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GROUP DECISION-MAKING WITH GENERALIZED AND PROBABILISTIC AGGREGATION OPERATORS

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Abstract. *The aim of this paper is to introduce a unified model between the generalized ordered weighted averaging (GOWA) operator and the generalized probabilistic aggregation. We present the generalized probabilistic OWA (GPOWA) operator. It is a new aggregation operator that unifies the probability with the OWA operator considering the degree of importance that each concept has in the analysis. It includes a wide range of particular cases including the GOWA operator and the probabilistic OWA (POWA) operator. We also study the applicability of this new approach and we see that it is very broad because all the previous studies that use the probability or the OWA operator can be revised with this new approach. We develop an application in multi-person decision making concerning the selection of the optimal strategies.*
Keywords: Decision-making, OWA operator, Probability, Aggregation operators

1. Introduction.

Aggregation operators are very common in the scientific literature [1-4]. One of the most popular one is the probabilistic aggregation [3] because it considers some kind of study that permits to make assumptions regarding the degree of importance that the information has. Another interesting type of aggregation operator is the ordered weighted averaging (OWA) operator [5]. It provides a parameterized family of aggregation operators between the minimum and the maximum. Since its introduction, it has been studied by a lot of authors [6-18].

A further interesting type of aggregation operators are those that provide a general formulation based on the use of generalized and quasi-arithmetic means. These types of functions are known as generalized aggregation operators [1,19-27]. Their main advantage is that they can provide a more complete representation of the problem considered by using different particular cases. For example, it is worth noting the generalized OWA (GOWA) operator [25] and its extensions [22-24,26,27].

Recently, Merigó [28] has suggested the probabilistic OWA (POWA) operator. It is a new aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum in a unified model between the probabilistic aggregation and the OWA operator. Thus, we are able to under- or over-estimate the probabilistic information or we can introduce probabilistic information in the OWA operator.

The aim of this paper is to present a new aggregation operator that provides a more complete representation of the GOWA operator [25] by using probabilistic information

[29-33]. Thus, we are able to introduce objective information (based on some kind of experiment) in the analysis. Note that our objective is to focus on the process of aggregating the information rather than focusing on probability theory. We present the generalized probabilistic OWA (GPOWA) operator. It is a new aggregation operator that unifies the probability with the OWA operator considering the degree of importance that each concept has in the analysis. A key advantage of the GPOWA is that it is able to consider a wide range of particular cases including the maximum, the minimum, the generalized probabilistic aggregation, the GOWA, the maximum probabilistic aggregation, the minimum probabilistic aggregation, the arithmetic probabilistic aggregation (A-PA), the arithmetic OWA (A-OWA), the probabilistic OWA (POWA), the geometric POWA, the quadratic POWA and the harmonic POWA operator.

We study the applicability of this approach and we see that it is very broad because all the previous studies that use the probability or the OWA operator can be revised and extended with this new approach. Its main advantage is that it deals with probabilistic information and with the attitudinal character of the decision maker in the same formulation considering the degree of importance of each concept in the analysis. For example, we can apply it in statistics, economics, engineering and physics. We study several examples such as the development of a probabilistic variance, covariance and a simple linear regression model.

We further extend this approach in a multi-person decision making problem where we are able to assess the information in a more complete way because we can consider the opinion of several persons (experts) in the analysis. We introduce the multi-person GPOWA (MP-GPOWA) operator. Its main advantage is that it can consider the opinion of several elements in the analysis and a wide range of particular cases in a unified model between the probability and the OWA operator. Among others, the MP-GPOWA operator includes the multi-person arithmetic mean (MP-AM), the multi-person generalized probabilistic aggregation (MP-GPA), the multi-person GOWA (MP-GOWA) and the multi-person POWA (MP-POWA) operator. We focus on a multi-person decision making application regarding the selection of strategies. We see that depending on the particular type of aggregation operator used, the results may lead to different decisions.

This paper is organized as follows. In Section 2, we briefly describe some basic preliminaries such as the probabilistic aggregation, the OWA operator, the GOWA operator and the POWA operator. In Section 3, we present the GPOWA operator. Section 4 analyzes a wide range of families of GPOWA operators. In Section 5, we give a general overview of the applicability of the GPOWA operator and we focus on the applicability in multi-person decision making. Section 6 presents an illustrative example and Section 7 summarizes the main conclusions of the paper.

2. **Preliminary Concepts.**

In this section, we briefly review some basic concepts regarding the probabilistic aggregation, the OWA operator, the GOWA operator and the POWA operator.

2.1. **Probabilistic aggregation.**

Probabilistic aggregation functions (or operators) are those functions that use probabilistic information in the aggregation process. Some examples are the aggregation with simple probabilities, the aggregation with belief structures [6,31], the concept of immediate probabilities [29,30,32] and the probabilistic OWA operator [28].

The POWA operator is an aggregation operator that provides a parameterized family of aggregation operators between the maximum and the minimum that unifies probabilities and OWAs in the same formulation [28]. Its main advantage is that it is able to include both concepts considering the degree of importance of each case in the problem. It is defined as follows.

Definition 2.1. *A POWA operator of dimension n is a mapping POWA: $R^n \to R$ that has an associated weighting vector W of dimension n such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:*

$$POWA(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{p}_j b_j, \tag{1}$$

where b_j is the jth largest of the a_i, each argument a_i has an associated probability p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, $\hat{p}_j = \beta w_j + (1-\beta)p_j$ with $\beta \in [0,1]$ and p_j is the probability p_i ordered according to the jth largest of the a_i.

2.2. **The OWA operator.** The OWA operator was introduced by Yager [5] and it provides a parameterized family of aggregation operators between the maximum and the minimum. It can be defined as follows.

Definition 2.2. *An OWA operator of dimension n is a mapping OWA: $R^n \to R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, such that:*

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{2}$$

where b_j is the jth largest of the a_i.

The OWA operator is a mean or averaging operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent. Different families of OWA operators can be used by choosing a different manifestation of the weighting vector [18,22-24,34].

2.3. **The GOWA operator.** The generalized OWA (GOWA) operator [25] is an aggregation that generalizes a wide range of aggregation operators that includes the OWA operator with its particular cases, the ordered weighted geometric (OWG) operator, the ordered weighted harmonic averaging (OWHA) operator and the ordered weighted quadratic averaging (OWQA) operator. It can be defined as follows.

Definition 2.3. *A GOWA operator of dimension n is a mapping GOWA: $R^n \to R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:*

$$GOWA(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}, \tag{3}$$

where b_j is the jth largest of the a_i, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

As it is demonstrated in [25], the GOWA operator is commutative, monotonic, bounded and idempotent. It can also be demonstrated that the GOWA operator has as special cases the maximum, the minimum, the generalized mean and weighted generalized mean. Other families of GOWA operators can be studied as shown in [16].

2.4. **The POWA operator.** The POWA operator is an aggregation operator that provides a parameterized family of aggregation operators between the maximum and the minimum that unifies probabilities and OWAs in the same formulation [28]. Its main advantage is that it is able to include both concepts considering the degree of importance of each case in the problem. It is defined as follows.

Definition 2.4. *A POWA operator of dimension n is a mapping POWA: $R^n \to R$ that has an associated weighting vector W of dimension n such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:*

$$POWA(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{p}_j b_j, \tag{4}$$

where b_j is the jth largest of the a_i, each argument a_i has an associated probability p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, $\hat{p}_j = \beta w_j + (1-\beta)p_j$ with $\beta \in [0,1]$ and p_j is the probability p_i ordered according to the jth largest of the a_i.

3. **The Generalized Probabilistic OWA Operator.** The generalized probabilistic OWA (GPOWA) operator is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum that uses probabilities and OWAs in the same formulation. Its main advantage is that it unifies these two concepts considering the degree of importance we want to give to each case depending on the specific problem considered. Moreover, it also uses generalized means providing a more complete representation that includes a wide range of particular cases. It can be defined as follows.

Definition 3.1. *A GPOWA operator of dimension n is a mapping GPOWA: $R^n \to R$ that has an associated weighting vector W of dimension n with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:*

$$GPOWA(a_1, \ldots, a_n) = \beta \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} + (1-\beta) \left(\sum_{i=1}^{n} p_i a_i^{\delta} \right)^{1/\delta}, \tag{5}$$

where b_j is the jth largest a_i, $\beta \in [0,1]$, and λ and δ are parameters such that λ and $\delta \in (-\infty, \infty) - \{0\}$.

Note that it is possible to distinguish between the descending GPOWA (DGPOWA) and the ascending GPOWA (AGPOWA) operator by using $w_j = w_{n-j+1}$, where w_j is the jth weight of the DGPOWA and w^*_{n-j+1} the jth weight of the AGPOWA operator. Note that this reordering is in the OWA aggregation. However, it is possible to consider a more general reordering process by using $\hat{p}_j = \hat{p}_{n-j+1}$. In this case, we consider descending and ascending orders in the OWA and in the probabilistic aggregation. Another interesting transformation can be developed [34] by using $w^*_i = (1 + w_i)/(m - 1)$. Furthermore, we can also analyze situations with buoyancy measures [34]. In this case, we assume that $w_i \geq w_j$, for $i < j$. Note that it is also possible to consider a stronger case known as buoyancy measure extensive where $w_i > w_j$, for $i < j$. Additionally, we can also consider the contrary case, that is, $w_i \leq w_j$, for $i < j$, and the extensive measure $w_i < w_j$, for $i < j$. The GPOWA operator is monotonic, bounded and idempotent.

Another interesting issue to analyze are the measures for characterizing the weighting vector W. Following a similar methodology as it has been developed for the OWA operator

[5,25] we could formulate the orness measure (attitudinal character) as follows:

$$\alpha(\hat{P}) = \beta\left(\sum_{j=1}^{n} w_j\left(\frac{n-j}{n-1}\right)^{\lambda}\right)^{1/\lambda} + (1-\beta)\left(\sum_{j=1}^{n} p_j\left(\frac{n-j}{n-1}\right)^{\delta}\right)^{1/\delta}. \quad (6)$$

where p_j represents the probabilistic weights reordered according to the values of the arguments b_j. Note that if $\beta = 1$, we get the orness measure of Yager [25] and if $\beta = 0$, the orness measure of the generalized probabilistic aggregation.

Note that other measures such as the entropy of dispersion, the divergence of W and the balance operator could be used in the analysis [24].

4. **Families of GPOWA Operators.** A further interesting issue is the analysis of different families of GPOWA operators by analyzing particular cases in the coefficient β, in the parameter λ and in the weighting vector W. If we analyze the coefficient β, we get the following:

- If $\beta = 1$, we get the GOWA operator.
- If $\beta = 0$, we get the generalized probabilistic approach.

The more β approaches to 1, the more importance we give to the GOWA operator, and vice versa. If we analyze different values of the parameter λ, we obtain another group of particular cases such as the usual POWA operator, the geometric POWA (POWGA) operator, the harmonic POWA (POWHA) operator and the quadratic POWA (POWQA) operator.

Remark 4.1. *When $\lambda = 1$, the GPOWA operator becomes the POWA operator.*

$$GPOWA(a_1, a_2, \ldots, a_n) = \beta\sum_{j=1}^{n} w_j b_j + (1-\beta)\sum_{i=1}^{n} p_i a_i. \quad (7)$$

Note that if $w_j = 1/n$, for all a_i, we get the arithmetic probabilistic aggregation (A-PA). Note also that if $p_i = 1/n$, for all a_i, we get the arithmetic OWA (A-OWA) operator.

Remark 4.2. *When $\lambda \to 0$, the GPOWA operator becomes the geometric probabilistic ordered weighted geometric averaging (GPOWGA) operator.*

$$GPOWA(a_1, a_2, \ldots, a_n) = \beta\prod_{j=1}^{n} b_j^{w_j} + (1-\beta)\prod_{i=1}^{n} a_i^{p_i}. \quad (8)$$

Note that if $w_j = 1/n$, for all a_i, we get the geometric probabilistic geometric aggregation (G-PGA). Note also that if $p_i = 1/n$, for all a_i, we get the geometric probability OWGA (G-OWGA).

Remark 4.3. *When $\lambda = -1$, we get the harmonic probabilistic ordered weighted harmonic averaging (GPOWHA) operator.*

$$GPOWA(a_1, a_2, \ldots, a_n) = \beta\frac{1}{\sum_{j=1}^{n}\frac{w_j}{b_j}} + (1-\beta)\frac{1}{\sum_{i=1}^{n}\frac{p_i}{a_i}}. \quad (9)$$

If $w_j = 1/n$, for all a_i, we get the harmonic probabilistic harmonic aggregation (H-PHA). Note also that if $p_i = 1/n$, for all a_i, we get the harmonic probability OWHA (H-OWHA) operator.

Remark 4.4. *When $\lambda = 2$, we get the quadratic probabilistic ordered weighted quadratic averaging (QPOWQA) operator.*

$$GPOWA(a_1, a_2, \ldots, a_n) = \beta \left(\sum_{j=1}^{n} w_j b_j^2 \right)^{1/2} + (1-\beta) \left(\sum_{i=1}^{n} p_i a_i^2 \right)^{1/2}. \quad (10)$$

Note that if $w_j = 1/n$, for all a_i, we get the quadratic probabilistic quadratic aggregation (Q-PQA). Note also that if $p_i = 1/n$, for all a_i, we get the quadratic probability OWQA operator.

Remark 4.5. *When $\lambda = 3$, we get the cubic probabilistic ordered weighted cubic averaging (CPOWCA) operator.*

$$GPOWA(a_1, a_2, \ldots, a_n) = \beta \left(\sum_{j=1}^{n} w_j b_j^3 \right)^{1/3} + (1-\beta) \left(\sum_{i=1}^{n} p_i a_i^3 \right)^{1/3}. \quad (11)$$

Note that if $w_j = 1/n$, for all a_i, we get the cubic probabilistic cubic aggregation (C-PCA). Note also that if $p_i = 1/n$, for all a_i, we get the cubic probability OWCA operator.

Remark 4.6. *When $\lambda \to \infty$ and $\delta \to \infty$, we get the maximum.*

Remark 4.7. *When $\lambda \to -\infty$ and $\delta \to -\infty$, we get the minimum.*

Remark 4.8. *Moreover, we can use different values in λ and δ. For example, if $\lambda = 2$ and $\delta = 3$, we form the probabilistic cubic ordered weighted quadratic averaging (PCOWQA) operator.*

$$GPOWA(a_1, a_2, \ldots, a_n) = \beta \left(\sum_{j=1}^{n} w_j b_j^2 \right)^{1/2} + (1-\beta) \left(\sum_{i=1}^{n} p_i a_i^3 \right)^{1/3}. \quad (12)$$

And if we look to the weighting vector W, we get, for example, the following ones:

- The probabilistic maximum ($w_1 = 1$ and $w_j = 0$, for all $j \neq 1$).
- The probabilistic minimum ($w_n = 1$ and $w_j = 0$, for all $j \neq n$).
- The generalized mean (GM) ($w_j = 1/n$, and $p_i = 1/n$, for all a_i).
- The arithmetic GOWA (A-GOWA) ($p_i = 1/n$, for all a_i).
- The arithmetic generalized probabilistic aggregation (A-GPA) ($w_j = 1/n$, for all a_i).
- The step-GPOWA ($w_k = 1$ and $w_j = 0$, for all $j \neq k$).
- The general olympic-GPOWA operator ($w_j = 0$ for $j = 1, 2, \ldots, k, n, n-1, \ldots, n-k+1$; and for all others $w_{j^*} = 1/(n-2k)$, where $k < n/2$).
- The centered-GPOWA (if it is symmetric, strongly decaying from the center to the maximum and the minimum, and inclusive).

Note that other families of GPOWA operators could be studied following Merigó and Gil-Lafuente [24] and Yager [25,34].

The GPOWA operator can be further generalized by using quasi-arithmetic means [19,24] forming the Quasi-POWA operator. It can be defined as follows.

Definition 4.1. *A Quasi-POWA operator of dimension n is a mapping $QPOWA: R^n \to R$ that has an associated weighting vector W of dimension n with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:*

$$QPOWA(a_1, \ldots, a_n) = \beta g^{-1} \left(\sum_{j=1}^{n} w_j g(b_j) \right) + (1-\beta) h^{-1} \left(\sum_{i=1}^{n} p_i h(a_i) \right), \quad (13)$$

where b_j is the jth largest a_i, $\beta \in [0,1]$, and $g(b)$ and $h(a)$ are strictly continuous monotonic functions.

Note that all the properties and particular cases commented in the GPOWA operator are also included in this generalization. Thus, we could study a wide range of particular cases as it has been done previously in this section.

5. **Group Decision Making with the GPOWA Operator.** In this Section we study the applicability of the GPOWA operator focussing on a multi-person decision making problem in strategic management.

5.1. **Applicability of the GPOWA operator.** The GPOWA operator can be applied in a wide range of disciplines because all the studies that use the probability or the OWA operator can be revised and extended with this new approach. The reason is that we can always reduce it to the classical case where we only use probabilities or OWA operators. Thus, all disciplines that use these types of statistical techniques can be revised with this new approach [35,36]. For example, we could mention statistics, economics, engineering, business, physics, biology, chemistry and medicine. Focusing in statistics, we could develop a new variance and covariance formula with the GPOWA operator as follows. For the variance, we obtain the following equation:

$$Var\text{-}GPOWA(X) = \beta\left(\sum_{j=1}^{n} w_j D_j^{\lambda}\right)^{1/\lambda} + (1-\beta)\left(\sum_{i=1}^{n} p_i\left((x_i-\mu)^2\right)^{\delta}\right)^{1/\delta}, \quad (14)$$

where D_j is the jth largest of the $(x_i - \mu)^2$, x_i is the argument variable, μ is the average (in this case, the GPOWA operator), $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, each argument $(x_i - \mu)^2$ has an associated probability (PA) p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, $\beta \in [0,1]$, and λ and δ are parameters such that λ and $\delta \in (-\infty,\infty) - \{0\}$.

In a similar way, we can represent the covariance by using the GPOWA operator as follows:

$$Cov\text{-}GPOWA(X,Y) = \beta\left(\sum_{j=1}^{n} w_j K_j^{\lambda}\right)^{1/\lambda} + (1-\beta)\left(\sum_{i=1}^{n} p_i[(x_i-\mu)(y_i-\nu)]^{\delta}\right)^{1/\delta}, \quad (15)$$

where K_j is the jth largest of the $(x_i - \mu)(y_i - \nu)$, x_i is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$ and y_i the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$, μ and ν are the average (in this case, the GPOWA operator) of the sets X and Y, respectively, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, each argument $(x_i - \mu)(y_i - \nu)$ has an associated probability (PA) p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, $\beta \in [0,1]$, and λ and δ are parameters such that λ and $\delta \in (-\infty,\infty) - \{0\}$.

Furthermore, we can formulate a linear regression process using the GPOWA operator. To construct the linear regression model $y_h = \alpha + \beta x_h$, we calculate β as follows:

$$\hat{\beta}_{GPOWA} = \frac{Cov\text{-}GPOWA(X,Y)}{Var\text{-}GPOWA(X)}. \quad (16)$$

Next, we calculate the $\hat{\alpha}_{GPOWA}$ value as follows: $\hat{\alpha}_{GPOWA} = \bar{y}_{GPOWA} - \beta_{GPOWA}\bar{x}_{GPOWA}$, where $\bar{x}_{GPOWA}$ and $\bar{y}_{GPOWA}$ are the average of the sets X and Y calculated by using a GPOWA operator. Once we have $\hat{\alpha}_{GPOWA}$ and $\hat{\beta}_{GPOWA}$, we can construct the linear regression model with the GPOWA operator as follows:

$$y_h = \hat{\alpha}_{GPOWA} + \hat{\beta}_{GPOWA} x_h. \quad (17)$$

Note that it is straightforward to extend this approach to a multiple linear regression model based on the use of the GPOWA operator [37].

5.2. **Multi-person decision making approach.** In the following, we are going to analyze the use of the GPOWA operator in multi-person problems. We are going to analyze a multi-person decision making problem in the selection of strategies. We focus on the selection of national strategies at a general level, that is, on the main directions that the government wants to follow in order to accomplish its objectives. Since this involves decisions that are very complex, the input of many experts is required to assess the problem. Thus, by using a multi-person analysis we can assess the information in the most efficient way.

The procedure to select national strategies with the GPOWA operator in multi-person decision-making is described in this section. Note that many other group decision-making models have been discussed in the literature [38-47].

Step 1: Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of finite alternatives, $S = \{s_1, s_2, \ldots, s_n\}$, a set of finite states of nature (or attributes), forming the payoff matrix $(a_{hi})_{m\times n}$. Let $E = \{e_1, e_2, \ldots, e_p\}$ be a finite set of decision-makers. Let $U = (u_1, u_2, \ldots, u_p)$ be the weighting vector of the decision-makers such that $\sum_{k=1}^{q} u_k = 1$ and $u_k \in [0, 1]$. Each decision-maker provides his own payoff matrix $\left(a_{hi}^{(k)}\right)_{m\times n}$.

Step 2: Calculate the weighting vector $\hat{P} = \beta \times W + (1-\beta) \times P$ to be used in the GPOWA aggregation. Note that $W = (w_1, w_2, \ldots, w_n)$ such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$ and $P = (p_1, p_2, \ldots, p_q)$ such that $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$.

Step 3: Aggregate the information of the decision-makers E using the weighting vector U. In this example, we use the weighted average (WA). The result is the collective payoff matrix $(\tilde{a}_{hi})_{m\times n}$. Thus, $a_{hi} = \sum_{k=1}^{q} u_k a_{hi}^k$. Note that we can use different types of aggregation operators instead of the WA to aggregate this information such as different types of GPOWA operators in case we have some probabilistic information to weight the information.

Step 4: Calculate the aggregated results using the GPOWA operator explained in Equation (5). Consider different families of GPOWA operators as described in Section 4.

Step 5: Adopt decisions according to the results found in the previous steps. Select the alternative (s) that provides the best result (s) and establish a ranking of the alternatives.

The previous multi-person decision process can be summarized using the following aggregation operator that we call the multi-person – GPOWA (MP-GPOWA) operator.

Definition 5.1. *A MP-GPOWA operator is a mapping MP-GPOWA:* $R^n \times R^p \to R$ *that has a weighting vector* U *of dimension* q *with* $\sum_{k=1}^{q} u_k = 1$ *and* $u_k \in [0, 1]$ *and a weighting vector* W *of dimension* n *with* $\sum_{j=1}^{n} w_j = 1$ *and* $w_j \in [0, 1]$*, such that:*

$$MP\text{-}GPOWA((a_1^1, \ldots, a_1^q), \ldots, (a_n^1, \ldots, a_n^q)) = \sum_{j=1}^{n} \hat{p}_j b_j, \tag{18}$$

where b_j *is the* j*th largest of the* a_i*, each argument* a_i *has an associated probability* p_i *with* $\sum_{i=1}^{n} p_i = 1$ *and* $p_i \in [0, 1]$*,* $\hat{p}_j = \beta w_j + (1-\beta) p_j$ *with* $\beta \in [0, 1]$ *and* p_j *is the probability* p_i *ordered according to* b_j*, that is, according to the* j*th largest of the* a_i*,* $a_i = \sum_{k=1}^{q} u_k a_i^k$*,* a_i^k *is the argument variable provided by each person (or expert).*

Note that the MP-GPOWA operator has similar properties to those explained in Section 3, such as the distinction between descending and ascending orders.

The MP-GPOWA operator includes a wide range of particular cases following the methodology explained in Section 4. Thus, it includes the multi-person – probabilistic aggregation (MP-PA) operator, the multi-person – OWA (MP-OWA) operator, the multi-person – arithmetic mean (MP-AM) operator, the multi-person – generalized probabilistic aggregation (MP-GPA), the multi-person – GOWA (MP-GOWA), the multi-person – generalized mean (MP-GM), the multi-person – arithmetic-GPA (MP-AGPA) operator and the multi-person – arithmetic-GOWA (MP-AGOWA) operator.

6. **Illustrative Example.** In the following, we present a numerical example of the new approach in a decision-making problem regarding the selection of national strategies. We analyze an economic problem regarding the general strategy of a country in national decision-making problems.

Step 1: Assume the government of a country has to decide on the type of general strategy to follow the next year. They consider five alternatives:

- A_1 = Develop strategy A.
- A_2 = Develop strategy B.
- A_3 = Develop strategy C.
- A_4 = Develop strategy D.
- A_5 = Develop strategy E.

In order to evaluate these strategies, the government has brought together a group of experts. This group considers that the key factor is the situation of the world economy for the next period. They consider 5 possible states of nature that could happen in the future:

- S_1 = Very bad economic situation.
- S_2 = Bad economic situation.
- S_3 = Regular economic situation.
- S_4 = Good economic situation.
- S_5 = Very good economic situation.

The experts are classified in 3 groups. Each group is led by one expert and gives different opinions than the other two groups. The results of the available strategies, depending on the state of nature S_i and the alternative A_k that the decision-maker chooses, are shown in Tables 1-3.

Step 2: In this problem, we assume the following weighting vector for the three groups of experts: $U = (0.5, 0.3, 0.2)$. The experts assume the following weighting vector for the OWA: $W = (0.1, 0.2, 0.2, 0.2, 0.3)$. The three groups assume the following probabilistic

TABLE 1. Group of experts 1

	S_1	S_2	S_3	S_4	S_5
A_1	60	40	70	80	20
A_2	30	70	50	60	80
A_3	60	80	70	40	30
A_4	40	60	70	50	60

TABLE 2. Group of experts 2

	S_1	S_2	S_3	S_4	S_5
A_1	40	80	60	50	70
A_2	40	70	60	60	70
A_3	50	70	70	40	40
A_4	40	60	50	50	80

Table 3. Group of experts 3

	S_1	S_2	S_3	S_4	S_5
A_1	40	70	60	80	30
A_2	50	70	60	60	80
A_3	40	70	70	60	30
A_4	40	60	80	50	60

Table 4. Collective results

	S_1	S_2	S_3	S_4	S_5
A_1	48	61	64	71	38
A_2	39	70	56	60	77
A_3	51	74	70	46	33
A_4	40	60	67	50	66

Table 5. Aggregated results

	AM	PA	OWA	POWA	A-PA	A-OWA
A_1	56.4	58	53.1	56.53	57.52	55.41
A_2	60.4	59.3	56.6	58.49	59.63	59.26
A_3	54.8	61.3	50.7	58.12	59.35	53.57
A_4	56.6	57.7	53.9	56.56	57.37	55.79

Table 6. Ranking of the national strategies

	Ordering		Ordering
AM	$A_2 \succ A_4 \succ A_1 \succ A_3$	POWA	$A_2 \succ A_3 \succ A_4 \succ A_1$
PA	$A_3 \succ A_2 \succ A_1 \succ A_4$	A-PA	$A_2 \succ A_3 \succ A_1 \succ A_4$
OWA	$A_2 \succ A_4 \succ A_1 \succ A_3$	A-OWA	$A_2 \succ A_4 \succ A_1 \succ A_3$

information for each state of nature: $P = (0.2, 0.3, 0.3, 0.1, 0.1)$. First, we aggregate the information of the three groups into one collective matrix that represents the information of all the experts of the problem. The results are shown in Table 4.

Step 3: Next, we calculate the attitudinal weights by mixing the weighting vectors W and P. Note that the OWA operator has an importance of 30% while the probabilistic one has 70% in this particular example.

Step 4: With this information, we can aggregate the expected results for each state of nature in order to make a decision. For this, we use Equation (4) or (5) to calculate the GPOWA aggregation. In Table 5, we present the results obtained using different types of GPOWA operators. Obviously, we get the same results with both methods.

Step 5: If we establish a ranking of the alternatives, then we get the results shown in Table 6. Note that the first alternative in each ordering is the optimal choice.

Obviously, the order preference for the national strategy to follow may be different depending on the aggregation operator used. Therefore, the decision about which strategy to select may be also different.

7. **Conclusions.**

We have presented the GPOWA operator. It is an aggregation operator that unifies the probability and the OWA operator in the same formulation and considering the degree of importance that each concept has in the aggregation. Moreover, it also uses generalized aggregation operators providing a general formulation that includes a wide range of particular cases including the POWA, the POWGA, the POWQA and the POWHA operator. We have studied some of its main properties and we have seen that the generalized probabilistic aggregation and the GOWA operator are particular cases of this generalization. Furthermore, the simple probabilistic aggregation, the OWA and the POWA operator are also particular types of GPOWA operators.

We have extended this analysis to multi-person decision making problems obtaining the MP-GPOWA operator. This operator provides a more general representation of the information because it permits us to deal with the opinion of several persons (elements) in the analysis. We have seen a wide range of particular cases of the MP-GPOWA operator including the MP-AM, the MP-PA, the MP-OWA, the MP-PGA, the MP-GOWA and the MP-POWA operator. We have applied this new approach in a decision making problem regarding the selection of strategies. We have seen that depending on the particular type of aggregation operator used, the results may lead to different decisions. Furthermore, we have seen how to deal with probabilistic information and with the attitudinal character of the decision maker in the same formulation.

In future research we expect to develop further extensions to this approach by adding more concepts in the GPOWA and MP-GPOWA operators such as the use of distance measures, uncertain information represented in the form of interval numbers, fuzzy numbers or linguistic variables. We will also consider other applications in other areas giving special attention in other statistical problems and in other decision making problems including political and juridical decision making.

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