\Pi_{i22} =& R_3 \otimes P_i + I_d \otimes (Q_{1i} + U_1^T Y_i U_1) + Sym(\beta_3 I_d \otimes (A_i H + B_i V) - (R_3 - \beta_6 I_d) \otimes H) \\
\Pi_{i23} =& \beta_4 I_d \otimes (A_i H + B_i V) - \beta_3 I_d \otimes H^T - \beta_2 I_d \otimes H
\end{aligned}$$

Proof: It is similar as the proof of Theorem 3.4.

4. **Numerical Example.**

In this section, some examples are presented to demonstrate the applicability of the proposed approach.

Example 4.1. *Considering the following continuous descriptor system:*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} a & 0 \end{bmatrix}$$

Our aim is to get the interval of a, in which the uncertain system is still admissible. In order to use the criteria Theorem 3.5 in this paper, we reformulate A as $A = \alpha_1 A_1 + (1 - \alpha_1) A_2$, where

$$A_1 = \begin{bmatrix} -1 + b_1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 - b_2 & 0 \\ 0 & 1 \end{bmatrix}$$

and b_1, b_2 denote the uncertainties of a in the positive and negative directions separately. Obviously, the descriptor system is admissible when $a < 1$. By Theorem 5 in [27]*, the result is $a \in [-1, 0.9999]$, the result of Theorem 3.3 in this paper is the same as* [27]*. However, by Theorem 3.5, the result is $a \in [-9999, 0.9999]$.*

Remark 4.1. *From the example, there is no improvement of the conservatism by Theorem 3.3. However, in Theorem 3.3, there is no square terms of A, and it is easy to deal the controller design problem. While in* [27] *square terms of A exist, and it could not be separated easily due to the non-definite of $Q - USYS^T U^T$.*

Example 4.2. *Consider the uncertain discrete-time descriptor system with the following parameters:*

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix},$$

$$M_a = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}, \quad N_a^T = \begin{bmatrix} 0.2 \\ 1 \\ 0.5 \end{bmatrix}, \quad M_b = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \quad N_b = 1.$$

Obviously, this descriptor system is irregular. Our objective is to design a controller such that the resultant closed-loop system is regular, casual and $D(0, 0.5)$ stable. Using the algorithm in Theorem 3.4, and choosing the tuning parameters as $\beta_1 = 0$, $\beta_2 = 0$, $\beta_3 = -0.4$, $\beta_4 = 2$, $\beta_5 = -0.1$, $\beta_6 = 0$, a feasible solution is given as:

$$P = \begin{bmatrix} 184.1587 & 21.2857 & -134.9267 \\ 21.2857 & 15.4431 & -34.4893 \\ -134.9267 & -34.4893 & 221.2272 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 19.1199 & 2.6252 & -14.6893 \\ 2.6252 & 2.1674 & -4.6126 \\ -14.6893 & -4.6126 & 110.0178 \end{bmatrix},$$

$$H = \begin{bmatrix} 161.5093 & 9.0365 & -64.1758 \\ 18.6375 & 27.1246 & -24.5257 \\ -176.1995 & -36.2104 & 129.2166 \end{bmatrix}, \quad V = \begin{bmatrix} -0.2705 \\ 0.4374 \\ -19.8444 \end{bmatrix}^T, \quad Y = -868.1321$$

and the corresponding state feedback control law is given as:

$$u(t) = \begin{bmatrix} -0.4341 & -0.4600 & -0.4374 \end{bmatrix} x(t)$$

Choosing $F = \delta I$ with $\delta = 0.02i$, $i = 0, \ldots, 50$, the finite eigenvalues of the uncertain descriptor system are shown as in Figure 3. It is obvious that the finite eigenvalues lay in the circle $D(0, 0.5)$.

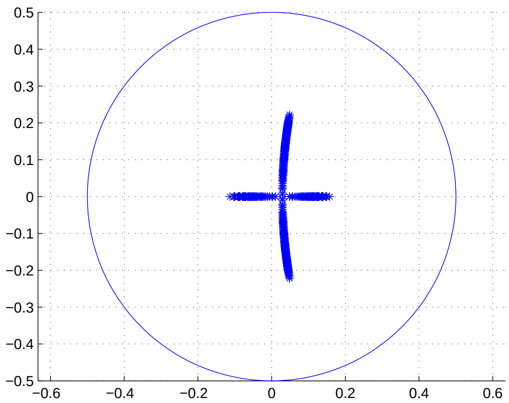

FIGURE 3. the roots of the systems in Example 4.1

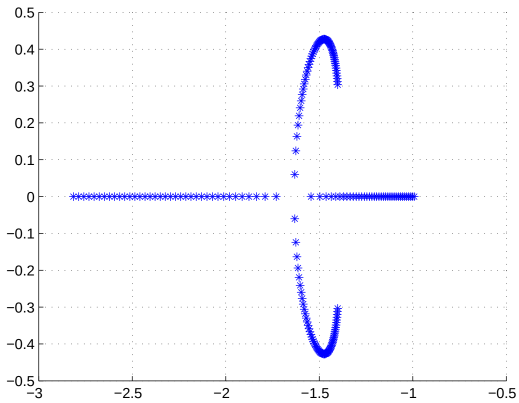

FIGURE 4. the roots of the systems in Example 4.2

Remark 4.2. *The tuning parameters β_i, $i = 1, \ldots, 6$ should be determined before our simulation. In this paper, they are chosen by the searching method. Another way is using the optimal function, such as* `fminsearch`*, see* [34, 35] *for details.*

Example 4.3. *Consider the uncertain continuous descriptor system with the polytopic uncertainties*

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

The region considered is shown as Figure 1 with $a = 0.5$. Using the algorithm in Theorem 3.6, and choosing the tuning parameters as $\beta_1 = 0$, $\beta_2 = 0$, $\beta_3 = 0$, $\beta_4 = 0.2$,

$\beta_5 = -1$, $\beta_6 = -1$, *a feasible solution is given as follows:*

$$P_1 = \begin{bmatrix} 0.7280 & 0.3803 & -0.4230 \\ 0.3803 & 0.3100 & -0.2833 \\ -0.4230 & -0.2833 & 0.9990 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.3544 & 0.4886 & -0.5228 \\ 0.4886 & 0.2521 & -0.1732 \\ -0.5228 & -0.1732 & 0.9237 \end{bmatrix},$$

$$Q_{11} = \begin{bmatrix} 0.0768 & 0.0243 & -0.0636 \\ 0.0243 & 0.0313 & -0.0186 \\ -0.0636 & -0.0186 & 0.5483 \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} 0.1546 & 0.0402 & -0.2681 \\ 0.0402 & 0.0361 & -0.1011 \\ -0.2681 & -0.1011 & 0.9110 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.3110 & 0.0484 & -0.1095 \\ 0.1324 & 0.0634 & -0.0648 \\ -0.2497 & -0.0804 & 0.2665 \end{bmatrix}, \quad V = \begin{bmatrix} -0.2323 \\ -0.0861 \\ -0.1140 \end{bmatrix}^T, \quad \begin{matrix} Y_1 = -1.3232, \\ Y_2 = -2.4248 \end{matrix}$$

and the corresponding state feedback controller is given by

$$u(t) = \begin{bmatrix} -0.7761 & -2.4745 & -1.3486 \end{bmatrix} x(t).$$

The finite eigenvalues of the uncertain descriptor system are shown as in Figure 4 and all the finite eigenvalues lay in the left of the line $x = -0.5$.

5. Conclusions.

In this paper, a new necessary and sufficient condition for the descriptor system to be $\mathcal{D}$ admissible is obtained by the free matrices technique. Then the robust $\mathcal{D}$ admissibility problem is considered for two classes of uncertain descriptor linear systems. Sufficient conditions for the system to be robust $\mathcal{D}$ admissible are proposed in terms of strict LMIs, based on which, the design problems are addressed. And the desired controllers are given in explicit expressions. Some numerical examples illustrate the efficiency of the proposed approach.

Acknowledgment. The author would like to thank the Associate Editor and the reviewers for their very helpful comments and suggestions for improving the paper. This work is supported by The National Creative Research Groups Science Foundation of China under Grant 60421002 and The National High Technology Research and Development Program of China under Grant 863 Program 2006AA04 Z182.

REFERENCES

[1] L. Dai, *Singular Control Systems*, Springer-Verlag, Berlin, 1989.
[2] M. Chaabane, O. Bachelier, M. Souissi and D. Mehdi, Stability and stabilization of continuous descriptor systems: An LMI approach, *Mathematical Problems in Engineering*, vol.2006, 2006.
[3] S. Y. Xu and J. Lam, Robust stability and stabilization of discrete singular systems: An equivalent characterization, *IEEE Transactions on Automatic Control*, vol.49, no.4, pp.568-574, 2004.
[4] K. Takaba, N. Morihira and T. Katayuam, A generalized Lyapunov theorem for descriptor system, *Systems and Control Letters*, vol.24, pp.49-51, 1995.
[5] V. L. Syrmos, P. Misra and R. Aripirala, On the discrete generalized Lyapunov equation, *Automatica*, vol.31, pp.297-301, 1995.
[6] J. Y. Ishihara and M. H. Terra, On the Lyapunov theorem for singular systems, *IEEE Transactions on Automatic Control*, vol.47, pp.1926-1930, 2002.
[7] L. Q. Zhang, J. Lam and S. Y. Xu, On positive realness of descriptor systems, *IEEE Transactions on Automatic Control*, vol.49, pp.401-407, 2002.
[8] L. Lee and J. L. Chen, Strictly positive real lemma and absolute stability for discrete-time descriptor systems, *IEEE Transactions on Circuits and Systems – I: Fundamental Theory and Applications*, vol.50, pp.788-794, 2003.
[9] Y. Li, Q. L. Zhang, X. D. Duan et al., Robust passive control of singular systems with uncertainties, *Proc. of the 2006 American Control Conference*, pp.3643-3647, 2006.
[10] S. Y. Xu and C. W. Yang, Stabilization of discrete-time singular systems: A matrix inequalities approach, *Automatica*, vol.35, pp.1613-1617, 1999.

[11] S. Y. Xu, C. W. Yang, Y. G. Niu and J. Lam, Robust stabilization for uncertain discrete singular systems, *Automatica*, vol.37, pp.769-774, 2001.
[12] L. Lee, J. L. Chen and C. H. Fang, On LMI approach to admissibilization of discrete-time descriptor systems, *The 5th Asian Control Conference*, pp.1001-1008, 2004.
[13] I. Masubuchi, Y. Kamitane, A. Ohara and N. Suda, H_∞ control for descriptor systems: A matrix inequalities approach, *Automatica*, vol.33, no.3, pp.669-673, 1997.
[14] U. Uezato and M. Ikeda, Strict LMI conditions for stability robust stabilization and control of descriptor systems, *Proc. of the 38th Conference on Decision and Control*, pp.4092-4097, 1999.
[15] C. Lin, J. L. Wang, G. H. Yang and J. Lams, Robust stabilization via state feedback for descriptor systems with uncertainties in the derivative matrix, *International Journal of Control*, vol.73, no.5, pp.407-415, 2000.
[16] C. Lin, J. Lams, J. L. Wang and G. H. Yang, Analysis on robust stability for interval descriptor systems, *Systems and Control Letters*, vol.42, pp.267-278, 2001.
[17] M. Chilali and P. Gahinet, H_∞ design with pole placement constraints: An LMI approach, *IEEE Transactions on Automatic Control*, vol.41, pp.358-367, 1996.
[18] M. Chilali, P. Gahinet and P. Apkarian, Robust pole placement in LMI regions, *IEEE Transactions on Automatic Control*, vol.44, pp.2257-2270, 1999.
[19] D. Peaucelle, D. Arzelier, O. Bachelier and J. Bernussou, A new robust $\mathcal{D}$ stability condition for real convex polytopic uncertainty, *Systems and Control Letters*, vol.40, pp.21-30, 2000.
[20] O. Bachelier and D. Mehdi, Robust matrix $\mathcal{D}_u$ stability analysis, *International Journal of Robust and Nonlinear Control*, vol.13, pp.533-558, 2003.
[21] P. Gahinet, P. Apkarian and M. Chilali, Affine parameter dependent Lyapunov functions and real parametric uncertainty, *IEEE Transactions on Automatic Control*, vol.41, pp.436-442, 1996.
[22] V. J. S. Leite and P. L. D. Peres, An improved LMI condition for robust $\mathcal{D}$ stability of uncertain polytopic systems, *IEEE Transactions on Automatic Control*, vol.48, pp.500-504, 2003.
[23] M. C. de Oliveira, J. Bernussou and J. C. Geromel, A new discrete-time robust stability condition, *Systems & Control Letters*, vol.37, pp.261-265, 1999.
[24] C. Lin, Q. G. Wang and T. H. Lee, Less conservative $\mathcal{D}$ stability test for polytopic systems using linearly parameter-dependent Lyapunov functions, *IEE Proc. of Control Theory and Applications*, vol.153, pp.665-670, 2006.
[25] J. L. Lin and S. J. Chen, LFT approach to robust $\mathcal{D}$ stability bounds of uncertain linear singular systems, *IEE Proc. of Control Theory and Applications*, vol.37, pp.127-134, 1998.
[26] G. Hu and J. M. Xu, Robust control for uncertain singular systems with disk pole contraints, *Proc. of the 42nd IEEE Conference on Decision and Control*, pp.288-293, 2003.
[27] C. H. Kuo and L. Lee, Robust $\mathcal{D}$ admissibility in generalized LMI regions for descriptor systems, *The 5th Asian Control Conference*, pp.1058-1065, 2004.
[28] B. Marx, D. Koenig and D. Georges, Robust pole-clustering for descriptor systems: A strict LMI characterization, *Proc. of the European Control Conference*, Cambridge, UK, 2003.
[29] O. Rejichi, O. Bachelier, M. Chaabane and D. Mehdi, Robustness bounds for uncertain descriptor systems in $\mathcal{D}_R$ admissibility, *The 7th International Conference STA*, Tunisia, 2006.
[30] O. Rejichi, O. Bachelier, M. Chaabane and D. Mehdi, $\mathcal{D}$ dissipativity analysis for descriptor systems, *2008 American Control Conference*, pp.424-428, 2008.
[31] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Philadelphia, PA, 1994.
[32] P. Gahinet, A. Nemirovski, A. J. Laub and M. Chilali, *LMI Control Toolbox for Use with Matlab*, Natick, MA, 1995.
[33] G. P. Lu and D. W. C. Ho, Generalized quadratic stabilization for discrete-time singular systems with time-delay and nonlinear perturbation, *Asian Journal of Control*, vol.7, pp.211-222, 2005.
[34] E. Fridman and U. Shaked, An improved stabilization method for linear time-delay systems, *IEEE Transactions on Automatic Control*, vol.47, pp.1931-1937, 2002.
[35] M. Wu, Y. He and J. H. She, New delay-dependent stability criteria and stabilizing method for neutral systems, *IEEE Transactions on Automatic Control*, vol.49, pp.2266-2271, 2004.
[36] I. R. Peterson, A stabilization algorithm for a class of uncertain linear systems, *System and Control Letters*, vol.8, pp.351-357, 1987.