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FUZZY RISK ANALYSIS BASED ON LINGUISTIC AGGREGATION OPERATORS

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Abstract. *The aim of fuzzy risk analysis is to evaluate the probability of failure of every component consisting of many sub-components and the probability of failure is the combination of estimations of severity of loss and probability of failure of sub-components which are vaguely known. In this paper, we present a new method for fuzzy risk analysis with linguistic evaluating values. Firstly, we propose the unbalanced linguistic weighted geometric operator, which can be used to deal with aggregation of unbalanced linguistic values with numerical weights. Then, we generalize the operator to deal with aggregation of unbalanced linguistic values with linguistic weights, and discuss some properties of the operator. Finally, we apply the operator to aggregate linguistic evaluating values of fuzzy risk analysis. A comparison is given between the new method in this paper and the one based on interval-valued fuzzy numbers in the same linguistic evaluating values. The advantages of our method are that the evaluating result is linguistic value which is no need of approximation processing and easier to communicate to decision- and policy-makers, no loss of information and no complex computation due to the linguistic aggregation operator and the 2-tuple fuzzy linguistic representations.*

Keywords: Fuzzy risk analysis, Aggregation operator, Linguistic aggregation operator, 2-tuple fuzzy linguistic representation

1. **Introduction.** From the system point of view, risk analysis means to combine the individual responses to one statement on the system's performance for the purpose of decision-making. It deals with the occurrence of individual failure events (e.g., changes in components or in relations among components as distinct points in space and time) and their possible consequences on the system level [1]. In the procedure of risk analysis, the estimation of the likelihood (e.g., frequencies) and the consequences of hazard occurrence are included. The estimation of the likelihood of hazard occurrence depends greatly on the reliability of the system's components, the interaction of the components taking the system as a whole and human-system interactions. Risk evaluation needs a systematic research of accidental scenarios, including failure rates for the component (e.g., safety barriers) as well as for operator behavior (human factor) within an evolving environment [3]. In practice, information of risk analysis stems from historical data of complex systems

or knowledge of decision makers, experts and regulators, accordingly, there exist qualitative and quantitative methods in risk analysis to identify the risk drivers, assess their likelihood of occurrence and their potential consequences, and find ways to monitor and then mitigate the risks [2]. The differences between qualitative and quantitative methods to risk analysis are focused on the evaluation of the likelihood of accident sequences. In the quantitative methods, probability and Bayesian networks are the main tools for risk analysis due to their ability to model probabilistic data with dependencies between events, compute the distribution probabilities in a set of variables according to the observation of some variables and the prior knowledge of the others, and quantify low probability events [3, 22, 23]. The qualitative methods to risk analysis are largely based on expert judgement under the assumed boundary conditions, in which statements and implications are considered in performing a risk analysis, and in which the result is expressed by linguistic value of assessment, e.g., safe or unsafe. Due to uncertainty and complexity, information about the probabilities of various risk items is vaguely known, which makes researchers consider risk analysis based on fuzzy logic, i.e., fuzzy risk analysis. In recent years, many methods to fuzzy risk analysis based on fuzzy numbers have been discussed, e.g., based on fuzzy arithmetic operations [4], the similarity measure of fuzzy numbers [6, 7], interval fuzzy numbers, the alpha level sets, the ranking of fuzzy numbers [5, 8, 9, 10, 12, 13, 14] and fuzzy partition tree [21]. In all of these fuzzy risk analysis, many shapes (membership functions) associated to fuzzy number are used to represent fuzziness of the evaluating value of the risk of each subcomponent, e.g., triangular fuzzy numbers, trapezoidal fuzzy numbers and interval fuzzy numbers, then fuzzy arithmetic operations, the similarity measure of fuzzy numbers and the ranking of fuzzy numbers could help us to evaluate the result of risk analysis system. To the best of our knowledge, there are following three drawbacks when we use membership functions associated to fuzzy numbers to represent fuzziness in risk analysis: 1) Fuzzy arithmetic operations, the similarity measure of fuzzy numbers and the ranking of fuzzy numbers depend on membership functions associated to fuzzy numbers. For example, triangular fuzzy numbers, trapezoidal fuzzy numbers and interval fuzzy numbers, different shapes of fuzzy numbers correspond to different results. In practice, which one is the best in risk analysis is a problem; 2) Fuzzy number is a fuzzy concept. For example, "about 3" is a fuzzy number, and its semantics (membership function) is triangular or trapezoidal. In risk analysis, linguistic values or fuzzy numbers rather than membership functions are used to represent fuzziness of the evaluating value, and the result of risk analysis does not depend on their membership functions; 3) In many cases, processing membership functions associated to linguistic values increase the computational complexity, and membership functions are no longer kept the same form after fuzzy arithmetic operations. As the results do not exactly match any of the initial linguistic values, an approximation process must be developed to express the results in the initial expression domain which induces the consequent loss of information and hence the lack of precision.

To overcome the above mentioned drawbacks, risk analysis based on Computing with Words (CWW) is an alternative method, which was proposed by Zadeh. It is a methodology for reasoning, computing and decision-making with information described in natural language [15, 16, 30], and a system of computation which adds to traditional systems of computation, including two important capabilities [31]: a) the capability to precisiate the meaning of words and propositions drawn from natural language; b) the capability to reason and compute with precisiated words and propositions. In CWW, linguistic values rather than membership functions or numerical values play an important or key role; i.e., linguistic values are computational variables, which makes the results of risk analysis no loss of information and easier to communicate to decision- and policy-makers. Informally, risk analysis based on CWW belongs to qualitative methods. In [17], we use fuzzy number indexes of linguistic evaluating values to deal with fuzzy risk analysis problems, in which the linguistic evaluating values are represented by their fuzzy number indexes, and in which the final evaluation value is obtained by linguistic information fusion based on fuzzy number indexes of linguistic values [19, 20]. In [18], we considered unbalanced linguistic information in fuzzy risk analysis, and proposed a new method for fuzzy risk analysis with unbalanced linguistic evaluating values.

In this paper, we provide an alternative method to risk analysis based on CWW, i.e., fuzzy risk analysis based on an unbalanced linguistic aggregation operator. The organization of this paper is as following: In Section 2, we make brief reviews of fuzzy risk analysis, the 2-tuple fuzzy linguistic representation model and unbalanced linguistic term sets. We propose the unbalanced linguistic weighted geometric operator in Section 3, and discuss some interesting properties of the operator. In Section 4, we use an example to illustrate our method to fuzzy risk analysis. Furthermore, we compare our method with the one based on interval-valued fuzzy numbers in the same linguistic evaluating values. The paper is concluded in Section 5.

2. **Preliminaries.**

In this section, we will review methods for fuzzy risk analysis, the 2-tuple fuzzy linguistic representation model and unbalanced linguistic term sets.

2.1. **Fuzzy risk analysis.**

The aim of fuzzy risk analysis is to evaluate the probability of failure of every component consisted of many sub-components, in which, estimations of severity of loss and probability of failure of sub-components are included. Formally, risk analysis can be described as follows: assume that there are r components P_1, P_2, $\cdots$ and P_r made by r manufactories M_1, M_2, $\cdots$ and M_r, respectively. Each component P_i consists of s sub-components p_{i1}, p_{i2}, $\cdots$ and p_{is}. Each sub-component p_{ij} $(1 \leq i \leq r$, $1 \leq j \leq s)$ is evaluated by severity of loss L_{ij} and probability of failure F_{ij}, each probability of failure F_i of component P_i are calculated by severity of loss L_{ij} and probability of failure F_{ij} $(1 \leq j \leq s)$, then the larger the value of F_i, the higher the probability of failure of component P_i made by manufactory M_i $(1 \leq i \leq r)$. The structure of fuzzy risk analysis can be shown in Figure 1 [9]. According to the evaluation of the likelihood of accident

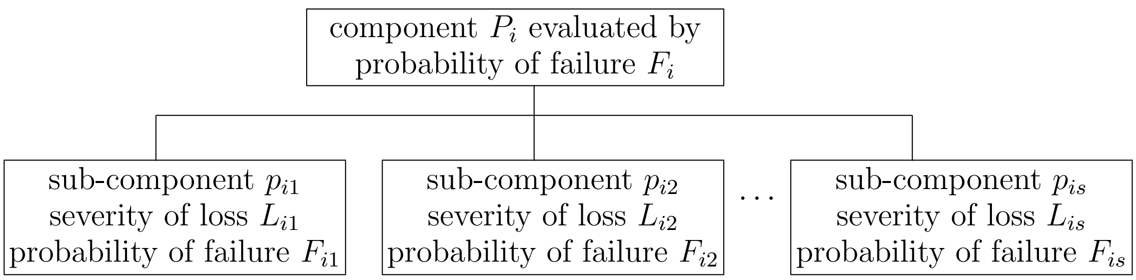

FIGURE 1. The structure of fuzzy risk analysis

sequences, qualitative and quantitative approaches to risk analysis have been discussed. In the quantitative approaches to risk analysis, every severity of loss L_{ij} and probability of failure F_{ij} of sub-components p_{ij} $(1 \leq i \leq r$, $1 \leq j \leq s)$ are described by probabilities, then Bayesian networks are used for obtaining final evaluation results. As a general modeling approach, Bayesian networks offers a compact presentation of the interactions in a stochastic system by visualizing system variables and their dependencies. Formally, a Bayesian network consists of two main parts: a qualitative part and a quantitative part.

The qualitative part is a directed acyclic graph mirrored the nodes of the system variables, the conditional dependence between variables are represented by the edges of the graph. Conditional probability functions are included in the quantitative part according to the relations between the nodes of the graph. According to Figure 1, the probability of failure F_i of component P_i can be calculated by the conditional probability distribution $P(F_i|Pa(L_{i1}, F_{i1}), Pa(L_{i2}, F_{i2}), \cdots, Pa(L_{is}, F_{is}))$, where $Pa(L_{ij}, F_{ij})$ $(1 \leq j \leq s)$ is joint distribution function of sub-component p_{ij}. Due to the complexity of systems and lack of data, risk assessments often rely on experts' opinion. The experts' opinion has its inherent vagueness, which is expressed by linguistic evaluation values, such as *very low, fairly low, pretty low, medium, almost high, more or less high, very high and absolutely high, etc.* According to the concept of linguistic variable proposed by Zadeh [29], membership functions can be used to represent linguistic evaluating values and handle risk analysis, generally, risk analysis based on fuzzy numbers is presented as follows: 1) Aggregate all evaluating values $\{(L_{ij}, F_{ij})|1 \leq j \leq s\}$ of component P_i by fuzzy number aggregation operators, e.g., fuzzy weighted mean method and the generalized fuzzy number arithmetic operator, where L_{i1} and F_{i1} are represented by corresponding fuzzy numbers (membership functions), i.e., $F_i = \frac{\sum_{j=1}^{s} F_{ij} \otimes L_{ij}}{\sum_{j=1}^{s} L_{ij}}$, in which, $\otimes$ is as generalized fuzzy numbers multiplication, F_i is a fuzzy number; 2) Rank fuzzy numbers $\{F_i|1 \leq i \leq r\}$ by the ranking values of the fuzzy numbers. The larger the value of P_i is, the higher the risk of the manufactory M_i is; 3) Approximate fuzzy number $\max\{F_i|1 \leq i \leq r\}$ to linguistic values, the final evaluation result is a linguistic evaluation value corresponding to the fuzzy number.

In [17, 18], alternative methods have been presented, i.e., CWW is used to handle risk analysis, compared the methods with risk analysis based on fuzzy numbers, evaluating values are linguistic evaluating values instead of fuzzy numbers (their membership functions), the final evaluations are directly represented by linguistic evaluating values, without approximation from a fuzzy number to a linguistic evaluation value. Generally, risk analysis based on CWW is presented as follows: 1) Aggregate all linguistic evaluating values $\{(L_{ij}, F_{ij})|1 \leq j \leq s\}$ of component P_i by linguistic aggregation operators, i.e., $F_i = f(\{(L_{ij}, F_{ij})|1 \leq j \leq s\})$, in which, f is a linguistic aggregation operator; 2) Rank linguistic evaluating values $\{F_i|1 \leq i \leq r\}$ by the ranking method of linguistic values. The larger the value of P_i is, the higher the risk of the manufactory M_i is.

2.2. **The 2-tuple fuzzy linguistic representation model.** The 2-tuple linguistic representation model be introduced by Herrera [24]. Let $S = \{s_0, \cdots, s_g\}$ be the initial finite linguistic value set. Formally, the 2-tuple linguistic representation model is formed by (s_i, α), in which, $s_i \in S$ $(i \in \{0, 1, \cdots, g\})$ and $\alpha \in [-0.5, 0.5)$, i.e., linguistic information is encoded in the space $S \times [-0.5, 0.5)$. Based on the representation (s_i, α), we can easily obtain the following symbolic translation of linguistic values from $\beta \in [0, g]$ to $S \times [-0.5, 0.5)$, i.e., $\Delta : [0, g] \to S \times [-0.5, 0.5)$, $\beta \longmapsto (s_i, \alpha)$, in which, $i = round(\beta)$ ($round(\cdot)$ is the usual round operation), $\alpha = \beta - i \in [-0.5, 0.5)$. Intuitively, $\Delta(\beta) = (s_i, \alpha)$ expresses that s_i is the closest linguistic value to β, and α is the value of the symbolic translation. Additionally, there is a Δ^{-1} function such that from a 2-tuple it returns its equivalent numerical value $\beta \in [0, g]$, i.e., $\Delta^{-1} : S \times [-0.5, 0.5) \longrightarrow [0, g]$, $\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta$.

In fact, it defines a set of transformation functions between linguistic values and 2-tuples linguistic representations as well as numeric values and 2-tuples linguistic representations. Evidently, an order relation on $S \times [-0.5, 0.5)$ can be deduced by Δ^{-1}, i.e., for any $(s_i, \alpha_i), (s_j, \alpha_j) \in S \times [-0.5, 0.5)$, $(s_i, \alpha_i) \leq (s_j, \alpha_j)$ if and only if $\Delta^{-1}(s_i, \alpha_i) \leq \Delta^{-1}(s_j, \alpha_j)$.

2.3. **Representation of unbalanced linguistic terms.** Unbalanced linguistic terms proposed in [25] are used to deal with scales for assessing preferences where the experts need to assess a number of terms in a side of reference domain higher than in the other one. Generally, an unbalanced linguistic term set S has a minimum label, a maximum label, and a central label, and the remaining labels are non-uniformly and non-symmetrically distributed around the central one on both left and right lateral sets, i.e., we can represent S as the form $S = S_l \cup S_c \cup S_r$, in which, S_l contains all left lateral labels but the central label, S_c just contains the central label, S_r contains all right lateral labels higher than the central label.

Example 2.1. [26] $S = \{$*none* (N), *low* (L), *medium* (M), *almost high* (AH), *high* (H), *quite high* (QH), *very high* (VH), *almost total* (AT), *total* $(T)\}$ *is an unbalanced linguistic term set, in which,* $S_l = \{N, L\}$, $S_c = \{M\}$ *and* $S_r = \{AH, H, QH, VH, AT, T\}$.

FIGURE 2. Scale with more values on the right of the midterm

To obtain 2-tuple fuzzy linguistic representations of unbalanced linguistic terms, we need the concept of linguistic hierarchies $LH = \bigcup_t l(t, n(t))$ [28], which takes into account a set of levels where each level is a linguistic term set with different granularity from the remaining levels of the hierarchy, where $l(t, n(t))$ is a linguistic hierarchy with t being a number that indicates the level of the hierarchy and $n(t)$ the granularity of the linguistic term set of t. The linguistic term set $S^{n(t+1)}$ of the level $t + 1$ is obtained from its predecessor $S^{n(t)}$ as $l(t, n(t)) \to l(t + 1, 2 \times n(t) - 1)$. Transformation function of LH is defined as follows [27]: for any t and t', $TF^t_{t'} : l(t, n(t)) \longrightarrow l(t', n(t'))$ such that

$$TF^t_{t'}(s_i^{n(t)}, \alpha^{n(t)}) = \triangle_{t'}\left(\frac{\triangle_t^{-1}(s_i^{n(t)}, \alpha^{n(t)}) \times (n(t') - 1)}{n(t) - 1}\right). \tag{1}$$

Example 2.2. *Let a linguistic hierarchies* LH *be* $LH = l(1, 3) \cup l(2, 5) \cup l(3, 9) \cup l(4, 17) = \{s_0^3, s_1^3, s_2^3\} \cup \{s_0^5, s_1^5, \cdots, s_4^5\} \cup \{s_0^9, s_1^9, \cdots, s_8^9\} \cup \{s_0^{17}, s_1^{17}, \cdots, s_{16}^{17}\}$. $(s_5^9, 0.3)$ *is a 2-tuple fuzzy linguistic representation of level 3, it's 2-tuple fuzzy linguistic representation in level 2 is* $TF_2^3(s_5^9, 0.3) = \triangle_2(\frac{\triangle_3^{-1}(s_5^9, 0.3) \times (5-1)}{9-1}) = \triangle_2(\frac{5.3 \times 4}{8}) = \triangle_2(2.65) = (s_3^5, -0.35)$.

By using linguistic hierarchies $LH = \bigcup_t l(t, n(t))$, we can obtain the 2-tuple fuzzy linguistic representation of each term of unbalanced linguistic term set in LH by using the algorithm presented in [25].

Example 2.3. *Continuing Example 2.1. For unbalanced linguistic term set* $S = \{N, L, M, AH, H, QH, VH, AT, T\}$, *1) due to* $n(2) = 5$ *and* $\frac{n(2)-1}{2} = |S_l| = |\{N, L\}| = 2$, *the representation of* S_l *is obtained from level 2 of* LH *as follows:* $\{L \leftarrow s_1^5, N \leftarrow s_0^5\}$; *2) due to* $n(3) = 9$, $n(4) = 17$ *and* $\frac{n(3)-1}{2} = 4 < |S_r| = |\{AH, H, QH, VH, AT, T\}| = 6 < \frac{n(4)-1}{2} = 8$, *we use level 3 and level 4 to represent* $S_r = \{AH, H, QH, VH, AT, T\}$, *according to* lab_3 *and* lab_4 *of* [25], $\{AH, H\}$ *and* $\{QH, VH, AT, T\}$ *are represented in level 3 and level 4, respectively,* $\{AH \leftarrow s_5^9, H \leftarrow s_6^9\}$, $\{QH \leftarrow s_{13}^{17}, VH \leftarrow s_{14}^{17}, AT \leftarrow s_{15}^{17}, T \leftarrow s_{16}^{17}\}$; *3) according to density and bridging representation gaps* [25], *the upside and the downside of the central label* M *are represented in level 2 and 3 of* LH *by means of* $\overline{s_2^5}$ *and* $\underline{s_4^9}$, *respectively. The upside and the downside of the label* H *are represented in level 3 and 4 of* LH *by means of* $\overline{s_6^9}$ *and* $\underline{s_{12}^{17}}$, *respectively; 4) the final 2-tuple fuzzy linguistic representations of* S *in* LH *are* $S_l : \{N \leftarrow s_0^5, L \leftarrow s_1^5\}$, $S_c : \{M \leftarrow \overline{s_2^5} \cup \underline{s_4^9}\}$, $S_r : \{AH \leftarrow s_5^9, H \leftarrow \overline{s_6^9} \cup \underline{s_{12}^{17}}, QH \leftarrow s_{13}^{17}, VH \leftarrow s_{14}^{17}, AT \leftarrow s_{15}^{17}, T \leftarrow s_{16}^{17}\}$.

Let S be an unbalanced linguistic term set. Formally, for any 2-tuple fuzzy linguistic representation (s_i, α_i) ($s_i \in S$ and $\alpha_i \in [-0.5, 0.5)$), (s_i, α) can be converted by the following unbalanced linguistic transformation functions in LH and vice versa, i.e., $\mathcal{LH}: S \times [-0.5, 0.5) \longrightarrow LH \times [-0.5, 0.5)$, $(s_i, \alpha_i) \longmapsto (s_{I(i)}^{G(i)}, \alpha_i)$ such that $s_{I(i)}^{G(i)} \in LH$; $\mathcal{LH}^{-1}: LH \times [-0.5, 0.5) \longrightarrow S \times [-0.5, 0.5)$, $(s_k^{n(t)}, \alpha_k) \longmapsto (s_i, \lambda)$, in which, $s_k^{n(t)} \in S^{n(t)} \subset LH$, (s_i, λ) is decided by cases as follows: Case 1: If there exists $s_i \in S$ such that $s_i \leftarrow s_k^{n(t)}$, then we consider two possible situations depending on s_i represented in LH: 1) if s_i is represented with only one label in LH, e.g., $L \leftarrow s_1^5$ in Example 2.3, then $(s_i, \lambda) = (s_i, \alpha_k)$; 2) if s_i is represented with two labels in LH from levels, e.g., $H \leftarrow \overline{s_6^9 \cup s_{12}^{17}}$ in Example 2.3, then (s_i, λ) depends on the localization of $s_i \in S$, and $\lambda = \alpha_k$ or $\frac{\triangle_t^{-1}(s_k^{n(t)}, \alpha_k) \times (n(t+1)-1)}{n(t)-1} - round(\frac{\triangle_t^{-1}(s_k^{n(t)}, \alpha_k) \times (n(t+1)-1)}{n(t)-1})$. Case 2: If there exists no $s_i \in S$ such that $s_i \leftarrow s_k^{n(t)}$, then $\mathcal{LH}^{-1}((s_k^{n(t)}, \alpha_k)) = \mathcal{LH}^{-1}(TF_{t'}^t(s_k^{n(t)}, \alpha_k))$, in which, t' is a level of LH such that $TF_{t'}^t(s_k^{n(t)}, \alpha_k) = (s_{k'}^{n(t')}, \alpha_{k'})$ and $\exists s_j \in S, s_j \leftarrow s_{k'}^{n(t')}$.

Example 2.4. *Continuing Example 2.3. We have* $\mathcal{LH}(H, 0.3) = (s_6^9, 0.3)$, $\mathcal{LH}^{-1}(s_{13}^{17}, -0.2) = (QH, -0.2)$, $\mathcal{LH}^{-1}(s_{12}^{17}, -0.2) = \mathcal{LH}^{-1}(TF_3^4(s_{12}^{17}, -0.2)) = \mathcal{LH}^{-1}(s_6^9, -0.1) = (H, -0.1)$, $\mathcal{LH}^{-1}(s_{12}^{17}, 0.4) = \mathcal{LH}^{-1}(TF_3^4(s_{12}^{17}, 0.4)) = \mathcal{LH}^{-1}(s_6^9, 0.2) = (H, 0.4)$.

3. **The Unbalanced Linguistic Weighted Geometric Operator.** To aggregate unbalanced linguistic values, we propose the unbalanced linguistic weighted geometric operator in this section.

Definition 3.1. *An weighted geometric operator of dimension n is a mapping $g : (R^+)^n \to R^+$ that has associated with it a weighting vector $W = (w_1, w_2, \cdots, w_n)$, with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, such that $g(a_1, a_2, \cdots, a_n) = \prod_{j=1}^n a_i^{w_i}$.*

Definition 3.2. *Assume that the set of unbalanced linguistic values $V = \{s_i | i = 1, 2, \cdots, n\}$ be aggregated, in which, $s_i \in S = \{s_0, s_1, \cdots, s_m\}$ $(m \geq n)$ is an unbalanced linguistic value, a weighting vector is $W = (w_1, w_2, \cdots, w_n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. Then the unbalanced linguistic weighted geometric operator (the ULWG operator) is defined as $f_{ULWG}(\{(w_i, s_i) | i = 1, \cdots, n\}) = f_{ULWG}(\{(w_i, TF_{t_0}^{t_i}(\mathcal{LH}(s_i))) | i = 1, 2, \cdots, n\})$ $= \mathcal{LH}^{-1}(\triangle_{t_0}(\prod_{i=1}^n(\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(s_i))))^{w_i})) = \mathcal{LH}^{-1}(s_k^{n(t_0)}, \alpha_k)$, where, t_i is the level of $\mathcal{LH}(s_i)$ in LH, t_0 is a level of LH fixed by users, $s_k^{n(t_0)} \in S^{n(t_0)} \subset LH$ and $\alpha_k \in [-0.5, 0.5)$ such that*

$$k + \alpha_k = \prod_{i=1}^n (\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(s_i))))^{w_i}. \tag{2}$$

Remark 3.1. *In Definition 3.3, $TF_{t_0}^{t_i}(\mathcal{LH}(s_i))$ means that unbalanced linguistic values are represented at level t_0 of LH, t_0 is decided by users. In fact, by using $TF_{t_0}^{t_i}(\cdot)$ and $\mathcal{LH}(\cdot)$, S is converted in $S^{n(t_0)}$ of LH. By using $\mathcal{LH}^{-1}$ $(s_k^{n(t_0)}, \alpha_k)$, $f_{ULWG}(V)$ is converted to the 2-tuple fuzzy linguistic representation of unbalanced linguistic value.*

Example 3.1. *Let $S = \{N, L, M, AH, H, QH, VH, AT, T\}$ be a set of unbalanced linguistic values. Suppose that $\{(0.25, AH), (0.35, QH), (0.4, H)\}$ will be aggregated. Here, we select $t_0 = 3$ and density$_{S_R}$=extreme* [25]*, hence, $TF_3^{t_i}(\mathcal{LH}(AH)) = TF_3^3(s_5^9) = s_5^9$, $TF_3^{t_i}(\mathcal{LH}(QH)) = TF_3^4(s_{13}^{17}) = (s_6^9, 0.5)$, $TF_3^{t_i}(\mathcal{LH}(H)) = TF_3^3(s_6^9) = s_6^9$, according to Equation (2), we have $k + \alpha_k = 5^{0.25} \times 6.5^{0.35} \times 6^{0.4} \doteq 5.93$, $f_{ULWG}(\{(0.25, AH), (0.35, QH), (0.4, H)\}) \doteq \mathcal{LH}^{-1}(s_6^9, -0.07) = (H, -0.07)$.*

In many cases, weight w_i is a linguistic weight rather that number in $[0,1]$, e.g., in fuzzy risk analysis, every severity of loss L_{ij} of sub-component p_{ij} ($1 \le i \le r$, $1 \le j \le s$) acts as the weight in aggregation, i.e., in Equation (2), w_i is a linguistic value. We use the following method to obtain numbers in $[0,1]$ from unbalanced linguistic values: 1) For any unbalanced linguistic values $\{l_1, \cdots, l_k\}$ and $LH = \bigcup_t l(t, n(t))$, we have $\{TF_{t_0}^{t_1}(\mathcal{LH}(l_1))), \cdots, TF_{t_0}^{t_k}(\mathcal{LH}(l_k)))\}$, where t_i is the level of $\mathcal{LH}(l_i)$ ($i \in \{1, \cdots, k\}$) in LH, t_0 is a level of LH fixed by users; 2) The function $f : \{l_1, \cdots, l_k\} \to [0,1]$ is defined by

$$f(l_i) = \frac{\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(l_i)))}{\sum_{j=1}^{k} \triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(l_j)))}. \tag{3}$$

Evidently, for any $l_i \in \{l_1, \cdots, l_k\}$, $f(l_i) \in [0,1]$ and $\sum_{i=1}^{k} \frac{\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(l_i)))}{\sum_{j=1}^{k} \triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(l_j)))} = 1$.

Definition 3.3. *Assume that the set of unbalanced linguistic values $V = \{(l_i, s_i)|i = 1, \cdots, n\}$ is aggregated, and every l_i is a linguistic weight corresponding to s_i. Then the $ULWG$ operator with linguistic weights (the $ULWGLW$ operator) is defined as $f_{ULWGLW}(\{(l_i, s_i)|i = 1, \cdots, n\}) = f_{ULWG}(\{(f(l_i), TF_{t_0}^{t_i}(\mathcal{LH}(s_i)))|i=1,2,\cdots,n\}) = \mathcal{LH}^{-1}(\triangle_{t_0}(\prod_{i=1}^{n} (\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(s_i))))^{f(l_i)})) = \mathcal{LH}^{-1}(s_k^{n(t_0)}, \alpha_k)$, where, t_i is the level of $\mathcal{LH}(s_i)$ in LH, t_0 is a level of LH fixed by users, $s_k^{n(t_0)} \in S^{n(t_0)} \subset LH$ and $\alpha_k \in [-0.5, 0.5)$ such that*

$$k + \alpha_k = \prod_{i=1}^{n} (\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(s_i))))^{f(l_i)}. \tag{4}$$

in which, every $f(l_i)$ is decided by Equation (3).

Proposition 3.1. *Let unbalanced linguistic values $V = \{(w_i, s_i)|i = 1, \cdots, n\}$ be aggregated, where $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. If for any $i \in \{1, \cdots, n\}$, $w_i = 1$, then $f_{ULWG}(\{(w_1, s_1), \cdots, (w_n, s_n)\}) = s_i$.*

Proof: According to Equation (2), if for any $i \in \{1, \cdots, n\}$, $w_i = 1$, then for any $r \ne i$, $w_r = 0$, $k + \alpha_k = \prod_{r=1}^{n} (\triangle_{t_0}^{-1}(TF_{t_0}^{t_r}(\mathcal{LH}(s_r))))^{w_r} = (\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(s_i))))^{w_i} \times (\prod_{r \ne i} (\triangle_{t_0}^{-1}(TF_{t_0}^{t_r} (\mathcal{LH}(s_r))))^{w_r}) = (\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(s_i)))) \times (\prod_{r \ne i} (\triangle_{t_0}^{-1}(TF_{t_0}^{t_r}(\mathcal{LH}(s_r))))^0) = \triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}\ (s_i)))$, hence, $f_{ULWG}(s_1, \cdots, s_n) = \mathcal{LH}^{-1}(\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(s_1)))) = s_i$.

As special cases, we have 1) Denote $s_j = max\{s_1, \cdots, s_n\}$, if $w_j = 1$, then $f_{ULWG}((w_1, s_1), \cdots, (w_n, s_n)) = s_j$; 2) Denote $s_k = min\{s_1, \cdots, s_n\}$, if $w_k = 1$, then $f_{ULWG}((w_1, s_1), \cdots, (w_n, s_n)) = s_k$; 3) If $w_j = w_k = 0$, then f_{ULWG} reduces to the linguistic Olympic operator, i.e., the smallest and largest linguistic values are deleted from linguistic evaluating values.

Proposition 3.2. *Let unbalanced linguistic values $V = \{(w_i, s_i)|i=1, ..., n\}$ be aggregated, where $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. The $ULWG$ operator satisfies: 1) $\min\{s_1, \cdots, s_n\} \le f_{ULWG}(\{(w_1, s_1), \cdots, (w_n, s_n)\}) \le \max\{s_1, \cdots, s_n\}$; 2) f_{ULWG} is idempotent, i.e., $f_{ULWG}(\{(w_1, s_1), \cdots, (w_n, s_n)\}) = s_1$ when $s_1 = \cdots = s_n$; 3) f_{ULWG} is monotone in relation to the input values s_i, i.e., for any $s_i' \ge s_i$, $f_{ULWG}((w_1, s_1), \cdots, (w_i, s_i'), \cdots, (w_n, s_n)) \ge f_{ULWG}((w_1, s_1), \cdots, (w_i, s_i), \cdots, (w_n, s_n))$; 4) f_{ULWG} is commutative; 5) f_{ULWG} reduces to the linguistic geometric mean if $w_i = \frac{1}{n}$ for all $i = 1, \cdots, n$, i.e., $f_{ULWG}(\{(w_1, s_1), \cdots, (w_n, s_n)\}) = (s_k, \alpha_k)$ and $k + \alpha_k = \sqrt[n]{\prod_{i=1}^{n} \triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(s_i)))}$.*

Proof: According to Equation (2), $k + \alpha_k = \prod_{i=1}^{n} (\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(s_i))))^{w_i} \le \prod_{i=1}^{n} (\triangle_{t_0}^{-1} (TF_{t_0}^{t_j}(\mathcal{LH}(s_j))))^{w_i} = (\triangle_{t_0}^{-1}(TF_{t_0}^{t_j}(\mathcal{LH}(s_j))))^{\sum_{i=1}^{n} w_i} = \triangle_{t_0}^{-1}(TF_{t_0}^{t_j}(\mathcal{LH}(s_j)))$. Hence, $f_{ULWG} (\{(w_1, s_1), \cdots, (w_n, s_n)\}) = \mathcal{LH}^{-1}(s_k^{n(t_0)}, \alpha_k) \le \mathcal{LH}^{-1}(\triangle_{t_0}^{-1}(TF_{t_0}^{t_j}(\mathcal{LH}(s_j)))) = s_j = \max\{s_1, \cdots, s_n\}$. Similarly, we can prove $f_{ULWG}(\{(w_1, s_1), \cdots, (w_n, s_n)\}) = \mathcal{LH}^{-1}\ (s_k^{n(t_0)}, \alpha_k) \ge$

$\mathcal{LH}^{-1}(\triangle_{t_0}^{-1}\ (TF_{t_0}^{t_k}(\mathcal{LH}(s_k)\ ))) = s_k = \min\{s_1, \cdots, s_n\}$. 2., 3., 4. and 5. can be proved similarly.

4. **Illustrative Example.** In this section, we apply the $ULWG$ operator to deal with fuzzy risk analysis problems, we firstly use an example to illustrate the fuzzy risk analysis process of our method, where linguistic evaluating values are unbalanced linguistic evaluating values, in the example, assume that the following unbalanced linguistic evaluating values are considered: 1) Unbalanced linguistic evaluating values for severity of loss: $L = \{$none (N), low (L), medium (M), almost high (AH), high (H), quite high (QH), very high (VH), almost total (AT), total $(T)\}$; 2) Unbalanced linguistic evaluating values for probability of failure: $F = \{$none (N), small (S), medium (M), almost big (AB), big (B), quite big (QB), very big (VB), almost total (AT), total $(T)\}$. The proposed fuzzy risk analysis algorithm is now presented as follows: 1) Unbalanced linguistic evaluating values for severity of loss and probability of failure are represented in the level t_0 of $LH = \bigcup_t l(t, n(t))$, where t_0 is fixed by decision makers; 2) According to transformation function (1), the transformed unbalanced linguistic evaluating values are 2-tuple fuzzy linguistic representations; 3) Aggregate all linguistic evaluating values $\{(L_{ij}, F_{ij})|1 \le j \le s\}$ of component P_i by the $ULWG$ operator with linguistic weights, i.e., $F_i = f_{ULWGLW}(\{(L_{ij}, F_{ij})|1 \le j \le s\}) = f_{ULWG}(\{(f(L_{ij}), TF_{t_0}^{t_i}(\mathcal{LH}(F_{ij})))|1 \le j \le s\}) = \mathcal{LH}^{-1}(\triangle_{t_0}(\prod_{i=1}^{n}\ (\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(F_{ij}))))^{f(L_{ij})})) = \mathcal{LH}^{-1}(s_k^{n(t_0)}, \alpha_k)$, $k + \alpha_k = \prod_{i=1}^{n}(\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(F_{ij}))))^{f(L_{ij})}$, $f(L_{ij}) = \frac{\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(L_{ij})))}{\sum_{j=1}^{k}\triangle_{t_0}^{-1}(TF_{t_0}^{t_i}(\mathcal{LH}(L_{ij})))}$; 4) Rank linguistic evaluating values $\{F_i|1 \le i \le r\}$ by the ranking method of linguistic values, i.e., $F_{i_1} \ge F_{i_2}$ if and only if $\triangle_{t_0}^{-1}(s_{k_1}^{n(t_0)}, \alpha_{k_1}) \ge \triangle_{t_0}^{-1}(s_{k_2}^{n(t_0)}, \alpha_{k_2})$. The larger the linguistic evaluation value F_i of P_i, the higher the risk of the manufactory M_i.

Example 4.1. *Assume that there are three manufactories M_1, M_2 and M_3 producing the components P_1, P_2 and P_3, respectively, where P_1, P_2 and P_3 are the same product made by different manufactories. Each component P_i consists of three sub-components P_{i1}, P_{i2} and P_{i3}, where $1 \le i \le 3$. Assume that there are two evaluating items L_{ij} and F_{ij} to derive the probability of failure F_i of component P_i made by manufactory M_i, where L_{ij} denotes the severity of loss of the sub-component P_{ij}, F_{ij} denotes the probability of failure of the sub-component P_{ij}, $1 \le i \le 3$ and $1 \le j \le 3$. The linguistic evaluation values of every sub-component P_{ij} are shown in Table 1. In the example, the*

Table 1. Linguistic evaluating values of the sub-components made by manufactories

manufactory	*sub-components*	*the severity of loss*	*the probability of failure*
	P_{11}	L	AT
M_1	P_{12}	QH	S
	P_{13}	AH	M
	P_{21}	M	QB
M_2	P_{22}	H	M
	P_{23}	N	T
	P_{31}	L	VB
M_3	P_{32}	M	B
	P_{33}	AT	S

linguistic hierarchies be $LH = l(1, 3) \cup l(2, 5) \cup l(3, 9) \cup l(4, 17)$ for the severity of loss and the probability of failure of sub-components, the levels t_0 of the severity of loss and *the probability of failure are fixed by 2 and 3, respectively. Transformations of linguistic evaluating values are shown in Table 2, we can obtain every F_i of component P_i according to (3), (4) and TABLE 2, respectively. For component P_1 with sub-components*

Table 2. Transformations of linguistic evaluating values

manufactory	*sub-components*	L_{ij}	$TF_2^{t_i}(\mathcal{LH}(L_{ij}))$	F_{ij}	$TF_3^{t_i}(\mathcal{LH}(F_{ij}))$
	P_{11}	L	$(s_1^5, 0)$	AT	$(s_8^9, -0.5)$
M_1	P_{12}	QH	$(s_3^5, 0.25)$	S	$(s_2^9, 0)$
	P_{13}	AH	$(s_3^5, -0.5)$	M	$(s_4^9, 0)$
	P_{21}	M	$(s_2^5, 0)$	QB	$(s_7^9, -0.5)$
M_2	P_{22}	H	$(s_3^5, 0)$	M	$(s_4^9, 0)$
	P_{23}	N	$(s_0^5, 0)$	T	$(s_8^9, 0)$
	P_{31}	L	$(s_1^5, 0)$	VB	$(s_7^9, 0)$
M_3	P_{32}	M	$(s_2^5, 0)$	B	$(s_6^9, 0)$
	P_{33}	AT	$(s_4^5, -0.25)$	S	$(s_2^9, 0)$

P_{11}, P_{12} *and* P_{13}, *we have* $f(L) = \frac{\triangle_2^{-1}(s_1^5,0)}{\triangle_2^{-1}(s_1^5,0)+\triangle_2^{-1}(s_3^5,0.25)+\triangle_2^{-1}(s_3^5,-0.5)} \doteq 0.15$, $f(QH) = \frac{\triangle_2^{-1}(s_3^5,0.25)}{\triangle_2^{-1}(s_1^5,0)+\triangle_2^{-1}(s_3^5,0.25)+\triangle_2^{-1}(s_3^5,-0.5)} \doteq 0.48$, $f(AH) = \frac{\triangle_2^{-1}(s_3^5,-0.5)}{\triangle_2^{-1}(s_1^5,0)+\triangle_2^{-1}(s_3^5,0.25)+\triangle_2^{-1}(s_3^5,-0.5)} \doteq 0.37$, $F_1 = f_{ULWGLW}(\{(L, AT), (QH, S), (AH, M)\}) = f_{ULWG}(\{(f(L), (s_8^9, -0.5)), (f(QH), (s_2^9, 0)), (f(AH), (s_4^9, 0))\}) = \mathcal{LH}^{-1}(\triangle_3((\triangle_3^{-1}(s_8^9, -0.5))^{f(L)} \times (\triangle_3^{-1}(s_2^9, 0))^{f(QH)} \times (\triangle_3^{-1}(s_4^9, 0))^{f(AH)})) = \mathcal{LH}^{-1}(\triangle_3(7.5^{0.15} \times 2^{0.48} \times 4^{0.37})) \doteq \mathcal{LH}^{-1}(\triangle_3(3.13)) = (M, -0.43)$. *According*

Table 3. The probability of failure of the component made by manufactory

manufactory	*the component*	*the probability of failure*
M_1	P_1	$F_1 = (M, -0.43)$
M_2	P_2	$F_2 = (AH, -0.15)$
M_3	P_3	$F_3 = (M, -0.325)$

to Table 3, the probability of failure F_2 of the component P_2 made by manufactory M_2 is $(AH, -0.15)$, i.e., almost big with the value of the symbolic translation -0.15, it is the largest linguistic evaluation value among the linguistic evaluating values of F_1, F_2 and F_3. Hence, the risk of the manufactory M_2 is the highest.

Example 4.2. [11] *Assume that the balanced linguistic values are {Absolutely-low, Very-low, Low, Fairly-low, Medium, Fairly-high, High, Very-high, Absolutely-high}, their corresponding interval valued fuzzy numbers are shown in Table 4. The linguistic evaluating values of every sub-component P_{ij} are shown in Table 5. In Table 5, w_{ij} denotes the degree of confidence of the decision-maker's opinion with respect to sub-components P_{ij}. The method proposed in* [11] *is presented as follows: 1) Aggregate the linguistic evaluating values of sub-components P_{ij} of each component P_i made by manufactory M_i based on the fuzzy weighted mean method and the interval-valued fuzzy numbers arithmetic operators proposed in* [11] *to get the probability of failure F_i, e.g., for F_1, according to Table 4 and Table 5, we have* $F_1 = ((low \otimes fairly\ low) \oplus (fairly\ high \otimes medium) \oplus (very\ low \otimes fairly\ high))/(low \oplus fairly\ high \oplus very\ low) = \frac{(A_3 \otimes A_4) \oplus (A_6 \otimes A_5) \oplus (A_2 \otimes A_6)}{A_3 \oplus A_6 \oplus A_2} = \frac{(A_3^* \otimes A_4^*) \oplus (A_6^* \otimes A_5^*) \oplus (A_2^* \otimes A_6^*)}{A_3^* \oplus A_6^* \oplus A_2^*} = [(0.240, 0.328, 0.656, 0.905; 0.288), (0.195, 0.306, 0.678, 0.949; 0.763)]$, *where,* A_2^*, A_3^*, A_4^*, A_5^* *and* A_6^* *are type-1 fuzzy numbers of* A_2, A_3, A_4, A_5 *and* A_6, *respectively, e.g.,* $A_2^* = (\frac{0.0075+0}{2}, \frac{0.0075+0}{2}, \frac{0.015+0.02}{2}, \frac{0.0525+0.07}{2}; \frac{0.5+1}{2}) = (0.00375, 0.00375,$

$0.0175, 0.06125; 0.75)$, $A_2^* \otimes A_6^* = (0.00375 \times 0.615, 0.00375 \times 0.65125, 0.0175 \times 0.77875, 0.06125 \times 0.825; \min(0.75, 0.75))$, $A_2^* \oplus A_6^* = (0.00375 \oplus 0.615, 0.00375 \oplus 0.65125, 0.0175 \oplus 0.77875, 0.06125 \oplus 0.825; \min(0.75, 0.75))$*; 2) Calculate the degree of similarity between the upper fuzzy numbers of the interval-valued fuzzy numbers F_i and every linguistic value shown in Table 4, respectively, e.g., for F_1 and A_1, we have* $S_X^U(F_1^U, A_1^U) = S_X^U((0.195, 0.306, 0.678, 0.949; 0.763), (0, 0, 0, 0; 1)) = 1 - \frac{|0.195-0|+|0.306-0|+|0.678-0|+|0.949-0|}{4} = 0.468$*; 3) Calculate the spread between the upper fuzzy numbers of the interval valued fuzzy numbers F_i and every linguistic value shown in Table 4, respectively, e.g., for F_1 and A_1, we have* $STD^U(F_1^U, A_1^U) = |STD_{F_1^U} - STD_{A_1^U}| = 0.347$, $x_1 = (0.195 + 0.306 + 0.678 + 0.949)/4 = 0.532$, $STD_{F_1^U} = \sqrt{\frac{(0.195-x_1)^2+(0.306-x_1)^2+(0.678-x_1)^2+(0.949-x_1)^2}{4-1}}$*; 4) Calculate the degree of similarity on the X-axis between the interval-valued fuzzy numbers F_i and every linguistic value shown in Table 4, respectively, e.g., for F_1 and A_1, we have* $S_X(F_1, A_1) = 1 - (|(0.195 - 0.24) - (0 - 0)| + |(0.306 - 0.328) - (0 - 0)| + |(0.678 - 0.656) - (0 - 0)| + |(0.949 - 0.905) - (0 - 0)|)/4 = 0.96675$*; 5) Calculate the degree of similarity on the Y-axis between the interval-valued fuzzy numbers F_i and every linguistic value shown in Table 4, respectively, e.g., for F_1 and A_1, we have* $S_Y(F_1, A_1) = 1 - |y_{F_1} - y_{A_1}| = 0.915$, *where y_{F_1} and y_{A_1} are associated to areas of F_1^U, F_1^L, A_1^U and A_1^L, respectively; 6) Calculate the degree of similarity between the interval-valued fuzzy numbers F_i and every linguistic value shown in Table 4, respectively, which is decided by the above mentioned degrees of similarity and the spread, e.g., for F_1 and A_1, we have* $S(F_1, A_1) = \frac{S_X^U(F_1^U, A_1^U) \times (1-|0.763-1|)}{1+STD^U(F_1^U, A_1^U)} \times S_X(F_1, A_1) \times S_Y(F_1, A_1) = 0.234$*. By using the method, the probability of failure of P_1 made by manufactory M_1 is "Medium", the probability of failure of P_2 is "fairly-high" and the probability of failure of P_3 is "fairly-high". Hence, the risk of manufactories M_2 and M_3 is the highest, see* [11] *for more detail.*

TABLE 4. Linguistic values and their corresponding interval-valued fuzzy numbers

Linguistic values	*Interval-valued fuzzy numbers*
Absolutely-low (s_0^9)	$A_1 = [(0, 0, 0, 0; 1), (0, 0, 0, 0; 1)]$
Very-low (s_1^9)	$A_2 = [(0.0075, 0.0075, 0.015, 0.0525; 0.5), (0, 0, 0.02, 0.07; 1)]$
Low (s_2^9)	$A_3 = [(0.0875, 0.12, 0.16, 0.1825; 0.5), (0.04, 0.1, 0.18, 0.23; 1)]$
Fairly-low (s_3^9)	$A_4 = [(0.2325, 0.255, 0.325, 0.3575; 0.5), (0.17, 0.22, 0.36, 0.42; 1)]$
Medium (s_4^9)	$A_5 = [(0.4025, 0.4525, 0.5375, 0.5675; 0.5), (0.32, 0.41, 0.58, 0.65; 1)]$
Fairly-high (s_5^9)	$A_6 = [(0.65, 0.6725, 0.7575, 0.79; 0.5), (0.58, 0.63, 0.8, 0.86; 1)]$
High (s_6^9)	$A_7 = [(0.7825, 0.815, 0.885, 0.9075; 0.5), (0.72, 0.78, 0.92, 0.97; 1)]$
Very-high (s_7^9)	$A_8 = [(0.9475, 0.985, 0.9925, 0.9925; 0.5), (0.93, 0.98, 1, 1; 1)]$
Absolutely-high (s_8^9)	$A_1 = [(1, 1, 1, 1; 1), (1, 1, 1, 1; 1)]$

In the following, we use our method to get the probability of failure F_i made by manufactory M_i ($i = 1, 2, 3$). In this example, because linguistic evaluating values are balanced, transformation function $TF_{t'}^t : l(t, n(t)) \longrightarrow l(t', n(t'))$ are unnecessary, or transformation function $TF_{t'}^t$ is such that $t = t' = 3$. According to (3), (4) and Table 5, we have $F_1 = f_{ULWG}(\{(low, fairly\ low), (fairly\ high, medium), (very\ low, fairly\ high)\}) = f_{ULWG}(\{(s_2^9, s_3^9), (s_5^9, s_4^9), (s_1^9, s_5^9)\}) = \mathcal{LH}^{-1}(\triangle_3((\triangle_3^{-1}(s_3^9, 0))^{f(s_2^9)} \times (\triangle_3^{-1}(s_4^9, 0))^{f(s_5^9)} \times (\triangle_3^{-1}(s_5^9, 0))^{f(s_1^9)}))$*, in which,* $f(s_2^9) = \frac{\triangle_3^{-1}(s_2^9)}{\triangle_3^{-1}(s_2^9)+\triangle_3^{-1}(s_5^9)+\triangle_3^{-1}(s_1^9)} = \frac{2}{2+5+1} = 0.25$, $f(s_5^9) = \frac{5}{2+5+1} = 0.625$ *and* $f(s_1^9) = \frac{1}{2+5+1} = 0.125$*. Hence,* $F_1 = \mathcal{LH}^{-1}(\triangle_{t_0}(3^{0.25} \times 4^{0.625} \times 5^{0.125})) \doteq \mathcal{LH}^{-1}(\triangle_{t_0}(3.83) = \mathcal{LH}^{-1}(s_4^9, -0.17) = (Medium, -0.17)$*. Similarly, F_2 and F_3 can be obtained as* $F_2 = f_{ULWG}(\{(s_2^9, s_7^9), (s_5^9, s_5^9), (s_1^9, s_4^9)\})$, $= \mathcal{LH}^{-1}((\triangle_3^{-1}(s_7^9, 0))^{f(s_2^9)} \times$

TABLE 5. Linguistic evaluating values of the sub-components made by manufactories

manufactory	*sub-components*	*the severity of loss*	*the probability of failure*
	P_{11}	*low*	*fairly-low* ($w_{11} = 0.9$)
M_1	P_{12}	*fairly-high*	*medium* ($w_{12} = 0.7$)
	P_{13}	*very-low*	*fairly-high* ($w_{13} = 0.8$)
	P_{21}	*low*	*very-high* ($w_{21} = 0.85$)
M_2	P_{22}	*fairly-high*	*fairly-high* ($w_{22} = 0.9$)
	P_{23}	*very-low*	*medium* ($w_{23} = 0.9$)
	P_{31}	*low*	*fairly-low* ($w_{31} = 0.95$)
M_3	P_{32}	*fairly-high*	*high* ($w_{32} = 0.8$)
	P_{33}	*very-low*	*fairly-high* ($w_{33} = 1.0$)

$(\triangle_3^{-1}(s_5^9, 0))^{f(s_5^9)} \times (\triangle_3^{-1}(s_4^9, 0))^{f(s_1^9)}) \doteq \mathcal{LH}^{-1}(s_5^9, 0.3) = (Fairly\text{-}high, 0.3)$, $F_3 = f_{ULWG}(\{(s_2^9, s_3^9), (s_5^9, s_6^9), (s_1^9, s_5^9)\}) = \mathcal{LH}^{-1}((\triangle_3^{-1}(s_3^9, 0))^{f(s_2^9)} \times (\triangle_3^{-1}(s_6^9, 0))^{f(s_5^9)} \times (\triangle_3^{-1}(s_5^9, 0))^{f(s_1^9)}) \doteq \mathcal{LH}^{-1}(s_5^9, -0.07) = (Fairly\text{-}high, -0.07)$. *Accordingly, the probability of failure of* P_1 *made by manufactory* M_1 *is "(Medium, –0.17)", the probability of failure of* P_2 *is "(Fairly-high, 0.3)" and the probability of failure of* P_3 *is "(Fairly-high, –0.07)". Hence, the risk of the manufactory* M_2 *is the highest.*

Compared our method with the method proposed in [11], *because interval valued fuzzy numbers are used to represent evaluating values, computation of the method is complex and the result of the method is the lack of precision, i.e.,* P_2 *and* P_3 *have the same probability of failure "fairly-high", this is loss of information due to the degree of similarity between the interval-valued fuzzy numbers. However, the result of our method is that the probability of failure of* P_2 *is "(Fairly-high, 0.3)" more than "(Fairly-high, –0.07)" of* P_3, *there is no loss of information due to 2-tuple fuzzy linguistic representation and no complex computation due to linguistic aggregation operator.*

5. **Conclusion.** Inspired by the weighted geometric operator, we proposed the unbalanced linguistic weighted geometric operator to deal with aggregation of unbalanced linguistic values with numerical weights as well as linguistic weights. Its properties showed that the result of the aggregation operator lies in the Min and Max operators. By the linguistic aggregation operator, we can directly represent the final linguistic evaluating values and there is no loss of information. Compared our method with the one based on interval-valued fuzzy numbers, it is concluded that our method overcomes the issues of complex computation, loss of information and the lack of precision.

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