International Journal of Innovative Computing, Information and Control
Volume **8**, Number **12**, December **2012**

ICIC International ©2012 ISSN 1349-4198
pp. 8049–8062

FAULT DETECTION FOR UNCERTAIN DELAYED SWITCHING DISCRETE-TIME SYSTEMS

Abdellah Benzaouia[1], Mustapha Ouladsine[2], Aziz Naamane[2] and Bouchra Ananou[2]

[1]LAEPT-URAC 28, Department of Physics
Faculty of Sciences Semlalia
University Cadi Ayyad
B.P. 2390, Marrakech 40000, Morocco
benzaouia@ucam.ac.ma

[2]LSIS-UMR 6168, University of Paul Cézanne-Aix-Marseille
Domaine Universitaire de Saint-Jérôme
Avenue Escadrille Normandie-Niemen
13397 Marseille Cedex 20, France
{ mustapha.ouladsine; aziz.naamane; bouchra.ananou }@lsis.org

Received July 2011; revised November 2011

Abstract. *In this paper, fault detection problem for uncertain discrete-time switching systems with delay is studied. Sufficient conditions of building an observer are obtained by using multiple Lyapunov function. These conditions are worked out in a new way to obtain new LMIs with slack variables and multiple weighted residual matrices. The obtained results are applied on a numerical example with all the entries of the same sizes.*
Keywords: Switching systems, Delay, Uncertain systems, Arbitrary switching sequence, Fault detection, Observer, Multiple Lyapunov function, LMI

1. **Introduction.** Switched systems are a class of hybrid systems encountered in many practical situations which involve switching between several subsystems depending on various factors. Generally, a switching system consists of a family of continuous-time subsystems and a rule that supervises the switching between them. For example, many processes in the chemical and pharmaceutical industries operate following batches, composed of different operations that are carried out in sequence. This changes discontinuously the dynamics of the operation [20]. In manufacturing, hybrid switched systems are found in steel rolling mills [15], used for producing thin metal sheets, following several steps based on pressing the metal strip with rolling cylinders: the dynamics are known to change at each pass due to the variation in thickness [11]. Many other examples can be found in the automotive industry, in aircraft and air traffic control, and many other fields.

Two main problems are widely studied in the literature according to the classification given in [8, 10]: the first one, which is the one solved in this work, looks for testable conditions that guarantee the asymptotic stability of a switching system under arbitrary switching rules [3, 4, 6, 7, 9, 12], while the second is to determine a switching sequence that renders the switched system asymptotically stable (see [16] and the reference therein).

A main problem which is always inherent to all dynamical systems is the possibility of the presence of faults. Fault detection and isolation (FDI) techniques present today an important topic in systems engineering for improving the system reliability. A fault represents any kind of malfunction in a plant, as actuator faults or sensor faults, that leads to the degradation of the performances of the overall system behavior. Without taking into account, by detection and isolation, this fault, the degradation may turn into instability of the system. Even for linear systems, this problem has been an active area of research for many years [1, 14, 17, 19, 22, 25, 26, 27, 28]. The study of this problem was extended to switching systems in [2, 18, 23, 24]. In [23], a switching discrete-time uncertain system with state delays is considered. The design method is based on the construction of a filter and a fault estimation. In the present work, the same class of systems as in [23] is studied but with a Luenberger observer. The obtained condition of asymptotic stability in presence of fault based on the H_∞ technique is then worked out in a simple way to obtain new LMIs. The proposed LMIs are different from the ones obtained in [18].

Besides the introduction of slack variables G_i, the contribution of this work consists in introducing a new residual signal with multiple matrices V_i and a new evaluation function based on a past receding window. The applicability of the obtained results on a numerical example shows the usefulness of the observer which can work with success even in presence of input control, bounded unknown perturbation and fault of the same sizes while filter based approaches may not succeed in this case.

This paper is organized as follows: Section 2 deals with the problem statement while Section 3 presents some preliminary results on fault detection problem. The main results of this paper are developed in Section 4 together with an illustrative example. Section 5 presents the conclusion of the paper.

2. **Problem Formulation.**

Consider the following delayed discrete-time switching system:

$$\begin{aligned} x(k+1) &= \mathcal{A}_\alpha x(k) + \mathcal{A}_{d\alpha} x(k-\tau) + \mathcal{B}_\alpha u(k) + E_\alpha d(k) + F_\alpha f(k), \\ y(k) &= C_\alpha x(k) + C_{d\alpha} x(k-\tau) + D_\alpha u(k) + N_\alpha d(k) + M_\alpha f(k), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $y \in \mathbb{R}^p$ is the output, τ is the delay, $d \in \mathbb{R}^g$ is an external unknown input, $f \in \mathbb{R}^q$ is the fault and α a switching rule which takes its values in the finite set $\mathcal{I} := \{1, \ldots, N\}$, $k \in \mathbb{Z}_+$. Each subsystem α is called a mode. Matrices $\mathcal{A}_i, \mathcal{A}_{di}, \mathcal{B}_i$ have the same following structure: $\mathcal{A}_i = A_i + \Delta A_i(k)$, $\mathcal{A}_{di} = A_{di} + \Delta A_{di}(k)$, $\mathcal{B}_i = B_i + \Delta B_i(k)$, with the uncertainty terms are norm-bounded according to:

$$[\Delta A_i(k) \;\; \Delta A_{di}(k) \;\; \Delta B_i(k)] = L_i W(k) [H_{1i} \;\; H_{2i} \;\; H_{3i}]. \tag{2}$$

Matrices A_i, A_{d_i}, B_i, E_i, F_i, C_i, C_{d_i}, D_i, N_i, M_i, L_i, H_{1i}, H_{2i}, H_{3i} are of appropriate size constant known matrices. It is assumed that:

- Each time only one subsystem is active.
- The switching rule is not known *a priori* but $\alpha(k)$ is available at each sampling time k.
- $W(k)^T W(k) \leq \mathbb{I}$.
- The delay τ is known.

The second assumption corresponds to practical implementations where the switched system is supervised by a discrete-event system or operator allowing for $\alpha(k)$ to be known in real time. Upon introducing the indicator function:

$$\xi(k) = [\xi_1(k), \ldots, \xi_N(k)]^T \tag{3}$$

where $\xi_i(k) = 1$ if the switching system is in mode i and $\xi_i(k) = 0$ if it is in a different mode, one can write the switching system (1) as follows:

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{N} \xi_i(k)[\mathcal{A}_i x(k) + \mathcal{A}_{di} x(k-\tau) + \mathcal{B}_i u(k) + E_i d(k) + F_i f(k)], \\ y(k) &= \sum_{i=1}^{N} \xi_i(k)[C_i x(k) + C_{di} x(k-\tau) + D_i u(k) + N_i d(k) + M_i f(k)]. \end{aligned} \tag{4}$$

In this work, we are interested by the synthesis of an observer of this class of systems in order to detect faults when they occur in the switching system. For this, consider the following switching observer:

$$\begin{aligned} \hat{x}(k+1) &= \sum_{i=1}^{N} \xi_i(k)[A_i \hat{x}(k) + A_{di} \hat{x}(k-\tau) + B_i u(k) + K_i(y(k) - \hat{y}(k))]. \\ \hat{y}(k) &= \sum_{i=1}^{N} \xi_i(k)[C_i \hat{x}(k) + C_{di} \hat{x}(k-\tau) + D_i u(k)] \end{aligned} \tag{5}$$

In this structure, only matrices K_i are to be designed.

Remark 2.1. *As assumed in the second assumption, the switching rule is not known* a priori *but $\alpha(k)$ is available at each sampling time k. This means that one can synchronize the switch of the observer with the switch of the system. In this case, the problem to have the system and the observer evolving in different modes cannot occur as it arises in the continuous case studied in* [2].

The residual of this observer is defined as:

$$r(k) = \sum_{i=1}^{N} \xi_i(k) V_i (y(k) - \hat{y}(k)) \tag{6}$$

Matrices V_i are to be computed. It is worth noting that the proposed structure of the residual is different from the classical one used in [2] where only one matrix V is looked for. Defining the observer error by $e_k = x_k - \hat{x}_k$ leads to:

$$\begin{aligned} e_{k+1} = \sum_{i=1}^{N} \xi_i(k)[&(A_i - K_i C_i) e_k + (A_{di} - K_i C_{di}) e_{k-\tau} \\ &+ (E_i - K_i N_i) d_k + (F_i - K_i M_i) f_k + \Delta A_i(k) x_k + \Delta A_{di}(k) x_{k-\tau} + \Delta B_i(k) u_k]. \end{aligned} \tag{7}$$

Define an augmented state vector and an augmented input vector as:

$$\tilde{x}_k = \begin{bmatrix} x_k \\ \hat{x}_k \\ e_k \end{bmatrix}, \quad w_k = \begin{bmatrix} u_k \\ d_k \\ f_k \end{bmatrix}. \tag{8}$$

The corresponding dynamical system is then derived.

$$\tilde{x}_{k+1} = \sum_{i=1}^{N} \xi_i(k) \left[\tilde{\mathcal{A}}_i \tilde{x}_k + \tilde{\mathcal{A}}_{di} \tilde{x}_{k-\tau} + \tilde{\mathcal{B}}_i w_k \right], \tag{9}$$

$$r_k = \sum_{i=1}^{N} \xi_i(k) \left[\tilde{C}_i \tilde{x}_k + \tilde{C}_{di} \tilde{x}_{k-\tau} + \tilde{D}_i w(k) \right], \tag{10}$$

where $\tilde{\mathcal{A}}_i = \tilde{A}_i + \Delta\tilde{A}_i(k)$, $\tilde{\mathcal{A}}_{di} = \tilde{A}_{di} + \Delta\tilde{A}_{di}(k)$ and $\tilde{\mathcal{B}}_i = \tilde{B}_i + \Delta\tilde{B}_i(k)$, with the following matrices:

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 & 0 \\ 0 & A_i & K_iC_i \\ 0 & 0 & A_i - K_iC_i \end{bmatrix}, \quad \tilde{A}_{di} = \begin{bmatrix} A_{di} & 0 & 0 \\ 0 & A_{di} & K_iC_{di} \\ 0 & 0 & A_{di} - K_iC_{di} \end{bmatrix},$$
$$\tilde{B}_i = \begin{bmatrix} B_i & E_i & F_i \\ B_i & -K_iN_i & -K_iM_i \\ 0 & E_i - K_iN_i & F_i - K_iM_i \end{bmatrix}, \tag{11}$$

$$\Delta\tilde{A}_i(k) = \begin{bmatrix} \Delta A_i(k) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta A_i(k) \end{bmatrix}, \quad \Delta\tilde{A}_{di}(k) = \begin{bmatrix} \Delta A_{di}(k) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta A_{di}(k) \end{bmatrix},$$
$$\Delta\tilde{B}_i(k) = \begin{bmatrix} \Delta B_i(k) & 0 & 0 \\ 0 & 0 & 0 \\ \Delta B_i(k) & 0 & 0 \end{bmatrix},$$
$$\tilde{C}_i = \begin{bmatrix} 0 & 0 & V_iC_i \end{bmatrix}, \quad \tilde{C}_{di} = \begin{bmatrix} 0 & 0 & V_iC_{di} \end{bmatrix},$$
$$\tilde{D}_i = \begin{bmatrix} 0 & V_iN_i & V_iM_i \end{bmatrix}. \tag{12}$$

The uncertain terms for the augmented system can be again developed as follows: $[\Delta\tilde{A}_i(k)\ \Delta\tilde{A}_{di}(k)\ \Delta\tilde{B}_i(k)] = \tilde{L}_i\tilde{W}[\tilde{H}_{1i}\ \ \tilde{H}_{2i}\ \tilde{H}_{3i}]$, where

$$\tilde{L}_i = \begin{bmatrix} L_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L_i \end{bmatrix}, \quad \tilde{H}_{3i} = \begin{bmatrix} H_{3i} & 0 & 0 \\ 0 & 0 & 0 \\ H_{3i} & 0 & 0 \end{bmatrix}, \quad \tilde{H}_{si} = \begin{bmatrix} H_{si} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & H_{si} \end{bmatrix},$$
$$\tilde{W}(k) = diag\{W(k), W(k), W(k)\}, \quad s = 1, 2. \tag{13}$$

The objective of this work is to develop a synthesis method based on the H_∞ tool to compute the unknown matrices representing the switching observer gains K_i and the residual gains V_i. One can note that the way followed in this paper is different from the one used in [23] where, a filter is completely characterized, while assuming that the dynamic of the weighted fault is known. With the use of a Luenberger switching observer, the number of unknown matrices is lower in our case.

3. **Preliminary Results.** As used in the literature of fault detection [2, 23, 28], the identification of the fault f_k is not necessary. One can use the following residual criterion:

$$J_k = \left(\sum_{i=k_0}^{k_0+k} r_i^T r_i \right)^{1/2}, \tag{14}$$

where k_0 and k define the interval of the evaluation window, k being the current sampled time. The size of this window is increasing until it is equal to the global horizon of observation. Then, this evaluation function can be compared to a threshold J_0 to conclude if a fault occurs or not, as follows:

$$J_k > J_0 \Rightarrow \text{Faults} \Rightarrow \text{Alarm}$$
$$J_k \leq J_0 \Rightarrow \text{No Faults}$$

The threshold criterion can be chosen as indicated in [23] as:

$$J_0 = \sup_{d \in l_2, u \in l_2, f=0, k} J_k \tag{15}$$

Nevertheless, another evaluation function based on a past receding window can be defined and used instead of the one of (14) as:

$$J_r = \left(\sum_{i=k-T}^{k} r_i^T r_i \right)^{1/2}, \quad k > T, \tag{16}$$

where T is the fixed window size and k is the current sampled time.

Now, the separation lemma is recalled.

Lemma 3.1. [21] *Given symmetric matrix S and matrices L, $W(k)$ and H of appropriate size, then*

$$S + L^T W(k) H + H^T W^T(k) L < 0,$$

holds for $W(k)^T W(k) \leq \mathbb{I}$ if and only if there exists a scalar $\epsilon > 0$ such that

$$S + \epsilon^{-1} L^T L + \epsilon H^T H < 0.$$

The technique of H_∞ problem for the augmented switching system (9) consists in ensuring the asymptotic stability of the system for $w_k = 0$ and $\tilde{x}_0 \neq 0$ while realizing the following condition for $w_k \neq 0$ and $\tilde{x}_0 = 0$:

$$\sup_{w_k \neq 0, w_k \in l_2[0,\infty)} \frac{\sqrt{r_k^T r_k}}{\sqrt{w_k^T w_k}} < \gamma, \gamma > 0. \tag{17}$$

Remark 3.1. *In the H_∞ theory, the input w_k is assumed to be only bounded without any knowledge of its bound. In our case, w_k contains the control u_k, the fault f_k and the perturbation d_k. No additional assumption is required on these inputs. The idea is to reduce the impact of this exogenous input w_k on the system by reducing as far as possible the scalar γ defined by (17).*

Condition (17) is realized if

$$J(\gamma) = \sum_{k=0}^{T_k - 1} [r_k^T r_k - \gamma^2 w_k^T w_k] < 0. \tag{18}$$

To realize condition (18), one has to use a Lyapunov function $V(\tilde{x})$ and look for the condition realizing

$$\begin{aligned} J(\gamma) &= \sum_{k=0}^{T_k - 1} [r_k^T r_k - \gamma^2 w_k^T w_k + \Delta V(\tilde{x})] - V(\tilde{x}(T_k)), \\ &\leq \sum_{k=0}^{T_k - 1} [r_k^T r_k - \gamma^2 w_k^T w_k + \Delta V(\tilde{x})] < 0 \end{aligned} \tag{19}$$

for any $w_k \in l_2[0,\infty)$ and with $\tilde{x}_0 = 0$. It is obvious that condition (19) is satisfied if $\Delta V(\tilde{x}) < 0$, that is the system is asymptotically stable for $w_k = 0$ and $\tilde{x}_0 \neq 0$.

The first result we recall is a sufficient condition of H_∞ fault detection for the augmented switching system (9) presented by [23].

Lemma 3.2. [23] *For a given scalar $\gamma > 0$, system (9) under arbitrary switching is asymptotically stable when $w_k = 0$ and under zero-initial conditions, guarantees the performance index (17) for all $w_k \in l_2[0,\infty)$, if there exists positive definite symmetric matrices P_i and*

Q_i, $i \in \mathcal{I}$, such that:

$$\Sigma = \begin{bmatrix} -P_j^{-1} & \tilde{\mathcal{A}}_i & \tilde{\mathcal{A}}_{di} & \tilde{\mathcal{B}}_i & 0 & 0 \\ * & -P_i & 0 & 0 & \tilde{C}_i^T & \mathbb{I} \\ * & * & -Q_s & 0 & \tilde{C}_{di}^T & 0 \\ * & * & * & -\gamma^2\mathbb{I} & \tilde{D}_i^T & 0 \\ * & * & * & * & -\mathbb{I} & 0 \\ * & * & * & * & * & -Q_i^{-1} \end{bmatrix} < 0, \quad \forall (i,j,s) \in \mathcal{I}^3, \tag{20}$$

where $*$ stands for the symmetrical term of the corresponding off-diagonal term.

Remark 3.2. *Even system (9) is given with similar notations as in* [23] *to have the possibility to use Lemma 3.2, the changes of variables (12) are completely different from those taken in* [23]*. Note that the proof of this result is based on the use of a multiple Lyapunov-Krasovskii functional given by:*

$$V(x_k) = \tilde{x}_k^T \left(\sum_{i=1}^{N} \xi_i(k+1) P_i \right) \tilde{x}_k + \sum_{s=k-\tau}^{k-1} \tilde{x}_s^T \left(\sum_{i=1}^{N} \xi_i(k) Q_i \right) \tilde{x}_s \tag{21}$$

With these preliminary results, we are now able to solve the problem of fault detection for switching discrete-time systems by using a switching observer.

4. Main Results.

The objective of this section is to work out Inequality (20) to obtain an LMI enabling one to synthesize the switching observer together with its corresponding residual, making possible fault detection of the delayed discrete-time switching system.

The first result we present is an equivalent lemma of Lemma 3.2 which takes account of the uncertainties by using the separation Lemma 3.1.

Lemma 4.1. *For a given scalar $\gamma > 0$, system (9) under arbitrary switching is asymptotically stable when $w_k = 0$ and under zero-initial conditions, guarantees the performance index (17) for all $w_k \in l_2[0, \infty)$, if there exists positive definite symmetric matrices P_i and Q_i, $i \in \mathcal{I}$ and positive scalars ϵ_i, such that:*

$$\begin{bmatrix} -P_j^{-1} & \tilde{A}_i & \tilde{A}_{di} & \tilde{B}_i & 0 & 0 & \tilde{L}_i^T \\ * & -\Gamma_{1i} & \Gamma_{4i} & \Gamma_{5i} & \tilde{C}_i^T & \mathbb{I} & 0 \\ * & * & -\Gamma_{2is} & \Gamma_{6i} & \tilde{C}_{di}^T & 0 & 0 \\ * & * & * & -\Gamma_{3i} & \tilde{D}_i^T & 0 & 0 \\ * & * & * & * & -\mathbb{I} & 0 & 0 \\ * & * & * & * & * & -Q_i^{-1} & 0 \\ * & * & * & * & * & * & -\epsilon_i\mathbb{I} \end{bmatrix} < 0, \quad \forall (i,j,s) \in \mathcal{I}^3, \tag{22}$$

where

$$\begin{aligned} \Gamma_{1i} &= P_i - \epsilon_i \tilde{H}_{1i}^T \tilde{H}_{1i} \\ \Gamma_{2is} &= Q_s - \epsilon_i \tilde{H}_{2i}^T \tilde{H}_{2i} \\ \Gamma_{3i} &= \gamma^2 \mathbb{I} - \epsilon_i \tilde{H}_{3i}^T \tilde{H}_{3i} \\ \Gamma_{4i} &= \epsilon_i \tilde{H}_{1i}^T \tilde{H}_{2i} \\ \Gamma_{5i} &= \epsilon_i \tilde{H}_{1i}^T \tilde{H}_{3i} \\ \Gamma_{6i} &= \epsilon_i \tilde{H}_{2i}^T \tilde{H}_{3i} \end{aligned} \tag{23}$$

Proof: Using the change of variables of the uncertain terms of the augmented system (13), one can rewrite matrix Σ as follows:

$$\Sigma = \Sigma_0 + \bar{L}_i^T \tilde{W}(k) \bar{H}_i + \bar{H}_i^T \tilde{W}^T(k) \bar{L}_i \tag{24}$$

with

$$\Sigma_0 = \begin{bmatrix} -P_j^{-1} & \tilde{A}_i & \tilde{A}_{di} & \tilde{B}_i & 0 & 0 \\ * & -P_i & 0 & 0 & \tilde{C}_i^T & \mathbb{I} \\ * & * & -Q_s & 0 & \tilde{C}_{di}^T & 0 \\ * & * & * & -\gamma^2\mathbb{I} & \tilde{D}_i^T & 0 \\ * & * & * & * & -\mathbb{I} & 0 \\ * & * & * & * & * & -Q_i^{-1} \end{bmatrix},$$

$$\bar{L}_i = \begin{bmatrix} \tilde{L}_i & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{H}_i = \begin{bmatrix} 0 & \tilde{H}_{1i} & \tilde{H}_{2i} & \tilde{H}_{3i} & 0 & 0 \end{bmatrix}. \tag{25}$$

Applying Lemma 3.1 for each subsystem and Schur complement, the Inequality (22) follows readily. In addition, it was shown in [5] that using the separation Lemma 3.1 with ϵ_i instead of ϵ leads to less conservative conditions. □

The following result is obtained by working out Inequality (22) to become an LMI by using a direct congruence transformation.

Lemma 4.2. *For a given scalar $\gamma > 0$, system (9) under arbitrary switching is asymptotically stable when $w_k = 0$ and under zero-initial conditions, guarantees the performance index (17) for all $w_k \in l_2[0,\infty)$, if there exist positive definite symmetric matrices P_i and Q_i, matrices G_i and positive scalars ϵ_i, $i \in \mathcal{I}$, such that:*

$$\begin{bmatrix} -G_i^T - G_i + P_j & G_i^T\tilde{A}_i & G_i^T\tilde{A}_{di} & G_i^T\tilde{B}_i & 0 & 0 & G_i^T\tilde{L}_i^T \\ * & -\Gamma_{1i} & \Gamma_{4i} & \Gamma_{5i} & \tilde{C}_i^T & \mathbb{I} & 0 \\ * & * & -\Gamma_{2is} & \Gamma_{6i} & \tilde{C}_{di}^T & 0 & 0 \\ * & * & * & -\Gamma_{3i} & \tilde{D}_i^T & 0 & 0 \\ * & * & * & * & -\mathbb{I} & 0 & 0 \\ * & * & * & * & * & -G_i^T - G_i + Q_i & 0 \\ * & * & * & * & * & * & -\epsilon_i\mathbb{I} \end{bmatrix} < 0,$$

$$\forall (i,j,s) \in \mathcal{I}^3, \tag{26}$$

where matrices Γ_* are given by (23).

Proof: Pre multiply Inequality (22) by $diag\left\{G_i^T, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}, G_i^T, \mathbb{I}\right\}$ and post multiply by $diag\left\{G_i, \mathbb{I}_i, \mathbb{I}_i, \mathbb{I}, \mathbb{I}, G_i, \mathbb{I}\right\}$ with matrix G_i is any nonsingular matrix, one can obtain:

$$\begin{bmatrix} -G_i^T P_j^{-1} G_i & G_i^T\tilde{A}_i & G_i^T\tilde{A}_{di} & G_i^T\tilde{B}_i & 0 & 0 & G_i^T\tilde{L}_i^T \\ * & -\Gamma_{1i} & \Gamma_{4i} & \Gamma_{5i} & \tilde{C}_i^T & \mathbb{I} & 0 \\ * & * & -\Gamma_{2is} & \Gamma_{6i} & \tilde{C}_{di}^T & 0 & 0 \\ * & * & * & -\Gamma_{3i} & \tilde{D}_i^T & 0 & 0 \\ * & * & * & * & -\mathbb{I} & 0 & 0 \\ * & * & * & * & * & -G_i^T Q_i^{-1} G_i & 0 \\ * & * & * & * & * & * & -\epsilon_i\mathbb{I} \end{bmatrix} < 0,$$

$$\forall (i,j,s) \in \mathcal{I}^3. \tag{27}$$

Using the following developments

$$(G_i^T - P_j)P_j^{-1}(G_i - P_j) = G_i^T P_j^{-1} G_i - G_i - G_i^T + P_j > 0. \tag{28}$$

Leads to

$$-G_i^T P_j^{-1} G_i < -G_i - G_i^T + P_j \tag{29}$$

$$-G_i^T Q_i^{-1} G_i < -G_i - G_i^T + Q_i. \tag{30}$$

Replacing (29) and (30) one obtains (26). □

Comment 4.1. *It is worth noting that if we restrict ourselves only to the observer problem for a switching system without delay, a different LMI was obtained in* [13, 18], *where the term* $(1,1)$ *of LMI (26) is* $-G_i^T - G_i + P_i^{-1}$. *This term was obtained with a non convincing proof based on a supposed small matrix* g_i *taken as* $G_i - P_i^{-1}$, *which is never guaranteed to be small. The proof presented in this work is direct and without any other assumption. Note also that the unknown matrices in our case are* P_i *and* Q_i.

Now we are able to present our main contribution with the following result which enables one to synthesize the observer and the residual.

Theorem 4.1. *For a given scalar* $\gamma > 0$, *system (9) under arbitrary switching is asymptotically stable when* $w_k = 0$ *and under zero-initial conditions, guarantees the performance index (17) for all* $w_k \in l_2[0,\infty)$, *if there exist matrices* $P_i = (P_{ls}^i)$, $Q_i = (Q_{ls}^i)$ $s,l = 1,2,3$, *matrices* X_i, Y_i, S_i, Z_i, V_i *and positive scalars* ϵ_i, $i \in \mathcal{I}$, *such that:*

$$\begin{bmatrix} -\Omega_1 & \Psi_a & \Psi_d & \Psi_b & 0 & 0 & \Omega_6^T \\ * & -\Gamma_{1i} & \Gamma_{4i} & \Gamma_{5i} & \Omega_3^T & \mathbb{I} & 0 \\ * & * & -\Gamma_{2is} & \Gamma_{6i} & \Omega_4^T & 0 & 0 \\ * & * & * & -\Gamma_{3i} & \Omega_5^T & 0 & 0 \\ * & * & * & * & -\mathbb{I} & 0 & 0 \\ * & * & * & * & * & -\Omega_2 & 0 \\ * & * & * & * & * & * & -\epsilon_i\mathbb{I} \end{bmatrix} < 0,$$

$$\forall (i,j,s) \in \mathcal{I}^3, \tag{31}$$

where matrices Ψ_*, Ω_*, Γ_* *are given by (33)-(35), with the observer gains* $K_i = Z_i^{-T} Y_i^T$.

Proof: In order to use the initial matrices of System (1) and Observer (5), matrix G_i is taken as $G_i = (G_{ls}^i)$, $s,l = 1,2,3$. By using (11)-(12), the terms $G_i^T \Gamma_*$ are then developed. The following terms are obtained: $G_{ls}^i K_i$, $s,l = 1,2,3$. To avoid redundancy in determination of K_i, the analysis of these terms leads to matrix G_i of the following form:

$$G_i = \begin{bmatrix} X_i & S_i & 0 \\ 0 & Z_i & 0 \\ 0 & 0 & Z_i \end{bmatrix}, \quad i \in \mathcal{I}. \tag{32}$$

With this choice, the following computations are obtained:

$$\begin{aligned} \Psi_a &= G_i^T \tilde{A}_i = \begin{bmatrix} X_i^T A_i & 0 & 0 \\ S_i^T A_i & Z_i^T A_i & Y_i^T C_i \\ 0 & 0 & Z_i^T A_i - Y_i^T C_i \end{bmatrix}, \\ \Psi_d &= G_i^T \tilde{A}_{di} = \begin{bmatrix} X_i^T A_{di} & 0 & 0 \\ S_i^T A_{di} & Z_i^T A_i & Y_i^T C_{di} \\ 0 & 0 & Z_i^T A_{di} - Y_i^T C_{di} \end{bmatrix}, \\ \Psi_b &= G_i^T \tilde{B}_i = \begin{bmatrix} X_i^T B_i & X_i^T E_i & X_i^T F_i \\ S_i^T B_i + Z_i^T B_i & S_i^T E_i - Y_i^T N_i & S_i^T F_i - Y_i^T N_i \\ 0 & Z_i^T E_i - Y_i^T N_i & Z_i^T F_i - Y_i^T M_i \end{bmatrix}, \end{aligned} \tag{33}$$

where $Y_i^T = Z_i^T K_i$. In addition, matrices P_i and Q_i are also taken as $P_i = (P_{ls}^i)$, $Q_i = (Q_{ls}^i)$ $s, l = 1, 2, 3$ leading to:

$$
\begin{aligned}
\Omega_1 &= \begin{bmatrix} X_i + X_i^T - P_{11}^j & S_i - P_{12}^j & -P_{13}^j \\ * & Z_i + Z_i^T - P_{22}^j & -P_{23}^j \\ * & * & Z_i + Z_i^T - P_{33}^j \end{bmatrix}, \\
\Omega_2 &= \begin{bmatrix} X_i + X_i^T - Q_{11}^i & S_i - Q_{12}^i & -Q_{13}^i \\ * & Z_i + Z_i^T - Q_{22}^i & -Q_{23}^i \\ * & * & Z_i + Z_i^T - Q_{33}^i \end{bmatrix}, \\
\Omega_3 &= \begin{bmatrix} 0 & 0 & V_i C_i \end{bmatrix}, \quad \Omega_4 = \begin{bmatrix} 0 & 0 & V_i C_{di} \end{bmatrix}, \\
\Omega_5 &= \begin{bmatrix} 0 & V_i N_i & V_i M_i \end{bmatrix}, \\
\Omega_6^T &= \begin{bmatrix} X_i^T L_i & 0 \\ S_i^T L_i & 0 \\ 0 & Z_i^T L_i \end{bmatrix},
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\Gamma_{1i} &= \begin{bmatrix} P_{11}^i - \epsilon_i H_{1i}^T H_{1i} & P_{12}^i & P_{13}^i \\ * & P_{22}^i & P_{23}^i \\ * & * & P_{33}^i - \epsilon_i H_{1i}^T H_{1i} \end{bmatrix}, \\
\Gamma_{2is} &= \begin{bmatrix} Q_{11}^s - \epsilon_i H_{2i}^T H_{2i} & Q_{12}^s & Q_{13}^s \\ * & Q_{22}^s & Q_{23}^s \\ * & * & Q_{33}^s - \epsilon_i H_{2i}^T H_{2i} \end{bmatrix}, \\
\Gamma_{3i} &= \begin{bmatrix} \gamma^2 \mathbb{I} - 2\epsilon_i H_{3i}^T H_{3i} & 0 & 0 \\ 0 & \gamma^2 \mathbb{I} & 0 \\ 0 & 0 & \gamma^2 \mathbb{I} \end{bmatrix}, \\
\Gamma_{4i} &= \begin{bmatrix} \epsilon_i H_{1i}^T H_{2i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon_i H_{1i}^T H_{2i} \end{bmatrix}, \\
\Gamma_{5i} &= \begin{bmatrix} \epsilon_i H_{1i}^T H_{3i} & 0 & 0 \\ 0 & 0 & 0 \\ \epsilon_i H_{1i}^T H_{3i} & 0 & 0 \end{bmatrix}, \\
\Gamma_{6i} &= \begin{bmatrix} \epsilon_i H_{2i}^T H_{3i} & 0 & 0 \\ 0 & 0 & 0 \\ \epsilon_i H_{2i}^T H_{3i} & 0 & 0 \end{bmatrix}.
\end{aligned}
\tag{35}
$$

With all these developments, LMI (31) follows. Finally, according to the form of LMI (31), if matrices (P_{ls}^i), (Q_{ls}^i) $s, l = 1, 2, 3$ are solution of (31), then necessarily, the global matrices P_i and Q_i are definite positive. To obtain the LMI (31), the following property of the Schur complement is used:

$$
\begin{bmatrix} M & R^T \\ * & N \end{bmatrix} < 0
$$

is equivalent obviously, for $R = 0$, to $M < 0$, for any definite negative matrix N. $\square$

Example 4.1. *Consider the following numerical example similar to the one studied in* [23] *with modified default matrices. The used matrices are divided by* 10 *to be closed to the perturbation matrices. Besides, the perturbation* d_k *is taken of size* 100 *times the one of* [23].

Mode 1:

$$A_1 = \begin{bmatrix} 0.2 & -0.1 \\ 0 & 0.4 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0.13 \\ 0.16 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \quad C_{d1} = \begin{bmatrix} 0 & 0.1 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix}, \quad H_{11} = \begin{bmatrix} 0.1 & 0.01 \end{bmatrix}, \quad H_{21} = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix},$$

$$D_1 = 1, \quad N_1 = 1.1, \quad M_1 = 1.4, \quad H_{31} = 0.01.$$

Mode 2:

$$A_2 = \begin{bmatrix} 0.4 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0.15 \\ 0.12 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0.1 \end{bmatrix}, \quad C_{d2} = \begin{bmatrix} 0.1 & 0 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad H_{12} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad H_{22} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix},$$

$$D_2 = 1.1, \quad N_2 = 1.2, \quad M_2 = 1.5, \quad H_{32} = 0.1, \quad W(k) = \frac{k}{k+1}, \quad \tau = 1.$$

The LMI (31) is feasible for $\gamma = 1.2$. It is worth noting that each time LMI (31) is feasible, the corresponding one with common matrices $P_i = P$ and $Q_i = Q$ is also feasible for this example. The obtained observer and residual gains are given by:

$$K_1 = \begin{bmatrix} 0.0785 \\ 0.0575 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.1071 \\ 0.2347 \end{bmatrix}, \quad V_1 = 0.2640, \quad V_2 = 0.0763.$$

The input is an unit step while the fault is generated between $k = 40$ and $k = 60$ with an unit magnitude. The bounded unknown input is $d_k = \exp(-0.04k).\cos(0.03\pi k)$.

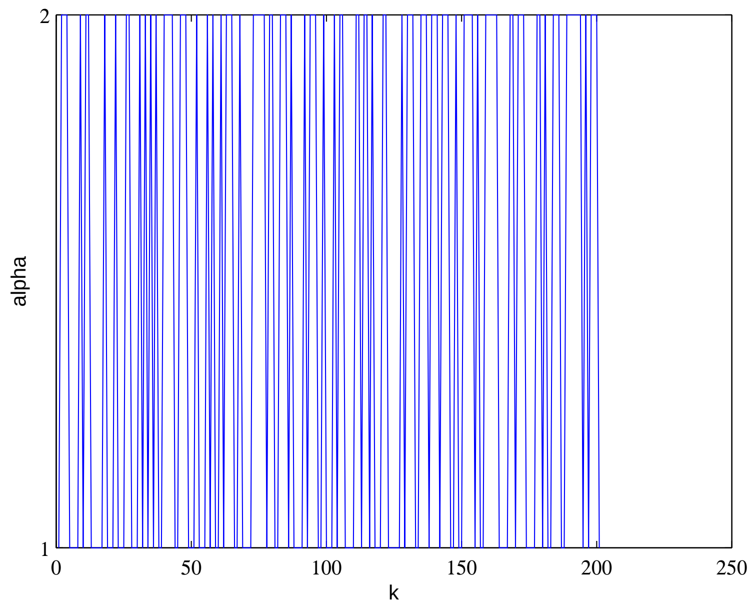

Figure 1. The evolution of the switching rule α

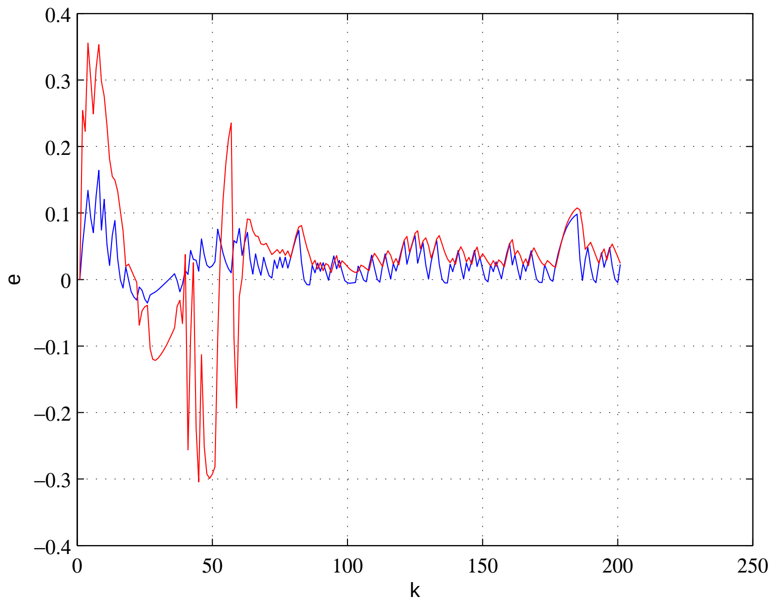

FIGURE 2. The evolution of the observer error e_k

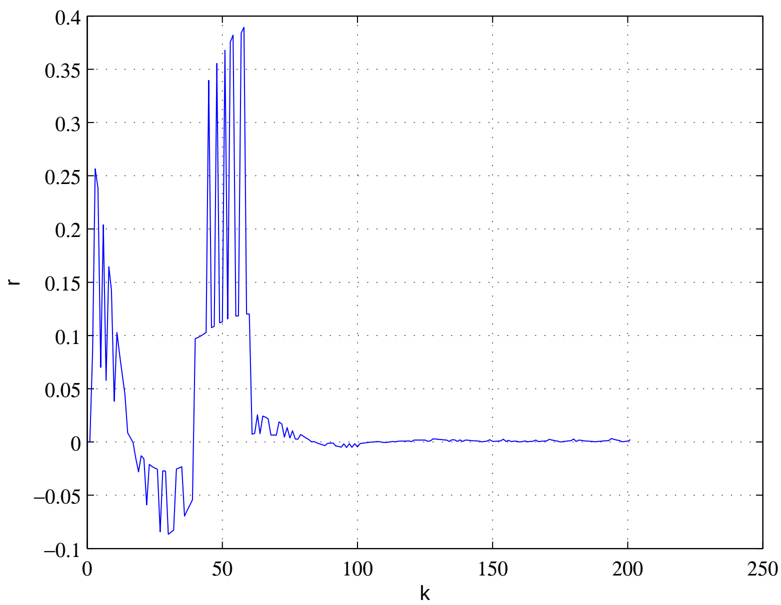

FIGURE 3. The evolution of the residual r_k

Figure 1 presents the switching sequences $\alpha(k)$, while Figure 2 plots the evolution of the observer error e_k. The evolution of the residual r_k is given in Figure 3 showing the exact interval time of presence of the fault in the switching system. Figure 4 presents the evaluation function J_k with fault and free fault together with the threshold function J_0 in dashed line. This function gives information about time occurrence and duration of the fault. Figure 5 plots the second evaluation function J_r with $T = 10$. One can notice that the index function J_r with fault case begin its "take-off" few time before that the

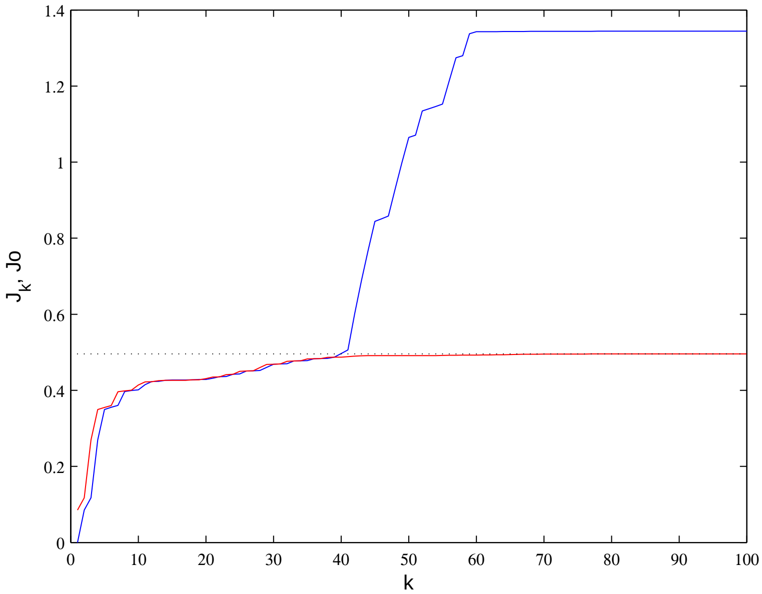

FIGURE 4. The evolution of the evaluation function J_k with free fault and fault case

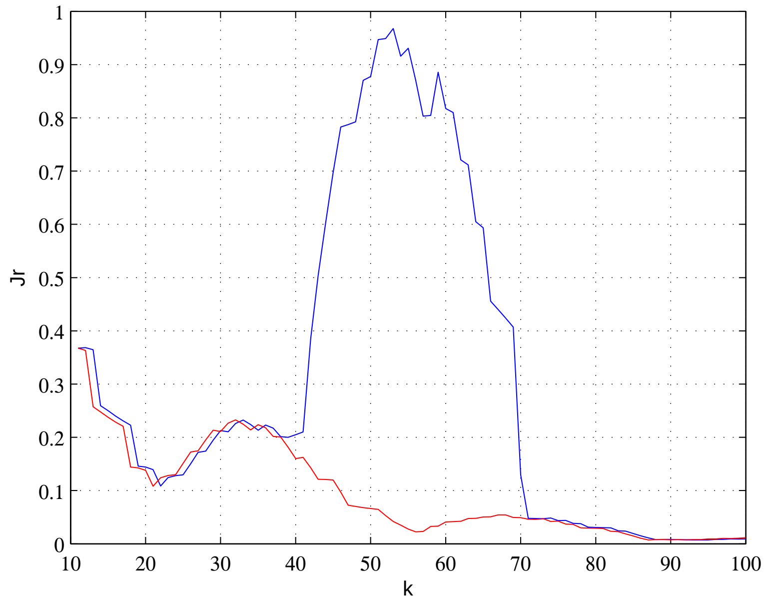

FIGURE 5. The evolution of the second evaluation function J_r

fault occurs, that is at $k = 39$. However, the duration of the fault occurrence is largely evaluated, that is a duration of 32 instead of 20.

Comment 4.2. *Another issue to be mentioned is that the proposed condition of Theorem 4.1 is only sufficient so the feasible solution of the LMIs (31) is just an upper bound of the true one. So a fault could be present but not detected. This would be not a performance problem (the cost would be guaranteed anyway) but a detection problem. This fact motivates one to look for less conservative conditions leading to small γ. In our case, we show upon an example used in* [23] *that our technique leads to the same* $\gamma = 1.2$ *but* *allowing fault detection among perturbation of size* 100 *times the one taken in* [23]*. The new introduced evaluation function* J_r *leads to an earlier fault detection time. However, the duration of the fault occurrence is largely evaluated, that is a duration of* 32 *instead of* 20*. In consequence, all these functions* r_k, J_k, J_r *can be complementarily used to detect the time and duration of the fault occurrence.*

5. **Conclusion.** In this paper, sufficient conditions for building an observer for discrete-time uncertain switching systems with delay are presented. The obtained condition is worked out in a new way to obtain new LMIs. Besides the introduction of slack variables G_i, the residual signal is designed with multiple matrices V_i. The observer is then used in fault detection. A numerical example is studied to illustrate the applicability of the obtained results. In particular, the proposed observer works with success even in presence of an input control u_k, a bounded unknown input d_k and a fault f_k of the same sizes. Filter based approaches may not success in this case.

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