International Journal of Innovative Computing, Information and Control
Volume **8**, Number **5(A)**, May **2012**
 ISSN 1349-4198
pp. 3361–3385

DYNAMIC POSITIONING CONTROL OF MARINE VEHICLE

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Received January 2011; revised May 2011

Abstract. *The paper deals with a novel fuzzy feedback linearization control of dynamic positioning marine vehicle with the almost disturbance decoupling performance. The main contribution of this study is to construct a controller such that the resulting dynamic positioning marine vehicle is valid for any initial and desired positions with the following characteristics: input-to-state stability with respect to disturbance inputs and almost disturbance decoupling. The dynamic positioning time based on our proposed novel nonlinear fuzzy feedback linearization control is better than some existing approaches. The proposed controller is constructed by feedback linearization controller and fuzzy controller. The feedback linearization controller guarantees the almost disturbance decoupling performance and the uniform ultimate bounded stability of the tracking error system. Once the tracking errors are driven to touch the global final attractor with the desired radius, the fuzzy logic controller immediately is applied via a human expert's knowledge to improve the convergence rate. Finally, the availability and feasibility of the proposed approach are investigated with a numerical simulation.*
Keywords: Marine vehicle, Dynamic positioning systems, Almost disturbance decoupling, Feedback linearization approach, Fuzzy logic control

1. **Introduction.** Offshore oilfield development has forwarded to a deeper and severe environment for new oil sources and maintaining the position of an offshore platform becomes a very challenging problem. Therefore, dynamic position of marine vehicle using thrusters is often used in many offshore oilfield operations, such as drilling, pipe-laying, tanking between marine vehicles and diving support. A dynamic positioning marine vehicle exhibits non-linear interaction in three degrees of freedom (sway, surge and yaw) by means of main propellers aft on the marine vehicle [8]. Dynamic positioning marine vehicle has usually been designed by assuming that the kinematic equations of motions can be linearized about predefined constant yaw angles, meaning that linear optimal control approach and gain-scheduling techniques can be applied [4,36,40]. It is obvious to

see that some modeling errors do exist during the overall linearization process. In order to solve the linearization problem, [6,14,49] apply the Takagi-Sugenno fuzzy model, which includes a set of IF-THEN rules, to investigating the non-linear dynamic positioning marine vehicle based on the parallel distributed compensation (PDC). Its designing procedure is as follows. First, representing the non-linear system as the famous Takagi-Sugeno fuzzy model offers an alternative to the conventional model. The control design is carried out based on an aggregation of linear controllers constructed for each local linear element of the fuzzy model via the parallel distributed compensation scheme [46]. For the stability analysis of fuzzy system, a lot of studies are reported (See, e.g., [27,41-43] and the references therein). The stability and controller design of fuzzy system can be mainly discussed by Tanaka-Sugeno's theorem [42]. However, it is difficult to find the common positive definite matrix P for linear matrix inequality (LMI) problem even if P is a second order matrix [24]. Moreover, the PDC will be subject to failure if the fuzzy model is decomposed by too many plant rules [10,45]. This is because there are many linear-matrix-inequality constraints involved in optimization process and many decision variables that need to be carried out simultaneously.

Many approaches to stabilizing and tracking tasks have been proposed including feedback linearization, variable structure control (sliding mode control), backstepping, regulation control, non-linear H^{∞} control, internal model principle and H^{∞} adaptive fuzzy control. [25] has proposed the use of variable structure control to deal with non-linear system. However, chattering behaviour caused by discontinuous switching and imperfect implementation that can drive the system into unstable regions is inevitable for variable structure control schemes. Backstepping has proven to be a powerful tool for synthesizing controllers for non-linear systems [17,52]. However, a disadvantage of this approach is an explosion in the complexity which is a result of repeated differentiations of non-linear functions [12,39,51]. An alternative approach is to utilize output regulation control [21] in which the outputs are assumed to be excited by an exosystem. However, the non-linear regulation approach requires the solution of difficult partial-differential algebraic equations. Another difficulty is that the exosystem states need to be switched to describe changes in the output and this creates transient tracking errors [34]. In general, non-linear H^{∞} control requires the solution of the Hamilton-Jacobi equation, which is a difficult non-linear partial-differential equation [2,22,44]. Only for some particular non-linear systems it is possible to derive a closed-form solution [20]. The control approach that is based on the internal model principle converts the tracking problem into a non-linear output regulation problem. This approach depends on solving a first-order partial-differential equation of the center manifold [21]. For some special non-linear systems and desired trajectories, the asymptotic solutions of this equation have been developed using ordinary differential equations [15,18]. Recently, H^{∞} adaptive fuzzy control has been proposed to systematically deal with non-linear systems [7]. The drawback with H^{∞} adaptive fuzzy control is that the complex parameter update law makes this approach impractical in real-world situations. During the past decade significant progress has been made in researching control approaches for non-linear systems based on the feedback linearization theory [19,25,33,38]. Moreover, feedback linearization approach has been applied successfully to many real control systems. These include the control of an electromagnetic suspension system [23], pendulum system [9], spacecraft [37], electrohydraulic servosystem [1], car-pole system [3] and bank-to-turn missile system [28]. The main contribution of this study is to solve the linearized and PDC shortcomings by using non-linear feedback linearization approach.

The almost disturbance decoupling problem, i.e., the design of a controller that attenuates the effect of the disturbance on the output terminal to an arbitrary degree of accuracy, was originally developed for linear and non-linear control systems by [30,48] respectively. The problem has attracted considerable attention and many significant results have been developed for both linear and non-linear control systems [13,31,35,47]. The almost disturbance decoupling problem of non-linear single-input single-output (SISO) systems was investigated in [30] by using a state feedback approach and solved in terms of sufficient conditions for systems with non-linearities that are not globally Lipschitz and disturbances bring linear but possibly actually bring multiples of non-linearities. The resulting state feedback control is constructed following a singular perturbation approach. The sufficient conditions in [30] require that the non-linearities multiplying the disturbances satisfy structural triangular conditions. [30] shows that for non-linear SISO systems the almost disturbance decoupling problem may not be solvable, as is case for

$$\begin{aligned}\dot{x}_1(t) &= \tan^{-1} x_2 + \theta(t), \quad |\theta(t)| > \frac{\pi}{2}\\ \dot{x}_2(t) &= u\\ y &= x_1\end{aligned}$$

where u, y denoted the input and output respectively and $\theta(t)$ was the disturbance of the system. On the contrary, this example can be easily solved via the proposed approach in this paper.

To overcome the problems for linearized technique and fuzzy-model approach, we will propose a new method to guarantee that the marine vehicle is stable and the almost disturbance decoupling performance is achieved. The designing structure is as follows. Firstly, based on the feedback linearization approach a tracking control is proposed in order to guarantee the almost disturbance decoupling property and the uniform ultimate bounded stability of the tracking error system. Once the tracking errors are driven to touch the global final attractor, the conventional fuzzy logic control is immediately applied via a human expert's knowledge to improve the convergence rate. In order to exploit the significant applicability, this paper has also successfully derived controller with almost disturbance decoupling for a marine vehicle. According to the simulation results, it is easy to see that the dynamic positioning time based on our proposed novel non-linear fuzzy feedback linearization control is shorter than the traditional fuzzy-model approach [5].

2. Feedback Linearization Controller Design.

The following non-linear uncertain control system with disturbances is considered.

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix} &= \begin{bmatrix}f_1(x_1,x_2,\ldots,x_n)\\ f_2(x_1,x_2,\ldots,x_n)\\ \vdots\\ f_n(x_1,x_2,\ldots,x_n)\end{bmatrix}\\ &+ [g_1(x_1,x_2,\ldots,x_n) \;\; g_2(x_1,x_2,\ldots,x_n) \;\; \cdots \;\; g_m(x_1,x_2,\ldots,x_n)] \begin{bmatrix}u_1(x_1,x_2,\ldots,x_n)\\ u_2(x_1,x_2,\ldots,x_n)\\ \vdots\\ u_m(x_1,x_2,\ldots,x_n)\end{bmatrix}\\ &+ \sum_{j=1}^{p} q_j^* \theta_{jd} + \begin{bmatrix}\Delta f_1(x_1,x_2,\ldots,x_n)\\ \Delta f_2(x_1,x_2,\ldots,x_n)\\ \vdots\\ \Delta f_n(x_1,x_2,\ldots,x_n)\end{bmatrix}\end{aligned} \tag{1a}$$

$$\begin{bmatrix} y_1(x_1, x_2, \ldots, x_n) \\ y_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ y_m(x_1, x_2, \ldots, x_n) \end{bmatrix} = \begin{bmatrix} h_1(x_1, x_2, \ldots, x_n) \\ h_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ h_m(x_1, x_2, \ldots, x_n) \end{bmatrix} \tag{1b}$$

that is

$$\dot{X}(t) = f(X(t)) + g(X(t))u + \sum_{j=1}^{p} q_j^* \theta_{jd} + \Delta f$$

$$y(t) = h(X(t))$$

where $X(t) \equiv [x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T \in \Re^n$ is the state vector, $u \equiv [u_1\ u_2\ \cdots\ u_m]^T \in \Re^m$ is the input vector, $y \equiv [y_1\ y_2\ \cdots\ y_m]^T \in \Re^m$ is the output vector, $\theta_d \equiv [\theta_{1d}(t)\ \theta_{2d}(t)\ \cdots\ \theta_{pd}(t)]^T$ is a bounded time-varying disturbances vector and $\Delta f \equiv [\Delta f_1\ \Delta f_2\ \cdots\ \Delta f_n] \in \Re^n$ is unknown non-linear function representing uncertainty such as modelling error. Let Δf be described as $\Delta f = \sum_{j=1}^{p} q_j^* \theta_{uj}$, where $\theta_u \equiv [\theta_{u1}(X,t)\ \theta_{u2}(X,t)\ \cdots\ \theta_{up}(X,t)]^T$ is a bounded time-varying vector and $q_j^* \in \Re^n$, $j = 1, 2, \ldots, p$, are coefficient vectors of disturbances. The vectors $f \in \Re^n$ and $h \in \Re^m$ are vectors of smooth real-valued functions $f_1, f_2, \cdots, f_n$ and $h_1, h_2, \cdots, h_m$, respectively. The matrix $g \in \Re^{n\times m}$ is a a matrix with columns being smooth vector fields $g_1, g_2, \cdots, g_m$. The nominal system is then defined as follows:

$$\dot{X}(t) = f(X(t)) + g(X(t))u \tag{2a}$$

$$y(t) = h(X(t)) \tag{2b}$$

The nominal system of the form (2) is assumed to have the vector relative degree $\{r_1, r_2, \cdots, r_m\}$ [16,19], i.e., the following conditions are satisfied for all $X \in \Re^n$:
$\langle 1 \rangle$

$$L_{g_j} L_f^k h_i(X) = 0 \tag{3}$$

for all $1 \le i \le m$, $1 \le j \le m$, $k < r_i - 1$, where the operator L is the Lie derivative [19] and $r_1 + r_2 + \cdots + r_m = r$.
$\langle 2 \rangle$ The $m \times m$ matrix

$$A \equiv \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(X) & \cdots & L_{g_m} L_f^{r_1-1} h_1(X) \\ L_{g_1} L_f^{r_2-1} h_2(X) & \cdots & L_{g_m} L_f^{r_2-1} h_2(X) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(X) & \cdots & L_{g_m} L_f^{r_m-1} h_m(X) \end{bmatrix} \tag{4}$$

is non-singular.

The desired output trajectory y_d^i, $1 \le i \le m$ and its first r_i derivatives are all uniformly bounded and

$$\left\| \left[y_d^i,\ y_d^{i(1)}, \cdots, y_d^{i(r_i)} \right] \right\| \le B_d^i, \quad 1 \le i \le m \tag{5}$$

where B_d^i is some positive constant.

The objective of the paper is to propose a control that includes a feedback linearization controller and a fuzzy logic controller to achieve the almost disturbance decoupling and tracking performances.

A. Feedback linearization controller design.

Under the assumption of well-defined vector relative degree, it has been shown [19] that the mapping

$$\phi : \Re^n \to \Re^n \tag{6}$$

defined as

$$\xi_i \equiv \begin{bmatrix} \xi_1^i \\ \xi_2^i \\ \vdots \\ \xi_{r_i}^i \end{bmatrix} \equiv \begin{bmatrix} \phi_1^i \\ \phi_2^i \\ \vdots \\ \phi_{r_i}^i \end{bmatrix} \equiv \begin{bmatrix} L_f^0 h_i(X) \\ L_f^1 h_i(X) \\ \vdots \\ L_f^{r_i-1} h_i(X) \end{bmatrix}, \quad i = 1, 2, \ldots, m \tag{7}$$

$$\phi_k(X(t)) \equiv \eta_k(t), \quad k = r+1, r+2, \cdots, n \tag{8}$$

and satisfying

$$L_{g_j}\phi_k(X(t)) = 0, \quad k = r+1, r+2, \cdots, n, \quad 1 \le j \le m \tag{9}$$

is a diffeomorphism onto image, if the following hold

⟨1⟩ The distribution

$$G \equiv span\{g_1, g_2, \cdots, g_m\} \tag{10}$$

is involutive.

⟨2⟩ The vector fields

$$Y_j^k, \quad 1 \le j \le m, \quad 1 \le k \le r_j \tag{11}$$

are complete, where

$$Y_j^k \equiv (-1)^{k-1} ad_{\tilde{f}}^{k-1}\tilde{g}_j, \quad 1 \le j \le m, \quad 1 \le k \le r_j \tag{12}$$

$$\tilde{f}(X) \equiv f(X) - g(X)A^{-1}(X)b(X) \tag{13}$$

$$b(X) \equiv \begin{bmatrix} L_f^{r_1} h_1(X) \\ L_f^{r_2} h_2(X) \\ \vdots \\ L_f^{r_m} h_m(X) \end{bmatrix} \tag{14}$$

$$\tilde{g} \equiv [\tilde{g}_1 \ \ \tilde{g}_2 \ \ \cdots \ \ \tilde{g}_m] \equiv g(X)A^{-1}(X) \tag{15}$$

$$ad_f^k g \equiv \lfloor f, \ \ ad_f^{k-1} g \rfloor \tag{16}$$

$$[f, \ \ g] \equiv \frac{\partial g}{\partial X} f(X) - \frac{\partial f}{\partial X} g(X) \tag{17}$$

For the sake of convenience, define the trajectory error to be

$$e_j^i \equiv \xi_j^i - y_d^{i(j-1)}, \quad i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, r_i \tag{18}$$

$$e^i \equiv \left[e_1^i \ \ e_2^i \cdots \ \ e_{r_i}^i\right]^T \in \Re^{r_i} \tag{19}$$

and the trajectory error to be multiplied with some adjustable positive constant ε

$$\overline{e_j^i} \equiv \varepsilon^{j-1} e_j^i, \quad i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, r_i \tag{20}$$

$$\overline{e^i} \equiv \left[\overline{e_1^i} \ \ \overline{e_2^i} \ \ \cdots \ \ \overline{e_{r_i}^i}(t)\right]^T \in \Re_i^r \tag{21}$$

$$\overline{e} \equiv \begin{bmatrix} \overline{e^1} \\ \overline{e^2} \\ \vdots \\ \overline{e^m} \end{bmatrix} \in \Re^r \tag{22}$$

and

$$\xi \equiv \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_r \end{bmatrix}^T \in \Re^r, \tag{23}$$

$$\eta(t) \equiv [\eta_{r+1}(t) \;\; \eta_{r+2}(t) \;\; \cdots \;\; \eta_n(t)]^T \in \Re^{n-r} \tag{24}$$

$$q(\xi(t), \eta(t)) \equiv [L_f\phi_{r+1}(t) \;\; L_f\phi_{r+2}(t) \;\; \cdots \;\; L_f\phi_n(t)]^T \equiv [q_{r+1} \;\; q_{r+2} \;\; \cdots \;\; q_n]^T \tag{25}$$

Define a phase-variable canonical matrix A_c^i to be

$$A_c^i \equiv \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_1^i & -\alpha_2^i & -\alpha_3^i & \cdots & -\alpha_{r_i}^i \end{bmatrix}_{r_i \times r_i}, \quad 1 \le i \le m \tag{26}$$

where α_1^i, α_2^i, $\cdots$, $\alpha_{r_i}^i$ are any chosen parameters such that A_c^i is Hurwitz and the vector B^i to be

$$B^i \equiv [0 \quad 0 \quad \cdots \quad 0 \quad 1]_{r_i \times 1}^T, \quad 1 \le i \le m \tag{27}$$

Let P^i be the positive definite solution of the following Lyapunov equation

$$(A_c^i)^T P^i + P^i A_c^i = -I, \quad 1 \le i \le m \tag{28}$$

$$\lambda_{\max}(P^i) \equiv \text{the maximum eigenvalue of } P^i, \quad 1 \le i \le m \tag{29}$$

$$\lambda_{\min}(P^i) \equiv \text{the minimum eigenvalue of } P^i, \quad 1 \le i \le m \tag{30}$$

$$\lambda_{\max}^* \equiv \min\left\{\lambda_{\max}(P^1),\ \lambda_{\max}(P^2), \cdots, \lambda_{\max}(P^m)\right\} \tag{31}$$

$$\lambda_{\min}^* \equiv \min\left\{\lambda_{\min}(P^1),\ \lambda_{\min}(P^2), \cdots, \lambda_{\min}(P^m)\right\} \tag{32}$$

Assumption 1. *For all $t \ge 0$, $\eta \in \Re^{n-r}$ and $\xi \in \Re^r$, there exists a positive constant M such that the following inequality holds*

$$\|q_{22}(\xi, \eta, \overline{e}) - q_{22}(t, \eta, 0)\| \le M\left(\|\overline{e}\|\right) \tag{33}$$

where $q_{22}(t, \eta, \bar{e}) \equiv q(\xi, \eta)$.

For the sake of stating precisely the investigated problem, define

$$d_{ij} \equiv L_{g_j} L_f^{r_i - 1} h_i(X), \quad 1 \le i \le m, \quad 1 \le j \le m \tag{34}$$

$$c_i \equiv L_f^{r_i} h_i(X), \quad 1 \le i \le m \tag{35}$$

and

$$\overline{\overline{e^i}} \equiv \alpha_1^i \overline{e_1^i} + \alpha_2^i \overline{e_2^i} + \cdots + \alpha_{r_i}^i \overline{e_{r_i}^i}, \quad 1 \le i \le m \tag{36}$$

Definition 2.1. *[25] Consider the system $\dot{x} = f(t, x, \theta)$, where $f : [0, \infty) \times \Re^n \times \Re^n \to \Re^n$ is piecewise continuous in t and locally Lipschitz in x and θ. This system is said to be input-to-state stable if there exists a class KL function β, a class K function γ and positive constants k_1 and k_2 such that for any initial state $x(t_0)$ with $\|x(t_0)\| < k_1$ and any bounded input $\theta(t)$ with $\sup\limits_{t \ge t_0} \|\theta(t)\| < k_2$, the state exists and satisfies*

$$\|x(t)\| \le \beta\left(\|x(t_0)\|, t - t_0\right) + \gamma\left(\sup_{t_0 \le \tau \le t} \|\theta(\tau)\|\right) \tag{37}$$

for all $t \ge t_0 \ge 0$.

Now we formulate the tracking problem with almost disturbance decoupling as follows:

Definition 2.2. *[31] The tracking problem with almost disturbance decoupling is said to be globally solvable by the state feedback controller u for the transformed-error system by a global diffeomorphism (6), if the controller u enjoys the following properties.*

$\langle 1 \rangle$ It is input-to-state stable with respect to disturbance inputs.

$\langle 2\rangle$ *For any initial value* $\bar{x}_{e0} \equiv [\bar{e}(t_0) \ \ \eta(t_0)]^T$, *for any* $t \geq t_0$ *and for any* $t_0 \geq 0$

$$|y(t) - y_d(t)| \leq \beta_{11}\left(\|x(t_0)\|, t - t_0\right) + \frac{1}{\sqrt{\beta_{22}}}\beta_{33}\left(\sup_{t_0 \leq \tau \leq t} \|\theta(\tau)\|\right) \tag{38}$$

and

$$\int_{t_0}^{t} \left[y(\tau) - y_d(\tau)\right]^2 d\tau \leq \frac{1}{\beta_{44}}\left[\beta_{55}\left(\|\bar{x}_{e0}\|\right) + \int_{t_0}^{t} \beta_{33}\left(\|\theta(\tau)\|^2\right) d\tau\right] \tag{39}$$

where β_{22}, β_{44} *are some positive constants,* β_{33}, β_{55} *are class* K *functions and* β_{11} *is a class* KL *function.*

Theorem 2.1. *Suppose that there exists a continuously differentiable function* V_0: $\Re^{n-r} \to \Re^{+}$ *such that the following three inequalities hold for all* $\eta \in \Re^{n-r}$:

$$\omega_1\|\eta\|^2 \leq V(\eta) \leq \omega_2\|\eta\|^2, \quad \omega_1, \omega_2 > 0 \tag{40a}$$

$$\nabla_t V + (\nabla_\eta V)^T q_{22}(t, \eta, 0) \leq -2\alpha_x V(\eta), \quad \alpha_x > 0 \tag{40b}$$

$$\|\nabla_\eta V\| \leq \varpi_3\|\eta\|, \quad \varpi_3 > 0, \tag{40c}$$

then the tracking problem with almost disturbance decoupling is globally solvable by the controller defined by

$$u_{feedback} = A^{-1}\{-b + v\} \tag{41}$$

$$b \equiv \begin{bmatrix} L_f^{r_1}h_1 & L_f^{r_2}h_2 & \cdots & L_f^{r_m}h_m \end{bmatrix}^T \tag{42}$$

$$v \equiv \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}^T \tag{43}$$

$$\begin{aligned} v_i \equiv{}& y_d^{i(r_i)} - \varepsilon^{-r_i}\alpha_1^i\left[L_f^0 h_i(X) - y_d^i\right] - \varepsilon^{1-r_i}\alpha_2^i\left[L_f^1 h_i(X) - y_d^{i(1)}\right] \\ &- \cdots - \varepsilon^{-1}\alpha_{r_i}^i\left[L_f^{r_i-1}h_i(X) - y_d^{i(r_i-1)}\right], \quad 1 \leq i \leq m \end{aligned} \tag{44}$$

Moreover, the influence of disturbances on the L_2 *norm of the tracking error can be arbitrarily attenuated by increasing the following adjustable parameter* $NN_2 > 1$:

$$\begin{aligned} H(\varepsilon) &\equiv \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} \\ &\equiv \begin{bmatrix} 2\alpha_x - \dfrac{4\omega_3^2}{\omega_1}\|\phi_\eta\|^2 & -\dfrac{1}{\sqrt{k(\varepsilon)}}\left[\dfrac{w_3 M}{\sqrt{2w_1\lambda_{\min}^*}}\right] \\ -\dfrac{1}{\sqrt{k(\varepsilon)}}\left[\dfrac{w_3 M}{\sqrt{2w_1\lambda_{\min}^*}}\right] & \dfrac{1}{\varepsilon\lambda_{\max}^*} - \dfrac{8k(\varepsilon)\|\phi_\xi^1\|^2\|P^1\|^2}{\varepsilon^2\lambda_{\min}(P^1)} - \cdots - \dfrac{8k(\varepsilon)\|\phi_\xi^m\|^2\|P^m\|^2}{\varepsilon^2\lambda_{\min}(P^m)} \end{bmatrix} \end{aligned} \tag{45a}$$

$$\alpha_s(\varepsilon) \equiv \frac{H_{11} + H_{22} - \left[(H_{11} - H_{22})^2 + 4H_{12}^2\right]^{1/2}}{4} \tag{45b}$$

$$N \equiv 2\alpha_s(\varepsilon) \tag{45c}$$

$$N_1 \equiv \frac{m+1}{4}\left(\sup_{t_0 \leq \tau \leq t}\|\theta_d(\tau) + \theta_u(\tau)\|\right)^2 \tag{45d}$$

$$N_2 \equiv \min\left\{\omega_1, \frac{k(\varepsilon)}{2}\lambda_{\min}^*\right\} \tag{45e}$$

$$\phi_\xi^i(\varepsilon) \equiv \begin{bmatrix} \varepsilon\frac{\partial}{\partial X}h_i q_1^* & \cdots & \varepsilon\frac{\partial}{\partial X}h_i q_p^* \\ \vdots & & \vdots \\ \varepsilon^{r_i}\frac{\partial}{\partial X}L_f^{r_i-1}h_i q_1^* & \cdots & \varepsilon^{r_i}\frac{\partial}{\partial X}L_f^{r_i-1}h_i q_q^* \end{bmatrix}, \quad 1 \leq i \leq m \tag{45f}$$

$$\phi_\eta(\varepsilon) \equiv \begin{bmatrix} \frac{\partial}{\partial X}\phi_{r+1}q_1^* & \cdots & \frac{\partial}{\partial X}\phi_{r+1}q_p^* \\ \vdots & & \vdots \\ \frac{\partial}{\partial X}\phi_n q_1^* & \cdots & \frac{\partial}{\partial X}\phi_n q_q^* \end{bmatrix} \tag{45g}$$

where H is positive definite matrix and $k(\varepsilon)$: $\Re^+ \to \Re^+$ is any continuous function satisfies

$$\lim_{\varepsilon\to 0} k(\varepsilon) = 0 \text{ and } \lim_{\varepsilon\to 0} \frac{\varepsilon}{k(\varepsilon)} = 0 \tag{45h}$$

Moreover, the output tracking error of system (2.1) is exponentially attracted into a sphere $B_{\underline{r}}$, $\underline{r} = \sqrt{\frac{N_1}{NN_2}}$, with an exponential rate of convergence

$$\frac{1}{2}\left(\frac{NN_2}{\Delta_{\max}} - \frac{N_1}{\Delta_{\max}\underline{r}^2}\right) \equiv \frac{1}{2}\alpha^* \tag{45i}$$

where

$$\Delta_{\max} = \max\left\{\omega_2, \frac{k}{2}\lambda^*_{\max}\right\} \tag{45j}$$

Proof: Applying the coordinate transformation (6) yields

$$\begin{aligned} \dot{\xi}_1^1 &= \frac{\partial\phi_1^1}{\partial X}\frac{dX}{dt} = \frac{\partial h_1}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^*\theta_{jd} + \Delta f\right] \\ &= \frac{\partial h_1}{\partial X}f + \frac{\partial h_1}{\partial X}g_1u_1 + \cdots + \frac{\partial h_1}{\partial X}g_mu_m + \sum_{j=1}^{p}\frac{\partial h_1}{\partial X}q_j^*\theta_{jd} + \sum_{j=1}^{p}\frac{\partial h_1}{\partial X}q_j^*\theta_{ju} \\ &= \frac{\partial h_1}{\partial X}f + \sum_{j=1}^{p}\frac{\partial h_1}{\partial X}q_j^*\left(\theta_{jd}+\theta_{ju}\right) = \xi_2^1 + \sum_{j=1}^{p}\frac{\partial h_1}{\partial X}q_j^*\left(\theta_{jd}+\theta_{ju}\right) \end{aligned} \tag{46}$$

$$\vdots$$

$$\begin{aligned} \dot{\xi}_{r_1-1}^1 &= \frac{\partial\phi_{r_1-1}^1}{\partial X}\frac{dX}{dt} = \frac{\partial L_f^{r_1-2}h_1}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^*\theta_{jd} + \Delta f\right] \\ &= \frac{\partial L_f^{r_1-2}h_1}{\partial X}f + \frac{\partial L_f^{r_1-2}h_1}{\partial X}g_1u_1 + \cdots + \frac{\partial L_f^{r_1-2}h_1}{\partial X}g_mu_m \\ &\quad + \sum_{j=1}^{p}\frac{\partial L_f^{r_1-2}h_1}{\partial X}q_j^*\theta_{jd} + \sum_{j=1}^{p}\frac{\partial L_f^{r_1-2}h_1}{\partial X}q_j^*\theta_{ju} \\ &= \frac{\partial L_f^{r_1-2}h_1}{\partial X}f + \sum_{j=1}^{p}\frac{\partial L_f^{r_1-2}h_1}{\partial X}q_j^*\left(\theta_{jd}+\theta_{ju}\right) \\ &= L_f^{r_1-1}h_1 + \sum_{j=1}^{p}\frac{\partial L_f^{r_1-2}h_1}{\partial X}q_j^*\left(\theta_{jd}+\theta_{ju}\right) \end{aligned} \tag{47}$$

$$
\begin{aligned}
\dot{\xi}_{r_1}^1 &= \frac{\partial \phi_{r_1}^1}{\partial X}\frac{dX}{dt} = \frac{\partial L_f^{r_1-1}h_1}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^*\theta_{jd} + \Delta f\right] \\
&= \frac{\partial L_f^{r_1-1}h_1}{\partial X}f + \frac{\partial L_f^{r_1-1}h_1}{\partial X}g_1u_1 + \cdots + \frac{\partial L_f^{r_1-1}h_1}{\partial X}g_mu_m + \sum_{j=1}^{p}\frac{\partial L_f^{r_1-1}h_1}{\partial X}q_j^*\theta_{jd} \\
&\quad + \sum_{j=1}^{p}\frac{\partial L_f^{r_1-1}h_1}{\partial X}q_j^*\theta_{ju} \\
&= L_f^{r_1}h_1 + L_{g_1}L_f^{r_1-1}h_1u_1 + \cdots + L_{g_m}L_f^{r_1-1}h_1u_m + \sum_{j=1}^{p}\frac{\partial L_f^{r_1-1}h_1}{\partial X}q_j^*\left(\theta_{jd} + \theta_{ju}\right) \\
&= c_1 + d_{11}u_1 + \cdots + d_{1m}u_m + \sum_{j=1}^{p}\frac{\partial L_f^{r_1-1}h_1}{\partial X}q_j^*\left(\theta_{jd} + \theta_{ju}\right)
\end{aligned} \tag{48}
$$

$$\vdots$$

$$
\begin{aligned}
\dot{\xi}_1^m &= \frac{\partial \phi_1^m}{\partial X}\frac{dX}{dt} = \frac{\partial h_m}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^*\theta_{jd} + \Delta f\right] \\
&= \frac{\partial h_m}{\partial X}f + \frac{\partial h_m}{\partial X}g_1u_1 + \cdots + \frac{\partial h_m}{\partial X}g_mu_m + \sum_{j=1}^{p}\frac{\partial h_m}{\partial X}q_j^*\theta_{jd} + \sum_{j=1}^{p}\frac{\partial h_m}{\partial X}q_j^*\theta_{ju} \\
&= L_f^1h_m + \sum_{j=1}^{p}\frac{\partial h_m}{\partial X}q_j^*\left(\theta_{jd} + \theta_{ju}\right) = \xi_2^m + \sum_{j=1}^{p}\frac{\partial h_m}{\partial X}q_j^*(\theta_{jd} + \theta_{ju})
\end{aligned} \tag{49}
$$

$$\vdots$$

$$
\begin{aligned}
\dot{\xi}_{r_m-1}^m &= \frac{\partial \phi_{r_m-1}^m}{\partial X}\frac{dX}{dt} = \frac{\partial L_f^{r_m-2}h_m}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^*\theta_{jd} + \Delta f\right] \\
&= \frac{\partial L_f^{r_m-2}h_m}{\partial X}f + \frac{\partial L_f^{r_m-2}h_m}{\partial X}g_1u_1 + \cdots + \frac{\partial L_f^{r_m-2}h_m}{\partial X}g_mu_m + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-2}h_m}{\partial X}q_j^*\theta_{jd} \\
&\quad + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-2}h_m}{\partial X}q_j^*\theta_{ju} \\
&= L_f^{r_m-1}h_m + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-2}h_m}{\partial X}q_j^*\left(\theta_{jd} + \theta_{ju}\right) = \xi_{r_m}^m + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-2}h_m}{\partial X}q_j^*\left(\theta_{jd} + \theta_{ju}\right)
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
\dot{\xi}_{r_m}^m &= \frac{\partial \phi_{r_m}^m}{\partial X}\frac{dX}{dt} = \frac{\partial L_f^{r_m-1}h_m}{\partial X}\left[f + g\cdot u + \sum_{j=1}^{p} q_j^*\theta_{jd} + \Delta f\right] \\
&= \frac{\partial L_f^{r_m-1}h_m}{\partial X}f + \frac{\partial L_f^{r_m-1}h_m}{\partial X}g_1u_1 + \cdots + \frac{\partial L_f^{r_m-1}h_m}{\partial X}g_mu_m + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-1}h_m}{\partial X}q_j^*\theta_{jd} \\
&\quad + \sum_{j=1}^{p}\frac{\partial L_f^{r_m-1}h_m}{\partial X}q_j^*\theta_{ju}
\end{aligned}
$$

$$\begin{aligned} &= L_f^{r_m} h_m + L_{g_1} L_f^{r_m-1} h_m u_1 + \cdots + L_{g_m} L_f^{r_m-1} h_m u_m + \sum_{j=1}^{p} \frac{\partial L_f^{r_m-1} h_m}{\partial X} q_j^* \left(\theta_{jd} + \theta_{ju}\right) \\ &= c_m + d_{m1} u_1 + \cdots + d_{mm} u_m + \sum_{j=1}^{p} \frac{\partial L_f^{r_m-1} h_m}{\partial X} q_j^* \left(\theta_{jd} + \theta_{ju}\right) \end{aligned} \tag{51}$$

$$\begin{aligned} \dot{\eta}_k(t) &= \frac{\partial \phi_k}{\partial X} \frac{dX}{dt} = \frac{\partial \phi_k}{\partial X} \left[f + g \cdot u + \sum_{j=1}^{p} q_j^* \theta_{jd} + \Delta f \right] \\ &= \frac{\partial \phi_k}{\partial X} f + \frac{\partial \phi_k}{\partial X} g_1 u_1 + \cdots + \frac{\partial \phi_k}{\partial X} g_m u_m + \sum_{j=1}^{p} \frac{\partial \phi_k}{\partial X} q_j^* \theta_{jd} + \sum_{j=1}^{p} \frac{\partial \phi_k}{\partial X} q_j^* \theta_{ju} \\ &= L_f \phi_k + \sum_{j=1}^{p} \frac{\partial \phi_k}{\partial X} q_j^* \left(\theta_{jd} + \theta_{ju}\right) \\ &= q_k + \sum_{j=1}^{p} \frac{\partial \phi_k}{\partial X} q_j^* \left(\theta_{jd} + \theta_{ju}\right), \quad k = r+1, r+2, \cdots, n \end{aligned} \tag{52}$$

Since

$$c_i(\xi(t), \eta(t)) \equiv L_f^{r_i} h_i(X(t)), \quad 1 \le i \le m \tag{53}$$

$$d_{ij} \equiv L_{g_j} L_f^{r_i-1} h_i(X), \quad 1 \le i \le m, \quad 1 \le j \le m \tag{54}$$

$$q_k(\xi(t), \eta(t)) = L_f \phi_k(X), \quad k = r+1, r+2, \cdots, n \tag{55}$$

the dynamic equations of system (2.1) in the new co-ordinates are as follows:

$$\dot{\xi}_i^1(t) = \xi_{i+1}^1(t) + \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^{i-1} h_1 q_j^* \left(\theta_{jd} + \theta_{ju}\right), \quad i = 1, 2, \cdots, r_1 - 1 \tag{56}$$

$$\begin{aligned} \dot{\xi}_{r_1}^1(t) &= c_1(\xi(t), \eta(t)) + d_{11}(\xi(t), \eta(t)) u_1 + \cdots + d_{1m}(\xi(t), \eta(t)) u_m \\ &\quad + \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^{r_1-1} h_1 q_j^* (\theta_{jd} + \theta_{ju}) \end{aligned} \tag{57}$$

$$\vdots$$

$$\dot{\xi}_i^m(t) = \xi_{i+1}^m(t) + \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^{i-1} h_m q_j^* \left(\theta_{jd} + \theta_{ju}\right), \quad i = 1, 2, \cdots, r_m - 1 \tag{58}$$

$$\begin{aligned} \dot{\xi}_{r_m}^m(t) &= c_m(\xi(t), \eta(t)) + d_{m1}\left(\xi(t), \eta(t)\right) u_1 + \cdots + d_{mm}(\xi(t), \eta(t)) u_m \\ &\quad + \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^{r_m-1} h_m q_j^* \left(\theta_{jd} + \theta_{ju}\right) \end{aligned} \tag{59}$$

$$\dot{\eta}_k(t) = q_k(\xi(t), \eta(t)) + \sum_{j=1}^{p} \frac{\partial}{\partial X} \phi_k(X) q_j^* \left(\theta_{jd} + \theta_{ju}\right), \quad k = r+1, \cdots, n \tag{60}$$

$$y_i(t) = \xi_1^i(t), \quad 1 \le i \le m \tag{61}$$

According to Equations (18), (44), (53) and (54), the tracking controller can be rewritten as

$$u_{feedback} = A^{-1}\left[-b + v\right] \tag{62}$$

Substituting Equation (62) into (57) and (59), the dynamic equations of system (2.1) can be shown as follows:

$$\begin{bmatrix} \dot{\xi}_1^i(t) \\ \dot{\xi}_2^i(t) \\ \vdots \\ \dot{\xi}_{r_i-1}^i(t) \\ \dot{\xi}_{r_i}^i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \xi_1^i(t) \\ \xi_2^i(t) \\ \vdots \\ \xi_{r_i-1}^i(t) \\ \xi_{r_i}^i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v_i + \begin{bmatrix} \sum_{j=1}^{p} \frac{\partial}{\partial X} h_i q_j^* \left(\theta_{jd} + \theta_{ju}\right) \\ \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^1 h_i q_j^* \left(\theta_{jd} + \theta_{ju}\right) \\ \vdots \\ \sum_{j=1}^{p} \frac{\partial}{\partial X} L_f^{r_i-1} h_i q_j^* \left(\theta_{jd} + \theta_{ju}\right) \end{bmatrix} \tag{63}$$

$$\begin{bmatrix} \dot{\eta}_{r+1}(t) \\ \dot{\eta}_{r+2}(t) \\ \vdots \\ \dot{\eta}_{n-1}(t) \\ \dot{\eta}_n(t) \end{bmatrix} = \begin{bmatrix} q_{r+1}(t) \\ q_{r+2}(t) \\ \vdots \\ q_{n-1}(t) \\ q_n(t) \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{p} \frac{\partial}{\partial X} \phi_{r+1} q_j^* \left(\theta_{jd} + \theta_{ju}\right) \\ \sum_{j=1}^{p} \frac{\partial}{\partial X} \phi_{r+2} q_j^* \left(\theta_{jd} + \theta_{ju}\right) \\ \vdots \\ \sum_{j=1}^{p} \frac{\partial}{\partial X} \phi_{n-1} q_j^* \left(\theta_{jd} + \theta_{ju}\right) \\ \sum_{j=1}^{p} \frac{\partial}{\partial X} \phi_n q_j^* \left(\theta_{jd} + \theta_{ju}\right) \end{bmatrix} \tag{64}$$

$$y_i = [1 \quad 0 \quad \cdots \quad 0 \quad 0]_{r\times 1} \begin{bmatrix} \xi_1^i(t) \\ \xi_2^i(t) \\ \vdots \\ \xi_{r_i-1}^i(t) \\ \xi_{r_i}^i(t) \end{bmatrix}_{r\times 1} = \xi_1^i(t), \quad 1 \le i \le m \tag{65}$$

Combining Equations (18), (20), (21), (26) and (44), it can be easily verified that Equations (63)-(65) can be transformed into the following form.

$$\dot{\eta}(t) = q(\xi(t), \eta(t)) + \phi_\eta(\theta_d + \theta_u) \equiv q_{22}(t, \eta(t), \bar{e}) + \phi_\eta(\theta_d + \theta_u) \tag{66a}$$

$$\varepsilon\, \dot{\overline{e^i}}(t) = A_c^i \overline{e^i} + \phi_\xi^i \left(\theta_d + \theta_u\right), \quad 1 \le i \le m \tag{66b}$$

$$y_i(t) = \xi_1^i(t), \quad 1 \le i \le m \tag{67}$$

We consider $L\left(\bar{e}, \eta\right)$ defined by a weighted sum of $V(\eta)$ and $W\left(\bar{e}\right)$,

$$L\left(\bar{e}, \eta\right) \equiv V(\eta) + k(\varepsilon)W(\bar{e}) \equiv V(\eta) + k(\varepsilon)\left(W^1\left(\overline{e^1}\right) + \cdots + W^m\left(\overline{e^m}\right)\right) \tag{68}$$

where

$$W(\bar{e}) \equiv W^1\left(\overline{e^1}\right) + \cdots + W^m\left(\overline{e^m}\right) \tag{69}$$

as a composite Lyapunov function of the subsystems (66a) and (66b) [26,29], where $W\left(\overline{e^i}\right)$ satisfies

$$W^i\left(\overline{e^i}\right) \equiv \frac{1}{2}\overline{e^i}^T P^i \overline{e^i} \tag{70}$$

In view of Equations (18), (33) and (40), the derivative of L along the trajectories of (66a) and (66b) is given by

$$\begin{aligned}
\dot{L} = &\left[\nabla_t V + (\nabla_\eta V)^T \dot{\eta}\right] + \frac{k}{2}\left[\left(\dot{\overline{e^1}}\right) P^1 \overline{e^1} + \left(\overline{e^1}\right)^T P^1 \left(\dot{\overline{e^1}}\right)\right.\\
&\left. + \cdots + \left(\dot{\overline{e^m}}\right) P^m \overline{e^m} + \left(\overline{e^m}\right)^T P^m \left(\dot{\overline{e^m}}\right)\right]\\
= &\left[\nabla_t V + (\nabla_\eta V)^T \dot{\eta}\right] + \frac{k}{2}\left[\left(\frac{1}{\varepsilon}A_c^1\overline{e^1} + \frac{1}{\varepsilon}\phi_\xi^1(\theta_d+\theta_u)\right)^T P^1\overline{e^1}\right.\\
&+ \left(\overline{e^1}\right)^T P^1\left(\frac{1}{\varepsilon}A_c^1\overline{e^1} + \frac{1}{\varepsilon}\phi_\xi^1(\theta_d+\theta_u)\right) + \cdots + \left(\frac{1}{\varepsilon}A_c^m\overline{e^m} + \frac{1}{\varepsilon}\phi_\xi^m(\theta_d+\theta_u)\right)^T P^m\overline{e^m}\\
&\left. + \left(\overline{e^m}\right)^T P^m\left(\frac{1}{\varepsilon}A_c^m\overline{e^m} + \frac{1}{\varepsilon}\phi_\xi^m(\theta_d+\theta_u)\right)\right]\\
= &\left[\nabla_t V + (\nabla_\eta V)^T\left(q_{22}\left(t,\eta(t),\bar{e}\right) + \phi_\eta(\theta_d+\theta_u)\right)\right] + \left\{\frac{k}{2\varepsilon}\left(\overline{e^1}\right)^T\left[(A_c^1)^T P^1 + P^1(A_c^1)\right]\overline{e^1}\right.\\
&+ \cdots + \frac{k}{2\varepsilon}\left(\overline{e^m}\right)^T\left[(A_c^m)^T P^m + P^m(A_c^m)\right]\overline{e^m}\\
&\left. + \frac{k}{\varepsilon}\left[(\theta_d+\theta_u)^T(\phi_\xi^1)^T P^1\overline{e^1} + \cdots + (\theta_d+\theta_u)^T(\phi_\xi^m)^T P^m\overline{e^m}\right]\right\}\\
\le &\left[\nabla_t V + (\nabla_\eta V)^T q_{22}\left(t,\eta(t),\bar{e}\right) + (\nabla_\eta V)^T\phi_\eta(\theta_d+\theta_u)\right] - \frac{k}{2\varepsilon}\left[\left(\overline{e^1}\right)^T\overline{e^1} + \cdots\right.\\
&\left. + \left(\overline{e^m}\right)^T\overline{e^m}\right] + \frac{k}{\varepsilon}\left[\|(\theta_d+\theta_u)\|\left\|\phi_\xi^1\right\|\left\|P^1\right\|\left\|\overline{e^1}\right\| + \cdots + \|(\theta_d+\theta_u)\|\|\phi_\xi^m\|\|P^m\|\|\overline{e^m}\|\right]\\
\le &\left[\nabla_t V + (\nabla_\eta V)^T q_{22}\left(t,\eta(t),0\right)\right] + (\nabla_\eta V)^T\left[q_{22}\left(t,\eta(t),\bar{e}\right) - q_{22}\left(t,\eta(t),0\right)\right]\\
&+ \|\nabla_\eta V\|\|\phi_\eta\|\|(\theta_d+\theta_u)\| - \frac{k}{\varepsilon}\left[\frac{W^1}{\lambda_{\max}(P^1)} + \cdots + \frac{W^m}{\lambda_{\max}(P^m)}\right]\\
&+ \frac{4k^2}{\varepsilon^2}\left\|\phi_\xi^1\right\|^2\left\|P^1\right\|^2\left\|\overline{e^1}\right\|^2 + \frac{1}{16}\|\theta_d+\theta_u\|^2 + \cdots\\
&+ \frac{4k^2}{\varepsilon^2}\|\phi_\xi^m\|^2\|P^m\|^2\|\overline{e^m}\|^2 + \frac{1}{16}\|(\theta_d+\theta_u)\|^2\\
\le &\left[\nabla_t V + (\nabla_\eta V)^T q_{22}(t,\eta(t),0)\right] + \|\nabla_\eta V\|\|q_{22}\left(t,\eta(t),\bar{e}\right) - q_{22}\left(t,\eta(t),0\right)\|\\
&+ \|\nabla_\eta V\|\|\phi_\eta\|\|(\theta_d+\theta_u)\| - \frac{k}{\varepsilon}\frac{1}{\lambda^*_{\max}}\left[W^1 + \cdots + W^m\right] + \frac{4k^2}{\varepsilon^2}\left\|\phi_\xi^1\right\|^2\left\|P^1\right\|^2\left\|\overline{e^1}\right\|^2\\
&+ \frac{1}{16}\|(\theta_d+\theta_u)\|^2 + \cdots + \frac{4k^2}{\varepsilon^2}\left\|\phi_\xi^m\right\|^2\left\|P^m\right\|^2\left\|\overline{e^m}\right\|^2 + \frac{1}{16}\|(\theta_d+\theta_u)\|^2\\
\le &-2\alpha_x V + \omega_3\|\eta\|M\|\bar{e}\| + \omega_3\|\eta\|\|\phi_\eta\|\|(\theta_d+\theta_u)\|\\
&- \frac{k}{\varepsilon}\frac{1}{\lambda^*_{\max}}W + \frac{4k^2}{\varepsilon^2}\|\phi_\xi^1\|^2\left\|P^1\right\|^2\left\|\overline{e^1}\right\|^2\\
&+ \frac{1}{16}\|(\theta_d+\theta_u)\|^2 + \cdots + \frac{4k^2}{\varepsilon^2}\|\phi_\xi^m\|^2\|P^m\|^2\|\overline{e^m}\|^2 + \frac{1}{16}\|(\theta_d+\theta_u)\|^2\\
\le &-2\alpha_x V + \omega_3\frac{1}{\sqrt{\omega_1}}\sqrt{V}M\sqrt{\frac{2}{\lambda^*_{\min}}}\sqrt{W} + \omega_3\frac{1}{\sqrt{\omega_1}}\sqrt{V}\|\phi_\eta\|\|\left(\theta_d+\theta_u\right)\|\\
&- \frac{k}{\varepsilon}\frac{1}{\lambda^*_{\max}}W + \frac{4k^2}{\varepsilon^2}\left\|\phi_\xi^1\right\|^2\left\|P^1\right\|^2\frac{W^1}{1/2\lambda_{\min}(P^1)}
\end{aligned}$$

$$
\begin{aligned}
&+\cdots+\frac{4k^2}{\varepsilon^2}\|\phi_\xi^m\|^2\|P^m\|^2\frac{W^m}{1/2\lambda_{\min}(P^m)}+\frac{m}{16}\|(\theta_d+\theta_u)\|^2\\
\le\ & -2\alpha_x\left(\sqrt{V}\right)^2+2\frac{\omega_3 M}{\sqrt{2k\lambda_{\min}^*\omega_1}}\sqrt{V}\sqrt{kW}+4\left(\frac{\omega_3}{\sqrt{\omega_1}}\|\phi_\eta\|\right)^2\left(\sqrt{V}\right)^2\\
&+\frac{1}{16}\|(\theta_d+\theta_u)\|^2-\frac{1}{\varepsilon}\frac{1}{\lambda_{\max}^*}\left(\sqrt{kM}\right)^2+\frac{4k^2}{\varepsilon^2}\|\phi_\xi^1\|^2\|P^1\|^2\frac{W^1}{1/2\lambda_{\min}(P^1)}\\
&+\cdots+\frac{4k^2}{\varepsilon^2}\|\phi_\xi^m\|^2\|P^m\|^2\frac{W^m}{1/2\lambda_{\min}(P^m)}+\frac{m}{16}\|(\theta_d+\theta_u)\|^2\\
\le\ & -\left(2\alpha_x-\frac{4\omega_3^2}{\omega_1}\|\phi_\eta\|^2\right)\left(\sqrt{V}\right)^2+2\left(\frac{\omega_3 M}{\sqrt{2\omega_1 k\lambda_{\min}^*}}\right)\sqrt{V}\sqrt{kW}\\
&-\left(\frac{1}{\varepsilon\lambda_{\max}^*}-\frac{4k\|\phi_\xi^1\|^2\|P^1\|^2}{\frac{1}{2}\varepsilon^2\lambda_{\min}(P^1)}-\cdots-\frac{4k\|\phi_\xi^m\|^2\|P^m\|^2}{\frac{1}{2}\varepsilon^2\lambda_{\min}(P^m)}\right)\left(\sqrt{kW}\right)^2\\
&+\frac{m+1}{16}\|(\theta_d+\theta_u)\|^2\\
=\ & -\begin{bmatrix}\sqrt{V} & \sqrt{kW}\end{bmatrix} H \begin{bmatrix}\sqrt{V}\\ \sqrt{kW}\end{bmatrix}+\frac{m+1}{16}\|(\theta_d+\theta_u)\|^2
\end{aligned}
$$

i.e.,

$$
\dot{L}\le-\lambda_{\min}(H)L+\frac{m+1}{16}\|(\theta_d+\theta_u)\|^2 \tag{72}
$$

where $\lambda_{\min}(H)$ denotes the minimum eigenvalue of the matrix H. Utilizing the fact that $\lambda_{\min}(H)=2\alpha_s$, we obtain

$$
\begin{aligned}
\dot{L}\le\ & -2\alpha_s L+\frac{m+1}{16}\|(\theta_d+\theta_u)\|^2\le-2\alpha_s(V+kW)+\frac{m+1}{16}\|(\theta_d+\theta_u)\|^2\\
\le\ & -2\alpha_s\left(\omega_1\|\eta\|^2+\frac{k}{2}\lambda_{\min}^*\|\bar{e}\|^2\right)+\frac{m+1}{16}\|(\theta_d+\theta_u)\|^2\\
\le\ & -NN_2\left(\|\eta\|^2+\|\bar{e}\|^2\right)+\frac{m+1}{16}\|(\theta_d+\theta_u)\|^2
\end{aligned} \tag{73}
$$

Define

$$
\bar{e}\equiv\begin{bmatrix}\overline{e^1}\\ \overline{e^2}\\ \vdots\\ \overline{e^m}\end{bmatrix}\equiv\begin{bmatrix}\overline{e_1^1}\\ \overline{e_{rem}^1}\end{bmatrix},\quad \overline{e_{rem}^1}\in\Re^{r-1} \tag{74}
$$

Hence,

$$
\dot{L}\le-NN_2\left(\|\eta\|^2+\left\|\overline{e_1^1}\right\|^2+\left\|\overline{e_{rem}^1}\right\|^2\right)+\frac{m+1}{16}\|(\theta_d+\theta_u)\|^2 \tag{75}
$$

Utilizing Equation (75) easily yields

$$
\int_{t_0}^{t}\left(y_1(\tau)-y_d^1(\tau)\right)^2d\tau\le\frac{L(t_0)}{NN_2}+\frac{m+1}{16NN_2}\int_{t_0}^{t}\|(\theta_d(\tau)+\theta_u(\tau))\|^2d\tau \tag{76}
$$

Similarly, it is easy to prove that

$$
\int_{t_0}^{t}\left(y_i(\tau)-y_d^i(\tau)\right)^2d\tau\le\frac{L(t_0)}{NN_2}+\frac{m+1}{16NN_2}\int_{t_0}^{t}\|(\theta_d(\tau)+\theta_u(\tau))\|^2d\tau,\quad 2\le i\le m \tag{77}
$$

so that Equation (39) is satisfied. From Equation (73), we get

$$\dot{L} \leq -NN_2 \left(\|y_{total}\|^2\right) + \frac{m+1}{16} \|(\theta_d + \theta_u)\|^2 \tag{78a}$$

where

$$\|y_{total}\|^2 \equiv \|\bar{e}\|^2 + \|\eta\|^2 . \tag{78b}$$

By virtue of Theorem 5.2 [25], Equation (78a) implies the input-to-state stability for the closed-loop system. Furthermore, it is easy to see that

$$\Delta_{\min} \left(\|\bar{e}\|^2 + \|\eta\|^2\right) \leq L \leq \Delta_{\max} \left(\|\bar{e}\|^2 + \|\eta\|^2\right) \tag{79}$$

that is

$$\Delta_{\min} \left(\|y_{total}\|^2\right) \leq L \leq \Delta_{\max} \left(\|y_{total}\|^2\right) \tag{80}$$

where $\Delta_{\min} \equiv \min\left\{\omega_1, \frac{k}{2}\lambda^*_{\min}\right\}$ and $\Delta_{\max} \equiv \max\left\{\omega_2, \frac{k}{2}\lambda^*_{\max}\right\}$. From Equation (73) and Equation (80) yield that

$$\dot{L} \leq -\frac{NN_2}{\Delta_{\max}} L + \frac{m+1}{16} \left(\sup_{t_0 \leq \tau \leq t} \|(\theta_d(\tau) + \theta_u(\tau))\|\right)^2 \tag{81}$$

Hence,

$$L(t) \leq L(t_0) e^{-\frac{NN_2}{\Delta_{\max}}(t-t_0)} + \frac{\Delta_{\max}(m+1)}{16NN_2} \left(\sup_{t_0 \leq \tau \leq t} \|(\theta_d(\tau) + \theta_u(\tau))\|\right)^2, \quad t \geq t_0 \tag{82}$$

which implies

$$\left|y_1(t) - y_d^1(t)\right| \leq \sqrt{\frac{2L(t_0)}{k\lambda^*_{\min}}} e^{-\frac{NN_2}{2\Delta_{\max}}(t-t_0)} + \sqrt{\frac{\Delta_{\max}(m+1)}{8k\lambda^*_{\min} NN_2}} \left(\sup_{t_0 \leq \tau \leq t} \|(\theta_d(\tau) + \theta_u(\tau))\|\right) \tag{83}$$

Similarly, it is easy to prove that

$$\begin{aligned} \left|y_i(t) - y_d^i(t)\right| \leq& \sqrt{\frac{2L(t_0)}{k\lambda^*_{\min}}} e^{-\frac{NN_2}{2\Delta_{\max}}(t-t_0)} \\ &+ \sqrt{\frac{\Delta_{\max}(m+1)}{8k\lambda^*_{\min} NN_2}} \left(\sup_{t_0 \leq \tau \leq t} \|(\theta_d(\tau) + \theta_u(\tau))\|\right), \quad 2 \leq i \leq m \end{aligned} \tag{84}$$

So that Equation (38) is proved and then the tracking problem with almost disturbance decoupling is globally solved. Finally, we will prove that the sphere $B_{\underline{r}}$ is a global attractor for the output tracking error of system (2.1). From Equation (78a) and Equation (45d), we get

$$\dot{L} \leq -NN_2 \left(\|y_{total}\|^2\right) + N_1 \tag{85}$$

For $\|y_{total}\| > \underline{r}$, we have $\dot{L} < 0$. Hence, any closed ball defined by

$$B_{\underline{r}} \equiv \left\{ \begin{bmatrix} \bar{e} \\ \eta \end{bmatrix} : \|\bar{e}\|^2 + \|\eta\|^2 \leq \underline{r} \right\} \tag{86}$$

is a global final attractor for the tracking error system of the non-linear control systems (2.1). Furthermore, it is easy routine to see that, for $y \notin B_{\bar{r}}$, we have

$$\begin{aligned} \frac{\dot{L}}{L} &\leq \frac{-NN_2 \|y_{total}\|^2 + N_1}{L} \leq \frac{-NN_2 \|y_{total}\|^2 + N_1}{\Delta_{\max} \|y_{total}\|^2} \leq \frac{-NN_2}{\Delta_{\max}} + \frac{N_1}{\Delta_{\max} \|y_{total}\|^2} \\ &\leq \frac{-NN_2}{\Delta_{\max}} + \frac{N_1}{\Delta_{\max} \underline{r}^2} \equiv -\alpha^* \end{aligned} \tag{87}$$

i.e.,

$$\dot{L} \leq -\alpha^* L$$

According to the comparison theorem [32], we get

$$L(t) \leq L(t_0) \exp\left[-\alpha^*(t - t_0)\right]$$

Therefore,

$$\begin{aligned} \Delta_{\min} \|y_{total}\|^2 \leq L(y_{total}(t)) \leq L(y_{total}(t_0)) \exp\left[-\alpha^*(t - t_0)\right] \\ \leq \Delta_{\max} \|y_{total}(t_0)\|^2 \exp\left[-\alpha^*(t - t_0)\right] \end{aligned} \tag{88}$$

Consequently, we get

$$\|y_{total}\| \leq \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}} \|y_{total}(t_0)\| \exp\left[-\frac{1}{2}\alpha^*(t - t_0)\right]$$

i.e., the convergence rate toward the sphere $B_{\underline{r}}$ is equal to $\alpha^*/2$. This completes our proof.
B. Fuzzy controller design.

After the feedback linearization control is utilized as a guarantee of uniform ultimate bounded stability, the multiple input/single output fuzzy control design can be technically applied via human expert's knowledge to improve the convergence rate of tracking error. The block diagram of the fuzzy control is shown in Figure 1.

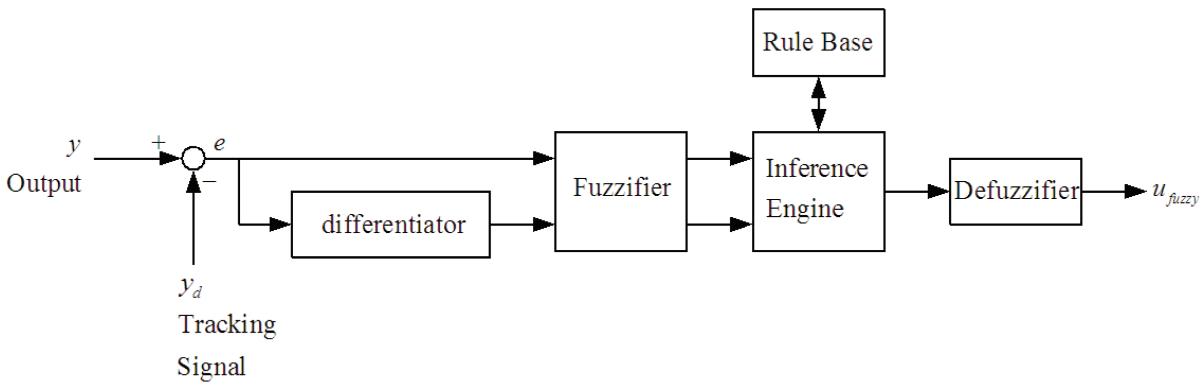
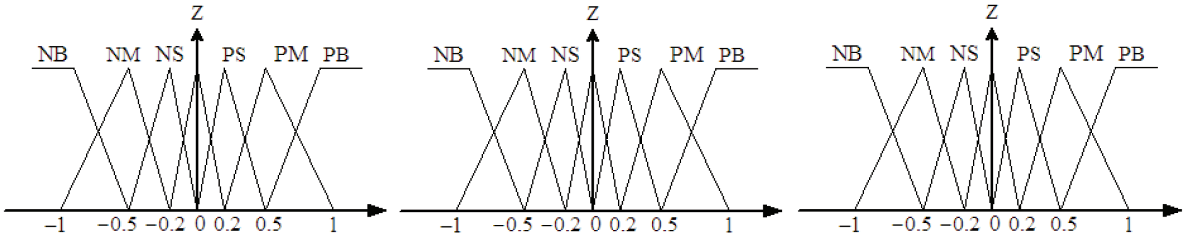


FIGURE 1. Fuzzy logic controller

In general, the tracking error $e(t)$ and its time derivative $\dot{e}(t)$ are utilized as the input fuzzy variables of the IF-THEN control rules and the output is the control variable u_{fuzzy}. For the sake of easy computation, the membership functions of the linguistic terms for $e(t)$, $\dot{e}(t)$ and u_{fuzzy} are all chosen to be the triangular shape function. We define seven linguistic terms: PB (Positive big), PM (Positive medium), PS (Positive small), ZE (Zero), NS (Negative small), NM (Negative medium) and NB (Negative big), for each fuzzy variable, as shown in Figure 2.

Fuzzy control rule table for u_{fuzzy} is shown in Table 1. The rule base is heuristically built by the standard Macvicar-Whelan rule base [50] for usual servo control systems. The Mamdani method is used for fuzzy inference. The defuzzification of the output set membership value is obtained by the centroid method. Therefore, we can combine the designs of feedback linearization control and fuzzy control to construct the overall controller as follows:

$$u_{fe+fu} \equiv \{-b + v\} u_s(t) + \begin{bmatrix} u_{fuzzy1} u_s(t - t_1) \\ u_{fuzzy2} u_s(t - t_2) \\ \vdots \\ u_{fuzzym} u_s(t - t_m) \end{bmatrix}, \quad 1 \leq i \leq m \tag{89}$$



(a) $e(t)$ (b) $\dot{e}(t)$ (c) u_{fuzzy}

FIGURE 2. Membership functions for (a) $e(t)$, (b) $\dot{e}(t)$ and (c) u_{fuzzy}

where $u_s(t)$ denotes the unit step function and t_i, $1 \le i \le m$ are the time variables that the tracking error of states touch the global final attractor $B_{\underline{r}}$.

TABLE 1. Fuzzy control rule base

u_{fuzzy}		$e(t)$ NB	NM	NS	ZE	PS	PM	PB
$\dot{e}(t)$	NB	PB	PB	PB	PB	PM	PS	ZE
	NM	PB	PB	PB	PM	PS	ZE	NS
	NS	PB	PB	PM	PS	ZE	NS	NM
	ZE	PB	PM	PS	ZE	NS	NM	NB
	PS	PM	PS	ZE	NS	NM	NB	NB
	PM	PS	ZE	NS	NM	NB	NB	NB
	PB	ZE	NS	NM	NB	NB	NB	NB

3. **Marine Vehicle in 6 Degrees of Freedom.** When investigating the behavior of marine vehicle in 6 degree of freedom, it is convenient to define body-fixed coordinate frame and earth-fixed coordinate frame as indicated in Figure 3.

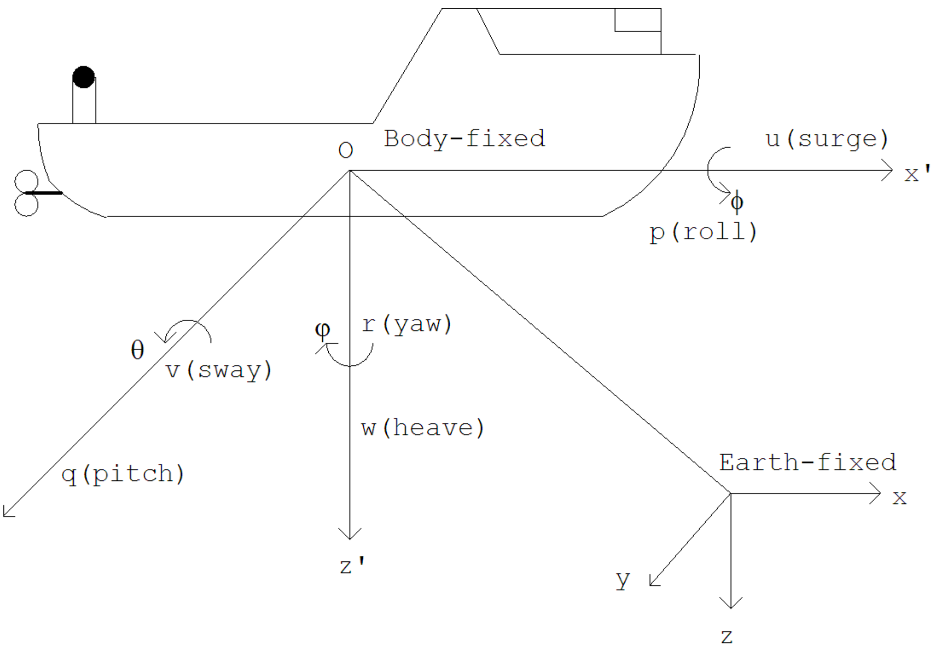


FIGURE 3. Body-fixed and earth-fixed reference frames for marine vehicle in 6 degrees of freedom

In general, the behavior of marine vehicle in 6 degree of freedom can be described by the following vectors: $\bar{\eta} \equiv \begin{bmatrix} \bar{\eta}_1^T & \bar{\eta}_2^T \end{bmatrix}^T$, $\bar{\eta}_1 \equiv [x \quad y \quad z]^T$, $\bar{\eta}_2 \equiv [\phi \quad \theta \quad \varphi]^T$, $\bar{\upsilon} \equiv \begin{bmatrix} \bar{\upsilon}_1^T & \bar{\upsilon}_2^T \end{bmatrix}^T$, $\bar{\upsilon}_1 \equiv [u \quad v \quad w]^T$, $\bar{\upsilon}_2 \equiv [p \quad q \quad r]^T$, $\bar{\tau} \equiv \begin{bmatrix} \bar{\tau}_1^T & \bar{\tau}_2^T \end{bmatrix}^T$, $\bar{\tau}_1 \equiv [\tau_x \quad \tau_y \quad \tau_z]^T$ and $\bar{\tau}_2 \equiv [\tau_K \quad \tau_M \quad \tau_N]^T$. Here $\bar{\eta}_1$ and $\bar{\eta}_2$ denote the position vector and orientation vector with coordinate in the earth-fixed frame, respectively, $\bar{\upsilon}_1$ and $\bar{\upsilon}_2$ are used to describe the linear velocity vector and angular velocity vector with coordinates in the body-fixed frame, respectively, $\bar{\tau}_1$ and $\bar{\tau}_2$ denote the force vector and moment vector acting on the marine vehicle in the body-fixed frame, respectively. Based on the derivation of [11], the kinematic equations relating the body-fixed reference frame to the earth-fixed reference frame and the 6 degree of freedom non-linear dynamic equations of motion can be expressed as

$$\begin{bmatrix} \dot{\bar{\eta}}_1 \\ \dot{\bar{\eta}}_2 \end{bmatrix} = \begin{bmatrix} \bar{J}_1(\bar{\eta}_2) & 0_{3\times3} \\ 0_{3\times3} & \bar{J}_2(\bar{\eta}_2) \end{bmatrix} \begin{bmatrix} \bar{\upsilon}_1 \\ \bar{\upsilon}_2 \end{bmatrix}, \tag{90}$$

$$\dot{\bar{\upsilon}} = \left[-\bar{M}^{-1}\bar{C}(\bar{\upsilon})\right]\bar{\upsilon} + \left[-\bar{M}^{-1}\bar{D}(\bar{\upsilon})\right]\bar{\upsilon} + \left[-\bar{M}^{-1}g(\bar{\eta})\right] + \left[\bar{M}^{-1}\bar{\tau}\right], \tag{91}$$

$$\bar{J}_1(\bar{\eta}_2) \equiv \begin{bmatrix} \cos\varphi\cos\theta & -\sin\varphi\cos\phi + \cos\varphi\sin\theta\sin\phi & \sin\varphi\sin\phi + \cos\varphi\cos\phi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi\cos\phi + \sin\phi\sin\theta\sin\varphi & -\cos\varphi\sin\phi + \sin\theta\sin\varphi\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}, \tag{92}$$

$$\bar{J}_2(\bar{\eta}_2) \equiv \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix}, \tag{93}$$

where $\bar{M}$ and $\bar{C}(\bar{\upsilon})$ denote the inertia matrix including added mass and the matrix of Coriolis and centripetal terms including added mass, respectively, $\bar{D}(\bar{\upsilon})$ denotes the damping matrix, $\bar{\tau}$ denotes the vector of control inputs and $g(\bar{\eta})$ is used to describe the vector of gravitational forces and moments. A dynamically positioned marine vehicle can be supplied with anchors if thruster-assisted mooring is to be investigated. The mooring forces are described as spring forces, $g(\bar{\eta}) = \bar{K}(\bar{\eta} - \bar{\eta}_0)$, where $\bar{\eta}_0$ is the equilibrium point of the mooring system. For the sake of simplicity it is assumed that $\bar{\eta}_0 = 0$. During station-keeping and low-speed application like dynamic positioning of marine vehicles, the linear velocities u, v and the angular velocity r are all small which contributes that a further simplification can be to neglect the term $-\bar{M}^{-1}\bar{C}(\bar{\upsilon})\bar{\upsilon}$. Hence the equations of motion in surge, sway and yaw can be written as follows according to $\eta \equiv [x \quad y \quad \varphi]^T$, $\upsilon \equiv [u \quad v \quad r]^T$, $\tau \equiv [\tau_x \quad \tau_y \quad \tau_N]^T$:

$$\dot{\eta} = J(\eta)\upsilon, \tag{94}$$

$$\dot{\upsilon} = \left[-M^{-1}D(\upsilon)\right]\upsilon + \left[-M^{-1}K\right]\eta + \left[M^{-1}\right]\tau, \tag{95}$$

where

$$J(\eta) \equiv \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{96}$$

is the rotation matrix in yaw, M is the inertia matrix including hydrodynamic added inertia, D is the damping matrix and τ are the control forces and moment provided by the thruster system. Starboard-port symmetry of marine vehicles implies that the matrices M and D are constructed as the following structure:

$$M \equiv \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{bmatrix}, \tag{97}$$

$$D \equiv \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{bmatrix}, \tag{98}$$

that is, there is no coupling item between the surge and sway-yaw subsystems. Hence, the anchor forces and moment are related by the diagonal matrix.

$$K \equiv \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix}. \tag{99}$$

We consider a tanker with the numerical system matrices as follows: (Bis-scaled values [11])

$$M \equiv \begin{bmatrix} 1.0852 & 0 & 0 \\ 0 & 2.0575 & -0.4087 \\ 0 & -0.4087 & 0.2153 \end{bmatrix}, \tag{100}$$

$$D \equiv \begin{bmatrix} 0.0865 & 0 & 0 \\ 0 & 0.0762 & 0.1510 \\ 0 & 0.0151 & 0.0031 \end{bmatrix}, \tag{101}$$

$$K \equiv \begin{bmatrix} 0.0389 & 0 & 0 \\ 0 & 0.0266 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{102}$$

After arranging Equations (94)-(96) in the state-space formulation, it is easy to obtain the following state equations:

$$\dot{x}_1 = x_4 (\cos x_3) - x_5 (\sin x_3), \tag{103}$$

$$\dot{x}_2 = x_4 (\sin x_3) - x_5 (\cos x_3), \tag{104}$$

$$\dot{x}_3 = x_6, \tag{105}$$

$$\dot{x}_4 = -0.0358x_1 - 0.0797x_4 + 0.9215u_1, \tag{106}$$

$$\dot{x}_5 = -0.0208x_2 - 0.0818x_5 - 0.1224x_6 + 0.7802u_2 + 1.4811u_3, \tag{107}$$

$$\dot{x}_6 = -0.0394x_2 - 0.2254x_5 - 0.2468x_6 + 1.4811u_2 + 7.4562u_3, \tag{108}$$

where

$$X \equiv \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T \equiv \begin{bmatrix} x & y & \varphi & u & v & r \end{bmatrix}^T, \tag{109}$$

$$u \equiv \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T \equiv \begin{bmatrix} \tau_x & \tau_y & \tau_z \end{bmatrix}^T. \tag{110}$$

It is assumed that only the position and yaw angle measurements are available, that is, the output equations are given by

$$y_1 = x_1 \tag{111}$$

$$y_2 = x_2 \tag{112}$$

$$y_3 = x_3 \tag{113}$$

Now we will show how to explicitly construct a controller that tracks the desired signals $y_d^1 = y_d^2 = y_d^3 = 0$ and attenuates the disturbance's effect on the output terminal to an arbitrary degree of accuracy. Let's arbitrarily choose $\alpha_1^1 = \alpha_2^1 = 100$, $\alpha_1^2 = \alpha_2^2 = 100$, $\alpha_1^3 = \alpha_2^3 = 10$, $A_c^1 = A_c^2 = A_c^3 = \begin{bmatrix} 0 & 1 \\ 100 & 100 \end{bmatrix}$. With the aid of Matlab, the solutions of Lyapunov equations are given by

$$P^1 = P^2 = P^3 = \begin{bmatrix} 1.005 & 0.005 \\ 0.005 & 0.005 \end{bmatrix} \tag{114}$$

Hence, $\lambda^*_{\min} = 0.05$ and $\lambda^*_{\max} = 1.005$. From Equation (89), we obtain the desired tracking controllers

$$u = A^{-1}\left(-\vec{b} + \vec{v}\right) u_s(t) + \begin{bmatrix} u_{fuzzy}u_s(t-t_1) \\ u_{fuzzy}u_s(t-t_2) \\ u_{fuzzy}u_s(t-t_3) \end{bmatrix} \tag{115}$$

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{116}$$

$$u \equiv \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T, \quad b \equiv \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T, \quad v \equiv \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T \tag{117}$$

where

$$a_{11} = 0.9215\cos x_3 \tag{118}$$

$$a_{12} = -0.7802\sin x_3 \tag{119}$$

$$a_{13} = -1.4811\sin x_3 \tag{120}$$

$$a_{21} = 0.9215\sin x_3 \tag{121}$$

$$a_{22} = -0.7802\cos x_3 \tag{122}$$

$$a_{23} = -1.4811\cos x_3 \tag{123}$$

$$a_{31} = 0 \tag{124}$$

$$a_{32} = 1.4811 \tag{125}$$

$$a_{33} = 7.4562 \tag{126}$$

$$\begin{aligned} b_1 = {} & (x_6)(-x_4\sin x_3 - x_5\cos x_3) + (-0.0358x_1 - 0.0797x_4)(\cos x_3) \\ & + (-0.0208x_2 - 0.0818x_5 - 0.1224x_6)(-\sin x_3) \end{aligned} \tag{127}$$

$$\begin{aligned} b_2 = {} & (x_6)(x_4\cos x_3 + x_5\sin x_3) + (-0.0358x_1 - 0.0797x_4)(\sin x_3) \\ & + (-0.0208x_2 - 0.0818x_5 - 0.1224x_6)(-\cos x_3) \end{aligned} \tag{128}$$

$$b_3 = (-0.0394x_2 - 0.2254x_5 - 0.2468x_6) \times 1 \tag{129}$$

$$v_1 = -100(\varepsilon)^{-2}x_1 - 100(\varepsilon)^{-1}(x_4(\cos x_3) - x_5(\sin x_3)) \tag{130}$$

$$v_2 = -100(\varepsilon)^{-2}x_2 - 100(\varepsilon)^{-1}(x_4(\sin x_3) - x_5(\cos x_3)) \tag{131}$$

$$v_3 = -100(\varepsilon)^{-2}x_3 - 100(\varepsilon)^{-1}(x_6) \tag{132}$$

It can be verified that the relative conditions of Theorem 2.1 are satisfied with $\varepsilon = 0.1$, $N = 9.95$, $N_2 = 0.79$ and $k = 100\sqrt{\varepsilon}$. Hence, the tracking controllers will steer the output tracking errors of the closed-loop system, starting from any initial value, to be asymptotically attenuated to zero by virtue of Theorem 2.1.

In the simulations, the ship is first controlled by the dynamic positioning control law (115) using three control inputs and initially steered to the point

$$\begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) & x_5(0) & x_6(0) \end{bmatrix}^T \equiv \begin{bmatrix} -10 & -10 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

The desired equilibrium point is

$$\begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) & x_5(0) & x_6(0) \end{bmatrix}^T \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

The simulated results are shown in Figures 4-9. We see that the ship eventually converges to a neighborhood of the desired equilibrium point. The dynamic positioning time based on our proposed novel non-linear fuzzy feedback linearization control is approximately equal to be 0.6. On the contrary, the dynamic positioning time for the same ship control system using the traditional fuzzy-model approach [5] is approximately equal to be 10. The simulation results show that our proposed control in this study is a marvelous approach for the control problem of non-linear dynamic positioning problem of ship.

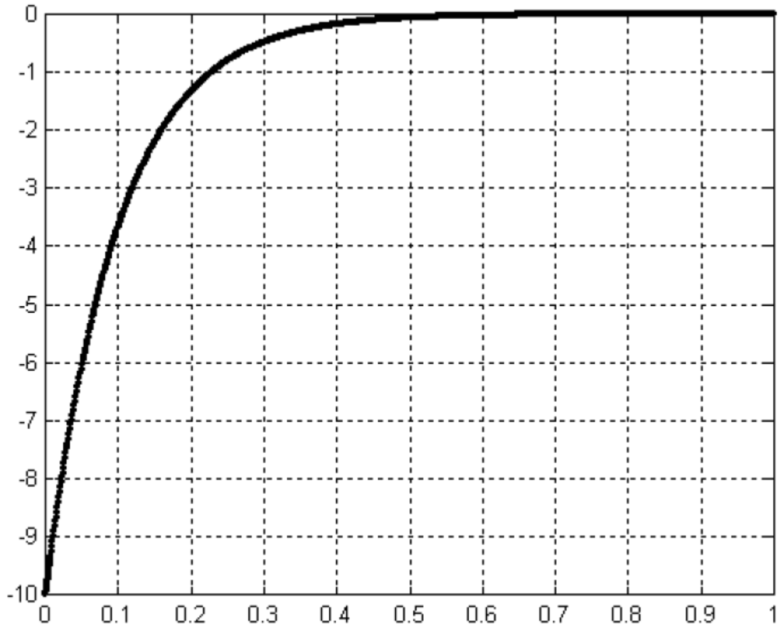


Figure 4. Measured x-position x_1

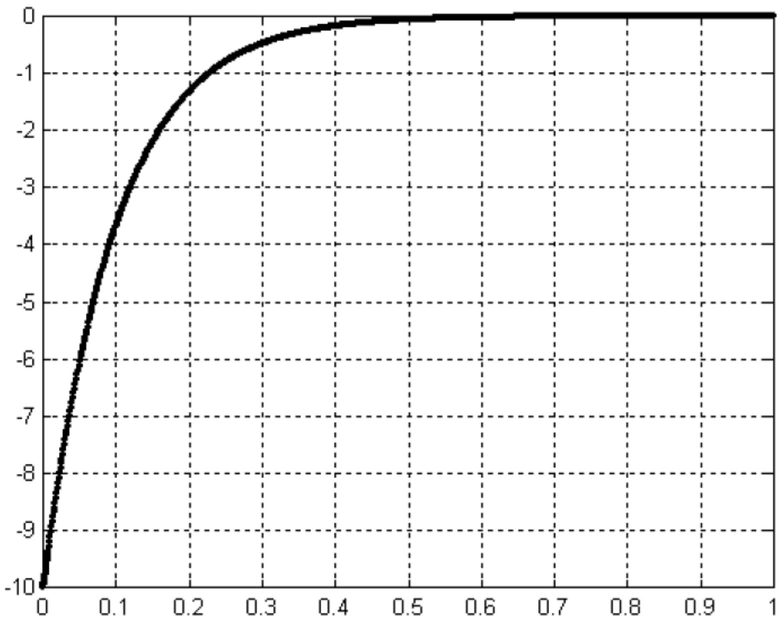


Figure 5. Measured y-position x_2

4. **Comparative Example to Existing Approach.** [30] exploited the fact that the almost disturbance decoupling problem could not be solvable for the following system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \tan^{-1}(x_2) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta(t) \tag{133}$$

$$y(t) = x_1(t) := h(X(t)) \tag{134}$$

where u, y denoted the input and output respectively, $\theta(t) := 2\sin t\,[u_s(t-1.1) - u_s(t-1.31)]$ and $u_s(t)$ denotes the unit step function. Assume that the desired tracking signal is equal to be $\sin t$. The almost disturbance decoupling problem can be easily solved via the proposed approach in this paper. Following the same procedures shown in

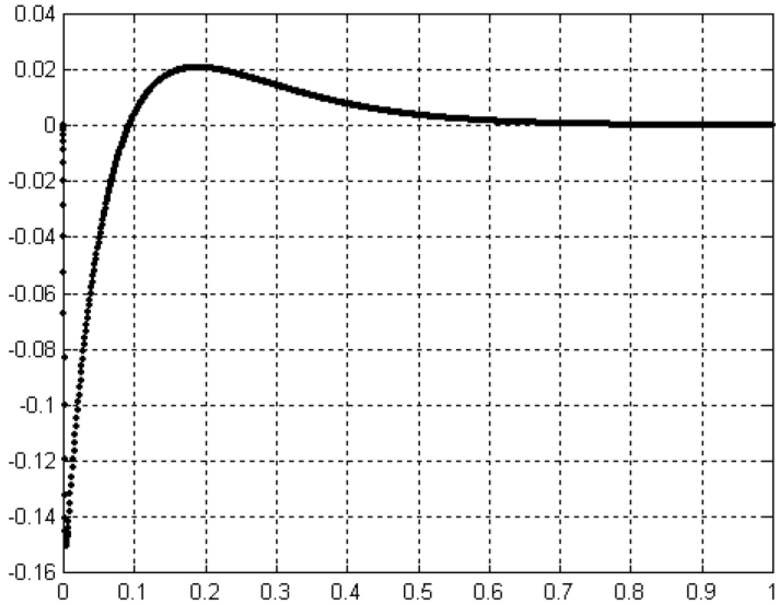


Figure 6. Measured yaw angle x_3

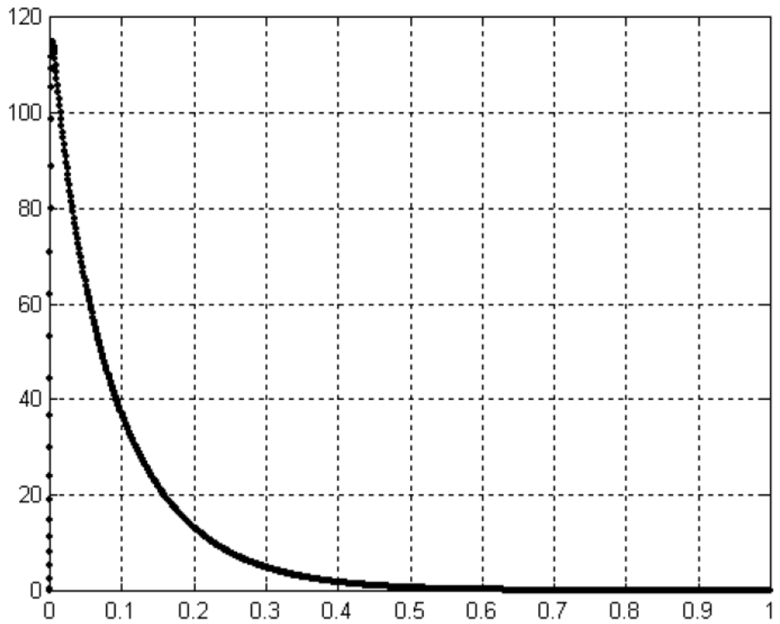


Figure 7. Surge velocity x_4

the demonstrated example, we can solve the tracking problem with almost disturbance decoupling problem by the controller u defined as

$$u = \left(1 + x_2^2\right)\left[-\sin t - 42(x_1 - \sin t) - 42(\tan^{-1} x_2 - \cos t)\right] u_s(t) + u_{fuzzy} u_s(t - t_1) \quad (135)$$

The tracking error dynamics for (133) is depicted in Figure 10.

It is worth noting that the sufficient conditions given in [30] (in particular the structural conditions on non-linearities multiplying disturbances) are not necessary in this study where a non-linear state feedback control is explicitly designed which solves the almost disturbance decoupling problem. For instance, the almost disturbance decoupling problem is solvable for the system (133) by our proposed approach, while the sufficient conditions

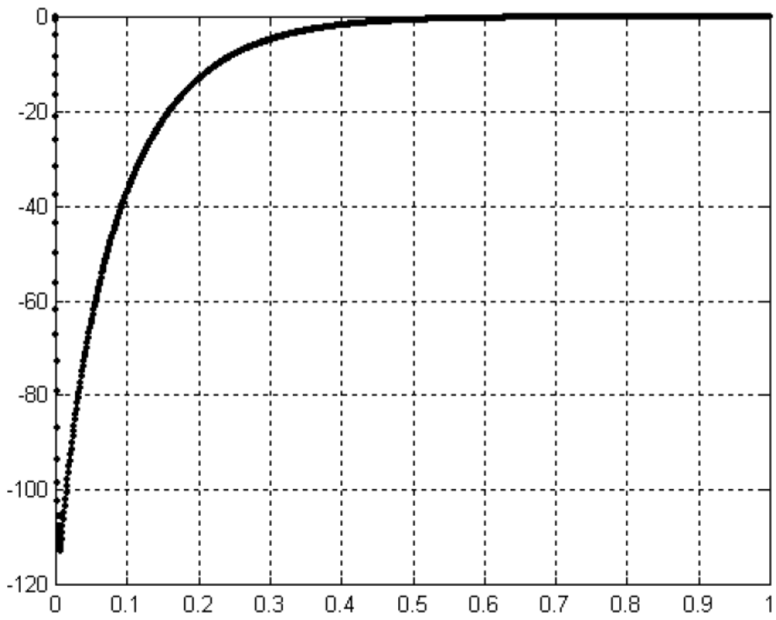


Figure 8. Sway velocity x_5

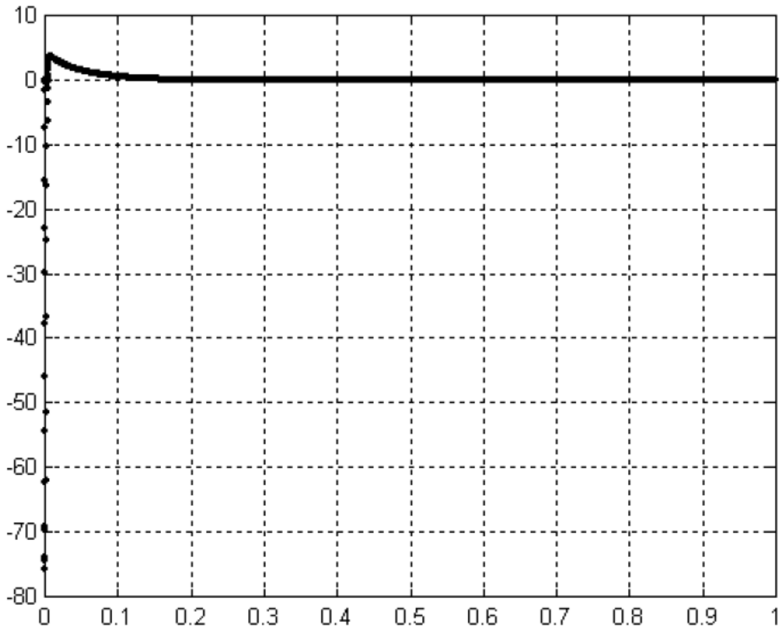


Figure 9. Yaw velocity x_6

given in [30] fail when applied to the system (133). The design techniques in this study are also entirely different than those in [30] since the singular perturbation tools are not used.

5. **Conclusions.** A novel fuzzy feedback control to globally solve the tracking problem with almost disturbance decoupling for dynamic positioning marine vehicle has been proposed. A discussion and a practical application of feedback linearization of non-linear control systems using a parameterized coordinate transformation have been presented. One comparative example is proposed to show the significant contribution of this paper with respect to existing approach. A practical simulation example has been used to

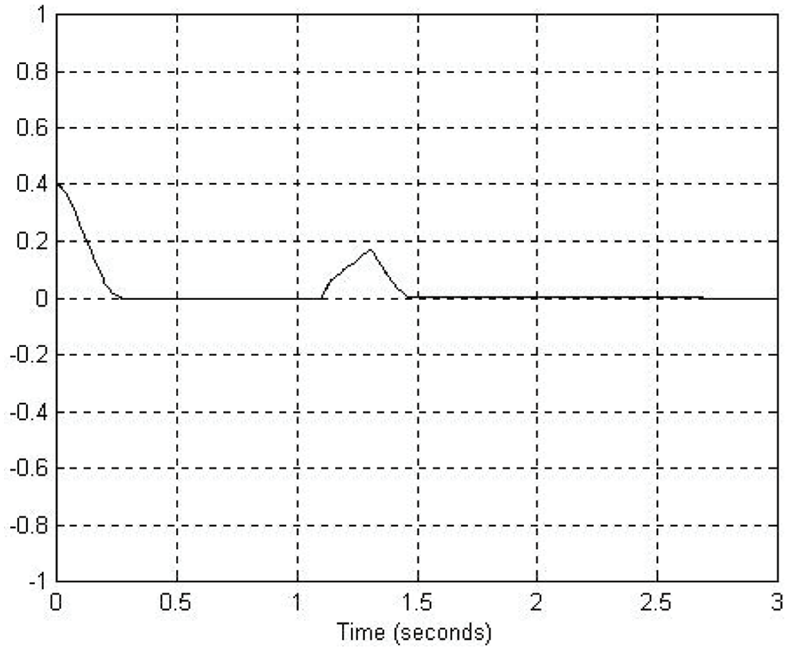


FIGURE 10. The tracking error for (133)

demonstrate the applicability of the proposed fuzzy feedback linearization approach and the composite Lyapunov approach. Simulation results have been presented to show that the proposed methodology can be successfully applied to feedback linearization problem and is able to achieve the desired tracking and almost disturbance decoupling performances of the controlled system. Moreover, it is easy to see that the dynamic positioning time based on our proposed novel non-linear fuzzy feedback linearization control is shorter than the traditional fuzzy-model approach.

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