GLOBAL STABILIZATION FOR A CLASS OF HIGH-ORDER TIME-DELAY NONLINEAR SYSTEMS

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ABSTRACT. This paper addresses the stabilization for a class of high-order time-delay nonlinear systems. Under some essential restriction on the system growth, by the method of adding a power integrator, a continuous state-feedback controller is successfully designed, and the global asymptotic stability of the resulting closed-loop system is proven with the help of an appropriate Lyapunov-Krasovskii functional. A numerical example is also provided to illustrate the effectiveness of the theoretical results.

Keywords: High-order time-delay nonlinear systems, Adding a power integrator, Stabilization, Lyapunov-Krasovskii functional

1. Introduction. Over the past decades, significant progress has been made on control design and stability analysis for time-delay linear systems (see e.g., [2,3,6] and references therein). However, for the time-delay nonlinear systems, there exist many open problems which are so important and interesting at least from the theoretical point of view and have been payed careful attention, see for example the lastly published papers [5,7-16]. Specifically, by backstepping method, [8] constructed the output feedback controller (independent of time-delay) and [10] presented a continuously differentiable state-feedback control design based on a Lyapunov-Krasovskii functional, both for time-delay nonlinear systems. In the presence of unknown uncertainties, merged with wavelet neural network, a time-delay independent adaptive controller is obtained in [5] for a class of time-delay nonlinear systems with triangular structure, and further research can be seen in [9].

In this paper, we consider a class of high-order time-delay nonlinear systems in the following form:

\[
\begin{align*}
\dot{x}_i(t) &= x_{i+1}^{p_i}(t) + f_i(x_i)(t), \quad x_i(t) \in \mathbb{R}, \\
\dot{x}_n(t) &= u^n(t) + f_n(x(t), x(t-\tau)),
\end{align*}
\]

where \( x = [x_1, \ldots, x_n] \in \mathbb{R}^n \) is the system state vector, and \( x_i = [x_1, \ldots, x_i] \in \mathbb{R}^i; \)
\( u \in \mathbb{R} \) is the control input; \( \tau \in \mathbb{R}^+ \) is the time-delay of the state; the system initial condition is \( x(\theta) = \xi_0(\theta), \theta \in [-\tau, 0], \xi_0 \) being specified continuous initial function; \( p_i \in \mathbb{R}_{\text{odd}}^+ := \left\{ \frac{p}{q} \mid p \text{ and } q \text{ are positive integers, and } p \ge q \right\}; f_i, i = 1, \ldots, n \) are unknown continuous functions satisfying \( f_i(0, 0) = 0, i = 1, \ldots, n. \)
For the case of $\tau = 0$ in system (1), with the help of the so-called adding a power integrator [17], fruitful results have been achieved over the past years (see e.g., [18-20,23-26,28] and [21,27] for the stabilization and the tracking, respectively). However, for the case of $\tau \neq 0$, some essential difficulties will inevitably be encountered in constructing the desired controller. For instance, the time-delay effect will make the common assumption on the high-order system nonlinearities infeasible, and what conditions should be placed to the nonlinearities remains unanswered. Secondly, due to the higher power, time-delay and assumptions on the nonlinearities, it is not easy to find a Lyapunov-Krasoviskii functional which can be behaved well in theoretical analysis. Thirdly, it is hard to effectively manipulate the nonlinearities and consequently to accomplish the desired control objective and obtain better closed-loop performance.

In this paper, motivated by the continuous control ideas in [18,22,28] and flexibly using the methods of adding a power integrator and backstepping, we give the recursive design procedure for the time-delay independent continuous controller. Then, we construct an appropriate Lyapunov-Krasovskii functional inspired by [5,9], and by means of it, we show that the controller designed guarantees globally asymptotic stability of the resulting closed-loop system. We also give a numerical example to illustrate the effectiveness of the theoretical results.

2. Design Procedure of Continuous State-feedback Controller. This paper is devoted to the stable control design for system (1) under the following assumption:

**Assumption 2.1.** For each $i = 1, \ldots, n$, there exists a known positive constant $C$ such that for any $(x_{i}[t], x_{i}(t - \tau)) \in \mathbb{R}^{i} \times \mathbb{R}^{i}$

$$
|f_{i}(x_{i}[t], x_{i}(t - \tau))| \leq C \left( |x_{1}(t - \tau)|^{1/p_{1} - 1} + \cdots + |x_{i-1}(t - \tau)|^{1/p_{i-1} - 1} + |x_{i}(t - \tau)|^{1/p_{i} - 1} \right)
\quad + C \left( |x_{1}(t)|^{1/p_{1} - 1} + \cdots + |x_{i-1}(t)|^{1/p_{i-1} - 1} + |x_{i}(t)|^{1/p_{i} - 1} \right).
$$

For Assumption 2.1, when $\tau = 0$, it is the same as that in [22]. Particularly, when $p_{i} = 1, i = 1, \ldots, n$, Assumption 2.1 shows that the system nonlinearities $f_{i}, i = 1, \ldots, n$ satisfy a linear growth condition, and are equivalent to those in [8,9].

We first introduce the following transformations:

$$
\begin{align*}
    z_{1} &= x_{1}, \\
    z_{i} &= x_{i}^{p_{1} \cdots p_{i-1}} - \alpha_{i-1}^{p_{1} \cdots p_{i-1}}(z_{i-1}), \quad i = 2, \ldots, n, \\
    u &= \alpha_{n}(z), \\
    \alpha_{i}^{p_{1} \cdots p_{i}} &= -g_{i}z_{i}, \quad i = 1, \ldots, n,
\end{align*}
$$

where $g_{i}$’s are positive constants to be specified later. From (1) and (2), we can easily deduce the new transformed system $z$ with the initial function $\tilde{z}_{0}(\theta), \theta \in [-\tau, 0]$, namely, $z(\theta) = \tilde{z}_{0}(\theta), \theta \in [-\tau, 0]$. For the sake of simplicity, we let $\alpha_{0} = 0$ and $p_{1} \cdots p_{i-1} = 0$ when $i = 1$, and hence $z_{1} = x_{1} = x_{1} - \alpha_{0}$.

Noting that the convergence of the system $z$ implies that of the original system $x$, and to design the desired controller, it suffices to find the appropriate design parameters $g_{i}$’s. Therefore, the next design and analysis proceed mostly toward the system $z$, and mainly determine what $g_{i}$’s is feasible to stabilize the closed-loop system.

We then define $W_{i} : \mathbb{R}^{k} \rightarrow \mathbb{R}, i = 1, \ldots, n$ as follows:

$$
W_{i}(x_{i}) = \int_{a_{i-1}}^{x_{i}} \left( s^{p_{1} \cdots p_{i-1}} - \alpha_{i-1}^{p_{1} \cdots p_{i-1}} \right)^{2-1/(p_{1} \cdots p_{i-1})} ds.
$$
Noting that \( \alpha_0 = 0 \) and \( p_1 \cdot p_0 = 0 \), we see that \( W_1 = \frac{1}{2} x_1^2 \). For these \( W_i \)'s, some useful properties are given by the following proposition whose proof can be found in [18] and hence omitted here.

**Proposition 2.1.** \( W_i \)'s are nonnegative, continuously differentiable and satisfy

\[
\begin{align*}
\frac{\partial W_i}{\partial x_i} &= \left\{ \begin{array}{ll}
\frac{2^{-1/p_i}(p_i \cdots p_{k-1})}{\alpha_{k-1}} & i = k, \\
1 - 2p_i \cdots p_{k-1} & i \neq k,
\end{array} \right.
\int_{\alpha_{k-1}}^{\alpha_k} \left( s^{p_i \cdots p_{k-1}} - \alpha_{k-1}^{-p_i \cdots p_{k-1}} \right) \frac{p_i \cdots p_{k-1}}{\alpha_{k-1}} ds, \quad i < k.
\end{align*}
\]

Furthermore, there holds \( 2^{-1/p_i \cdots p_{k-1}}(x_k - \alpha_{k-1})^{2p_i \cdots p_{k-1}} \leq W_k(\cdot) \leq 2z_i^2 \).

The next turns to determining the appropriate design parameters \( g_i \)'s, which will proceed in \( n \) steps. Although the design procedure below is not compact in nature, the design/choice is explicit and intuitive, and the detailed techniques adopted can be well displayed.

**Step 1:** Define \( V_1 = W_1 + n \int_{t-\tau}^t z_i^2(s)ds \) as the candidate Lyapunov functional for this step, where \( W_1 = \frac{z_i^2}{2} \). Then, by Assumption 2.1 and (2), we have

\[
\begin{align*}
\dot{V}_1 &= z_1^2(x_1^p + f_1) + 2z_1^2(t - \tau) \\
&\leq z_1^2(x_1^p - \alpha_{k-1}^p) + z_1(\alpha_{k-1}^p + C |z_1| + |z_1(t - \tau)|) + n z_i^2 - (n - 1) z_i^2(t - \tau) \\
&\leq -n z_i^2 + z_1 z_2 + z_1 \left( \alpha_{k-1}^p + \left( C + \frac{C^2}{4} + 2n \right) z_1 \right) - (n - 1) z_i^2(t - \tau).
\end{align*}
\]

From this and the following choice:

\[
\alpha_{k-1}^p(x_1) = -g_1 z_1 := - \left( C + \frac{C^2}{4} + 2n \right) z_1,
\]

we obtain

\[
\dot{V}_1 \leq -n z_i^2 + z_1 z_2 - (n - 1) z_i^2(t - \tau).
\]

This completes Step 1.

**Step 2:** Define \( V_2 = V_1 + W_2 + (n - 1) \int_{t-\tau}^t z_i^2(s)ds \) as the candidate Lyapunov functional of this step. Then, by (1), (2), (6) and Proposition 2.1, we have

\[
\begin{align*}
\dot{V}_2 &\leq -n z_i^2 + z_1 z_2 - (n - 1) z_i^2(t - \tau) + z_2 \left( \frac{\partial \alpha_{k-1}^p}{\partial x_1} (x_1^p + f_1) + (n - 1) z_i^2 - (n - 1) z_i^2(t - \tau) \right) \\
&\quad - \omega_2(x_2) \frac{\partial \alpha_{k-1}^p}{\partial x_1} (x_1^p + f_1) + (n - 1) z_i^2 - (n - 1) z_i^2(t - \tau) \\
&= -n z_i^2 - (n - 1)(z_i^2(t - \tau) + z_i^2(t - \tau)) + (n - 1) z_i^2 + z_2^{-1/p_i} (x_3^p - \alpha_{p_i}^2) \\
&\quad + z_2^{-1/p_i} \alpha_{p_i}^2 + z_1 z_2 + z_2^{-1/p_i} f_2 + g_1 \omega_2(x_2) f_1 + g_1 \omega_2(x_2) x_2 \alpha_{p_i}^2,
\end{align*}
\]

where \( \omega_2(x_2) = (2 - \frac{1}{p_i}) \int_{x_2}^{x_2} (s^p - \alpha_{p_i}^1)^{-1/p_i} ds \).

To give the explicit form of \( \omega_2 \), we should find the appropriate estimations for the last four terms on the right-hand side of (7). First, it is very easy to get

\[
z_1 z_2 \leq z_i^2 + \frac{1}{4} z_1^2.
\]

Second, by Lemmas 5.1 and 5.2 in Appendix, we have

\[
\begin{align*}
z_2^{-1/p_i} f_2 &\leq C |z_i|^{2/p_i} \left( |x_2| + |x_1|^{1/p_i} + |x_2(t - \tau)| + |x_1(t - \tau)| \right)^{1/p_i} \\
&= C |z_i|^{2/p_i} \left( |z_2 + \alpha_{p_i}^2|^{1/p_i} + |z_1|^{1/p_i} + |z_2(t - \tau)| + |z_1(t - \tau)| \right)^{1/p_i} \\
&\quad + \alpha_{p_i}^2 \left( t - \tau \right)^{1/p_i} + |z_1(t - \tau)|^{1/p_i} \\
&\leq C |z_i|^{2/p_i} \left( |z_2|^{1/p_i} + \left( 1 + g_1 \right) |z_1|^{1/p_i} \right)
\end{align*}
\]
This completes Step 2. We conclude that by Lemma 5.2 in Appendix, we have

$$C|z_2|^{2-1/p_1} \left( |z_2(t-\tau)|^{1/p_1} + \left( 1 + \frac{1}{p_1} \right) |z_1(t-\tau)|^{1/p_1} \right)$$

$$\leq \left( C + \frac{2p_1 - 1}{2p_1} \left( \left( \frac{2}{p_1} \right)^{1/(p_1-1)} + \left( \frac{1}{p_1} \right)^{1/(2p_1-1)} \right) \right) \left( 1 + \frac{1}{p_1} \right)^{2p_1/(2p_1-1)}$$

$$+ \frac{2p_1 - 1}{2p_1} \left( \frac{1}{2p_1} \right)^{1/(2p_1-1)} C^{2p_1/(2p_1-1)} z_2^2 + \frac{1}{4} z_1^2 + z_2^2(t-\tau) + \frac{1}{2} z_2^2(t-\tau).$$

Therefore, substituting (8) into (7), and by the following choice:

$$\alpha_2 p_1 p_2 (x_{[g]}) = -g_2 z_2 := -(2n - 1 + r_{21} + r_{22} + 24g_1^2 C^2)^{p_1} z_2,$$

we conclude that

$$V_2 \leq -(n - 1)(z_1^2 + z_2^2) - (n - 2)(z_1(t-\tau) + z_2(t-\tau)) + z_2^{2-1/p_1} (x_{p_1 - 2} - \alpha_2^2).$$

This completes Step 2.

The first two steps can be viewed as the initialization of the whole recursive design procedure. From Step 3, we turn to the recursive steps.

**Step k (k = 3, \ldots, n):** Suppose that $V_{k-1} (\cdot)$ is the candidate Lyapunov functional for step $k - 1$ and such that

$$V_{k-1} \leq -(n - k + 2) \sum_{i=1}^{k-1} z_i^2(t - \tau) + z_k^{2-1/(p_1 \cdot p_k - 2)} (x_{p_1 \cdot p_k - 2} - \alpha_{k-1}^{p_k-1}).$$

Define $\tilde{V}_k = V_{k-1} + W_k + (n - k + 1) \int_{t-\tau}^t z_k^2(s) ds$ as the candidate Lyapunov functional for this step. Then, by (1), (2) and (13), we have

$$\tilde{V}_k \leq -(n - k + 2) \sum_{i=1}^{k-1} z_i^2(t - \tau) + z_k^{2-1/(p_1 \cdot p_k - 2)} (x_{p_1 \cdot p_k - 2} - \alpha_{k-1}^{p_k-1})$$

$$+ \frac{2}{p_1 \cdot p_k - 2} \left( x_{p_1 \cdot p_k - 2} + f_k \right) - \omega_k(\cdot) \sum_{i=1}^{k-1} \frac{\partial \alpha_k (x_{p_1 \cdot p_k - 2})}{\partial x_i} (x_{p_1 \cdot p_k - 2} + f_i)$$

$$+ (n - k + 1) z_k^2 - (n - k + 1) z_k^2(t - \tau)$$

$$= -(n - k + 2) \sum_{i=1}^{k-1} z_i^2(t - \tau) + z_k^{2-1/(p_1 \cdot p_k - 2)} \alpha_k^{p_k}.$$
Similarly, to give the explicit form of $\alpha_k$, we should find the appropriate estimations of the last four terms on the right-hand side of (14). First, from (2) and Lemma 5.1 in Appendix, it follows that

$$|f_k| \leq C \left( \sum_{i=1}^{k-1} |x_i|^{1/(p_1 \cdots p_{k-1})} + |x_k| \right) + C \left( \sum_{i=1}^{k-1} |x_i(t - \tau)|^{1/(p_1 \cdots p_{k-1})} + |x_k(t - \tau)| \right)$$

$$= C \left( \sum_{i=1}^{k-1} |x_i|^{1/(p_1 \cdots p_{k-1})} + |x_k|^{1/(p_1 \cdots p_{k-1})} \right) + C \left( \sum_{i=1}^{k-1} |x_i|^{1/(p_1 \cdots p_{k-1})} + |x_k|^{1/(p_1 \cdots p_{k-1})} \right)$$

$$\leq C |z_k|^{1/(p_1 \cdots p_{k-1})} + C \sum_{i=1}^{k-1} \left( 1 + g_i^{1/(p_1 \cdots p_{k-1})} \right) |z_i|^{1/(p_1 \cdots p_{k-1})}$$

$$+ C |z_k(t - \tau)|^{1/(p_1 \cdots p_{k-1})} + C \sum_{i=1}^{k-1} \left( 1 + g_i^{1/(p_1 \cdots p_{k-1})} \right) |z_i(t - \tau)|^{1/(p_1 \cdots p_{k-1})}$$

$$=: \sum_{i=1}^{k} C_i |z_i|^{1/(p_1 \cdots p_{k-1})} + \sum_{i=1}^{k} C_i |z_i(t - \tau)|^{1/(p_1 \cdots p_{k-1})}, \quad (15)$$

and hence by Lemma 5.2 in Appendix

$$\frac{z_k^{2 - \frac{1}{p_1 \cdots p_{k-1}}} f_k}{\sum_{i=1}^{k-1} \left( \sum_{i=1}^{k} C_i |z_i|^{1/(p_1 \cdots p_{k-1})} + \sum_{i=1}^{k} C_i |z_i(t - \tau)|^{1/(p_1 \cdots p_{k-1})} \right)^{2 - \frac{1}{p_1 \cdots p_{k-1}}} + \sum_{i=1}^{k-1} \frac{z_k^{2 - \frac{1}{p_1 \cdots p_{k-1}}}}{C_k}} \leq \left( C + \frac{2 p_1 \cdots p_{k-1} - 1}{2 p_1 \cdots p_{k-1}} \left( 1 \frac{3}{2 p_1 \cdots p_{k-1}} \right) \sum_{i=1}^{k-2} C_i \frac{2 p_1 \cdots p_{k-1} - 1}{C_k} \right) z_k^2$$

$$+ \frac{2 p_1 \cdots p_{k-1} - 1}{2 p_1 \cdots p_{k-1}} \left( \frac{2}{p_1 \cdots p_{k-1}} \right) z_k^2 + \frac{2 p_1 \cdots p_{k-1} - 1}{2 p_1 \cdots p_{k-1}} \left( \frac{1}{p_1 \cdots p_{k-1}} \right) \sum_{i=1}^{k-1} C_i \frac{2 p_1 \cdots p_{k-1} - 1}{C_k} \right) z_k^2$$

$$+ \frac{2 p_1 \cdots p_{k-1} - 1}{2 p_1 \cdots p_{k-1}} \left( \frac{1}{2 p_1 \cdots p_{k-1}} \right) \sum_{i=1}^{k-1} C_i \frac{2 p_1 \cdots p_{k-1} - 1}{C_k} \right) z_k^2$$

$$+ \sum_{i=1}^{k-1} \frac{z_k^{2 - \frac{1}{p_1 \cdots p_{k-1}}}}{3} + \sum_{i=1}^{k-1} \frac{z_k^{2 - \frac{1}{p_1 \cdots p_{k-1}}}}{4} + \sum_{i=1}^{k-1} \frac{z_k^{2 - \frac{1}{p_1 \cdots p_{k-1}}}}{2} + z_k^{2 - \frac{1}{p_1 \cdots p_{k-1}}}, \quad (16)$$
where positive constants $m_j$ and $\bar{m}_j$ are defined as follows:

\[
\begin{align*}
    m_j &= \frac{\hat{g}_i \left( \frac{1}{1 + g_{i-1}} \right)}{p_1 \cdots p_{i-1}} C_j, \quad j = 1, \ldots, i, \\
    \bar{m}_j &= m_j, \quad j = 1, \ldots, i - 2, \\
    \bar{m}_{i-1} &= m_{i-1} + \frac{2\hat{g}_i g_{i-1} \left( \frac{1}{1 + g_{i-1}} \right)}{p_1 \cdots p_{i-1}} \sum_{j=1}^{i} C_j, \\
    \bar{m}_i &= m_i + \frac{2\hat{g}_i (1 - \frac{1}{p_1 \cdots p_{i-1}})}{p_1 \cdots p_{i-1}} \sum_{j=1}^{i} C_j.
\end{align*}
\]

Furthermore, by Lemma 5.4 in Appendix and $|x_k - \alpha_{k-1}|^{p_1 \cdots p_{k-1}} \leq 2^{p_1 \cdots p_{k-1}-1} |x_k^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{k-1}}| = 2^{p_1 \cdots p_{k-1}-1} |z_k|$, we have

\[
\omega_k \leq \left( 2 - \frac{1}{p_1 \cdots p_{k-1}} \right) |x_k - \alpha_{k-1}| \cdot |x_k^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{k-1}}|^{1/(p_1 \cdots p_{k-1})}
\leq 2^{1/(p_1 \cdots p_{k-1})} \left( 2 - \frac{1}{p_1 \cdots p_{k-1}} \right) |z_k|^{1/(p_1 \cdots p_{k-1})} \cdot |x_k^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{k-1}}|^{1/(p_1 \cdots p_{k-1})}
\leq 4 |z_k|.
\]

Using the above inequalities about $\frac{\partial \alpha_{k-1}^{p_1 \cdots p_{k-1}}}{\partial x_i} f_i$ and $\omega_k$ together with Lemma 5.2 in Appendix, we have

\[
-\omega_k \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}^{p_1 \cdots p_{k-1}}}{\partial x_i} f_i \leq 4 |z_k| \sum_{i=1}^{k-1} \left( \sum_{j=1}^{i} m_j |z_j(t - \tau)| + \sum_{j=1}^{i} \bar{m}_j |z_j| \right)
\]
This completes Step k.

Substituting this into (21), we obtain

\[ r_k^2z_k^2 + \sum_{i=1}^{k-2} \frac{z_i^2}{3} + \sum_{i=1}^{k-1} \frac{z_{k-1}^2(t - \tau)}{2} =: r_k z_k^2 + \sum_{i=1}^{k-2} \frac{z_i^2}{3} + \sum_{i=1}^{k-1} \frac{z_{k-1}^2(t - \tau)}{2}. \]

Moreover,

\[ -\omega_k \sum_{i=1}^{k-1} \frac{\partial x_i}{\partial x_i} x_i^{p_i} \leq 4|z_k| \left| \sum_{i=1}^{k-1} \tilde{g}_i (|x_i|^{p_i-1} - |x_{i+1}|^{p_i}) \right| \leq 4|z_k| \left( \tilde{g}_1 (|x_1| + |x_2|^{p_1}) + \sum_{i=2}^{k-1} \tilde{g}_i (|x_i|^{p_i-1} + |x_{i+1}|^{p_i}) \right) \leq \left( 4\tilde{g}_{k-1} + 12 \left( \tilde{g}_1 + \tilde{g}_1g_1 + \tilde{g}_2g_1 \right)^2 + \sum_{i=2}^{k-2} \left( \tilde{g}_{i-1} + \tilde{g}_i + \tilde{g}_i \right)^2 \right) + 16(\tilde{g}_{k-2} + \tilde{g}_{k-1} + \tilde{g}_{k-1}g_{k-1})^2 \right) z_k^2 + \sum_{i=1}^{k-2} \frac{z_i^2}{3} + \frac{z_{k-1}^2}{4} =: r_k z_k^2 + \sum_{i=1}^{k-2} \frac{z_i^2}{3} + \frac{z_{k-1}^2}{4}. \]

At last, by Lemma 5.1 in Appendix and (2), we can get the upper bound estimate of the last term on the right-hand side of (14) as follows:

\[ \frac{2}{(p_1 \cdots p_k - 1)} \left( x_k^{p_k-1} - \alpha_k^{p_k-1} \right) \leq |z_k|^{2 - \frac{1}{p_1 \cdots p_k - 1}} \cdot \frac{1}{2} \sum_{i=1}^{k-2} \frac{z_i^2}{3} + \frac{z_{k-1}^2}{4} \leq 2\left( \sum_{i=1}^{p_1 \cdots p_k - 2} \frac{1}{2} \right) \frac{1}{p_1 \cdots p_k - 1} \sum_{i=1}^{p_1 \cdots p_k - 2} \frac{z_i^2}{3} + \frac{z_{k-1}^2}{4} \leq r_k z_k^2 + \sum_{i=1}^{k-2} \frac{z_i^2}{3} + \frac{z_{k-1}^2}{4}. \]

Defining \( r_k = r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} \), and substituting (16), (18)~(20) into (14), after some simple calculations, we can obtain

\[ \dot{V}_k \leq -(n - k + 1) \sum_{i=1}^{k-1} z_i^2 - (n - k) \sum_{i=1}^{k} z_i^2(t - \tau) + z_k^{2 - \frac{1}{p_1 \cdots p_k - 1}} (x_k p_k - \alpha_k^{p_k}) \]

\[ + z_k^{2 - \frac{1}{p_1 \cdots p_k - 1}} \alpha_k^{p_k} + (n - k + 1 + r_k) z_k^2. \]

Now, we choose the virtual controller \( \alpha_k \) as

\[ \alpha_k^{p_1 \cdots p_k}(x_k) = -g_k z_k := -(2(n - k + 1) + r_k)^{p_1 \cdots p_k - 1} z_k. \]

Substituting this into (21), we obtain

\[ \dot{V}_k \leq -(n - k + 1) \sum_{i=1}^{k} z_i^2 - (n - k) \sum_{i=1}^{k} z_i^2(t - \tau) + z_k^{2 - \frac{1}{p_1 \cdots p_k - 1}} (x_k p_k - \alpha_k^{p_k}). \]

This completes Step k.
3. Main Results. From above recursive design steps, we can see that when \( k = n \), the candidate Lyapunov functional \( V_n \), which is clearly expressed as

\[
V_n(\cdot) = \sum_{i=1}^{n} W_i(x_{[i]}) + \sum_{i=1}^{n} (n - i + 1) \int_{t-\tau}^{t} z_i^2(s)ds,
\]

under the following actual controller:

\[
u = \alpha_n = -(2 + r_n)z_n^{1/p_n-1} =: -g_nz_n^{1/p_n-1}, \quad (23)
\]
satisfies

\[
\dot{V}_n \leq -\sum_{i=1}^{n} z_i^2.
\]

Remark 3.1. By (17), we know that

\[
z_n = x_n^{p_1 \cdots p_n-1} + \sum_{i=1}^{n-1} \prod_{j=i}^{n-1} g_j x_i^{p_i \cdots p_{i-1}},
\]

by which and (23), we can obtain another expression of controller \( u \) as a function of \( x \). From this and the fact that \( p_i \in \mathbb{R}_{odd}^+ \), we easily see that the power of \( u \) is at most 1 with respect to \( x_i, i = 1, \ldots, n \), and hence \( u \) is a continuous function of \( x \), rather than a smooth one if \( p_i > 1 \) for at least one \( i \).

We are now in a position to present the following theorem to summarize the main results of the paper.

Theorem 3.1. For the high-order time-delay nonlinear system (1) under Assumption 2.1, the continuous state-feedback controller (23) designed in the above section, which preserves the equilibrium at the origin, renders that the closed-loop system state converges to zero.

Proof: From the expressions of \( \alpha_i, i = 1, \ldots, n \) and (23), we can easily see that the control \( u \) preserves the equilibrium at the origin.

We next prove that \( \lim_{t \to \infty} x(t) = 0 \). From the above definitions of \( W_i \)'s and Proposition 2.1, we easily see that \( \sum_{i=1}^{n} W_i \), as the function of \( z \), is positive definite and radially unbounded. Then, by (24) and using Lemma 4.3 in [4], we know that there exist \( K_\infty \) functions \( \beta(\cdot), \gamma(\cdot) \) and \( \omega(\cdot) \) such that

\[
\begin{align*}
\beta(\|z(t)\|) &\leq \sum_{i=1}^{n} W_i(x_{[i]}(t)) \leq V_n(t, z_t(\theta)) \leq \gamma(\max_{-\tau \leq \theta \leq 0} \|z(t + \theta)\|), \\
V_n(t, z_t(\theta)) &\leq -\omega(\|z(t)\|),
\end{align*}
\]

(25)

where \( \| \cdot \| \) denotes the Euclidean norm of vector, \( z_t(\theta) = z(t + \theta), \theta \in [-\tau, 0] \). Then by (24) and Lyapunov-Krasovskii stability theorem (see [1,6]), we have \( \lim_{t \to \infty} z(t) = 0 \). This together with the definitions of \( \alpha_i \)'s directly concludes that \( \lim_{t \to \infty} x(t) = 0 \).

4. Simulations. To illustrate the correctness of theoretic results and effectiveness of the control design, we consider the following second-order time-delay nonlinear system:

\[
\begin{align*}
\dot{x}_1 &= x_3^2 + x_1(t - 1) + x_1 \sin x_2, \\
\dot{x}_2 &= u^3 - x_2 \sin x_1(t - 1) - x_2(t - 1) \cos 2x_1.
\end{align*}
\]

It is easy to verify that Assumption 2.1 holds as follows:

\[
\begin{align*}
f_1 &= x_1(t - 1) + x_1 \sin x_2 \leq |x_1| + |x_1(t - 1)|, \\
f_2 &= -x_2 \sin x_1(t - 1) - x_2(t - 1) \cos 2x_2 \leq |x_2| + |x_2(t - 1)|.
\end{align*}
\]
We next estimate the last second term on the right-hand side of (29). By Lemma 5.4 and bound estimates of the last four terms on the right-hand side of (29). First, it is very easy to check that
\[ V_1 = z_1 \left( x_1^3 + f_1 \right) + 2z_2^2 - 2z_1^2(t - 1) \leq z_1(2^3 - \alpha_1^3) + z_1\alpha_1^3 + |z_1|(|z_1| + |z_1(t - 1)|) + 2z_1^2 - 2z_1(t - 1) \leq z_1z_2 + z_1\alpha_1^3 + \frac{13z_1^2}{4} - z_1^2(t - 1) \leq -2z_1^2 + z_1z_2 + z_1 \left( \alpha_1^3 + \frac{21z_1}{4} \right) - z_1^2(t - 1). \] (26)

So far, we choose the first virtual controller \( \alpha_1 \) as
\[ \alpha_1^3(x_1) = -\frac{21}{4} z_1 =: g_1z_1. \] (27)

Substituting (27) into (26), we finally get
\[ \dot{V}_1 \leq -2z_1^2 + z_1z_2 - z_1^2(t - 1). \] (28)

This completes Step 1.

**Step 2:** Choose \( V_2 = V_1 + W_2 + \int_{t-1}^t z_2^2(l) dl \). Taking the time derivative of \( V_2 \), and substituting (28) into it, we get
\[
\dot{V}_2 \leq -2z_1^2 + z_1z_2 - z_2(t - 1) + z_2^{5/3} (u_3^3 + f_2) - \omega_2 \frac{\partial \alpha_2^3}{\partial x_1} (x_3^3 + f_1) + z_2^2 - z_2^2(t - 1) \]
\[
= -2z_1^2 - (z_1^2(t - 1) + z_1^2(t - 1)) + z_2^2 + z_2^{5/3} u_3^3 \]
\[
+ z_1z_2 + z_2^{5/3} f_2 + g_1 \omega_2 f_1 + g_1 \omega_2 x_3^2, \]
(29)

where \( \omega_2(x_2) = \frac{5}{6} \int_{t_0}^{x_2} (s^3 - \alpha_1^3)^{2/3} ds. \)

Next, to design the second virtual controller \( \alpha_2 \), we have to find the appropriate upper bound estimates of the last four terms on the right-hand side of (29). First, it is very easy to get
\[ z_1z_2 \leq z_2^2 + \frac{1}{4} z_1^2. \] (30)

Second, by Lemmas 5.1 and 5.2, we have
\[
z_2^{5/3} f_2 \leq |z_2|^{5/3} (|x_2| + |x_2(t - 1)|) \]
\[
= |z_2|^{5/3} \left( |x_2 + \alpha_1^3|^{1/3} + |z_2(t - 1) + \alpha_1^3(t - 1)|^{1/3} \right) \leq |z_2|^{5/3} \left( |x_2|^{1/3} + g_1 \alpha_1^3 z_1^{1/3} \right) \]
\[
+ |z_2|^{5/3} \left( |z_2(t - 1)|^{1/3} + g_1 \alpha_1^3 z_1(t - 1) \right) \]
\[
\leq \left( 1 + \frac{5}{6} \left( \frac{2}{3} \right)^{1/5} + \left( \frac{1}{3} \right)^{1/5} \right) g_1^{2/5} \left[ \frac{5}{6} \left( \frac{1}{3} \right)^{1/5} \right] z_2^2 + \frac{1}{4} z_1^2 + z_2^2(t - 1) + \frac{1}{2} z_1^2(t - 1) \]
\[
=: r_2z_2^2 + \frac{1}{4} z_1^2 + z_2^2(t - 1) + \frac{1}{2} z_1^2(t - 1). \] (31)

We next estimate the last second term on the right-hand side of (29). By Lemma 5.4 and \(|x_2 - \alpha_1^3| \leq 2^{3-1}|x_2^3 - \alpha_1^3| = 2^2|z_2|\), we get
\[
\omega_2 \leq \frac{5}{3} |x_2 - \alpha_1| \cdot |x_2^3 - \alpha_1^3|^{2/3} \leq 2^{2/3} \cdot \frac{5}{3} |x_2^3 - \alpha_1^3|^{1/3} \cdot |x_2^3 - \alpha_1^3|^{2/3} \leq 4|z_2|, \]

By the above inequalities of \( \omega_2 \) and Lemma 5.2, we have
\[
g_1 \omega_2 f_1 \leq 4g_1 |z_2|(|x_1| + |x_1(t - 1)|) \leq 24g_1 z_2^2 + \frac{1}{4} z_1^2 + \frac{1}{2} z_1^2(t - 1). \] (32)
Moreover, by Lemma 5.2, we have
\[
g_1 \omega_2 x_2^3 \leq 4g_1 |z_2| (|z_2| + g_1 |z_1|) \leq (4g_1 + 16g_1^4) z_2^2 + \frac{1}{4} z_1^2 =: r_{22} z_2^2 + \frac{1}{4} z_1^2. \tag{33}
\]
Now, substituting (30)~(33) into (29), and after some straightforward calculations, we have
\[
\dot{V}_2 \leq -z_2^2 + z_2^{5/3} u^3 + \left(2 + r_{21} + r_{22} + 24g_1^2\right) z_2^2. \tag{34}
\]
Based on (34), we choose the controller \( u \) as
\[
u(x[2]) = -\left(3 + r_{21} + r_{22} + 24g_1^2\right) z_2^{\frac{1}{3}} \approx -23.42 z_2^{\frac{1}{3}}. \tag{35}
\]
Finally, substituting (35) into (34) concludes that
\[
\dot{V}_2 \leq -(z_1^2 + z_2^2). \tag{36}
\]

![Figure 1. Response of the system states \( x_1 \) and \( x_2 \)](image)

By Theorem 3.1, we know that \( x_1 \) and \( x_2 \) will converge to zero. To see this, in simulation, we set \( x_1(\theta) \equiv 0.5, x_2(\theta) \equiv -1.4, \forall \theta \in [-1, 0] \), and obtain Figure 1, from which we can see that the constructed controller (35) indeed renders the resulting closed-loop system states \( x_1 \) and \( x_2 \) asymptotically converge to zero. The simulation demonstrates that the high-order time-delay nonlinear system can be stabilized with a satisfactory response by the state feedback controller designed recursively.

5. **Concluding Remarks.** In this paper, under Assumption 2.1 which formulates the growth of the system nonlinearities, we present a new approach to design a continuous state-feedback stabilizing controller independent of time-delay for a class of high-order time-delay nonlinear systems. The designed controller preserves the equilibrium at the origin, and guarantees the globally asymptotic stability of the system. It is necessary to point out that only one class of the systems is investigated in the paper, and there are many other or more general classes of the systems need to investigate.
Our future effort will be devoted to the systems in the form of (1) and whose nonlinearities satisfy the following more general assumption:

$$|f_i(x_i(t), x_i(t-\tau))| \leq C \left( \sum_{j=1}^{i} |x_j(t)|^{\delta_i} + \sum_{j=1}^{i} |x_j(t-\tau)|^{\delta_i} \right), \quad i = 1, \ldots, n,$$

where $C$ is a known or unknown positive constant, and $\delta_i$'s take values in certain interval, rather than precisely known constants.

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REFERENCES

Lemma 5.1. For $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $p \in \mathbb{R}_{\text{odd}}^{\geq 1}$, there hold
\[
\begin{cases}
|x + y|^{1/p} \leq |x|^{1/p} + |y|^{1/p}, \\
|x - y| \leq 2^{(p-1)/p}|x^p - y^p|^{1/p}.
\end{cases}
\]

Lemma 5.2. For $x \in \mathbb{R}$, $y \in \mathbb{R}$ and positive numbers $a$, $m$, $n$, there holds
\[
|ax^m y^n| \leq c(x, y)|x|^{m+n} + \frac{n}{m+n} \left( \frac{m}{c(x, y)(m+n)} \right)^{\frac{m}{n}} a^{\frac{m+n}{n}} |y|^{m+n},
\]
where $c(x, y) > 0$ for any $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

Lemma 5.3. For $x \in \mathbb{R}^+$, $y \in \mathbb{R}^+$ and $p > 1$, $q > 1$, there holds:
\[
x^{p-1}y^q \leq x^p + y^p.
\]

Lemma 5.4. Let $f : [a, b] \rightarrow \mathbb{R}$ ($a \leq b$) be a continuous function, which is monotone and satisfies $f(a) = 0$. Then $\left| \int_a^b f(x)dx \right| \leq |f(b)| \cdot |b - a|$.