International Journal of Innovative
Computing, Information and Control
Volume **8**, Number **10(B)**, October **2012**

 ISSN 1349-4198
pp. 7095–7102

A NEW APPROXIMATION BASED ON THE DIFFERENTIAL EVOLUTION ALGORITHM FOR THE GAUSSIAN *Q*-FUNCTION

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Received August 2011; revised December 2011

Abstract. *This paper presents a new approach that combines two well-known approximations in order to improve the approximation accuracy of the Gaussian Q-function. Differential evolution, which can be considered as a simple and efficient evolutionary algorithm for optimizing real-valued optimization problems, is applied to estimate the parameters of the new approximation. The results produced by the presented approximation are compared with some other Q-function approaches from the literature. It is shown that the new approximation outperforms previously reported approximations and can provide an accurate representation of the Gaussian Q-function, especially for small arguments.*
Keywords: Digital communications, Gaussian Q-function, Differential evolution, Absolute relative error

1. **Introduction.** The one-dimensional Gaussian Q-function, $Q(x)$, is described as the complement of the cumulative distribution function corresponding to the Gaussian random variable X with zero mean and unit variance [1]. This function plays an important role in analyzing the error probability of various communication systems when the noise is Gaussian [2]. However, a simple and exact closed-form expression that can be practically used for this analysis is not available. Therefore, over the years, a notable amount of research has been conducted on the issue of the derivation of approximations for the $Q(x)$. It is worth noting that the birth of approximations was inevitable to reduce computational complexity while paying the price with a loss in accuracy.

To reduce complexity at the expense of a tolerable loss of accuracy, infinite series of representations were proposed in [3, 4]. Rapid convergence was reported as the main advantage of the convergent series proposed in [3]. However, poor convergence can be seen as the main disadvantage of this approach especially for large argument x. Tellambura et al. claimed that the infinite series representation for the error function presented in [4] becomes more accurate and efficient as the argument x increases. In fact, instead of using infinite series of representations, most recent earlier works on this subject in the current literature have mainly focused on developing various approximations that are both computationally efficient as well as being sufficiently accurate [5-13]. The most valuable feature of the approximations proposed in [6, 9, 11] is their high level of accuracy while those in [7, 8] are mainly noted for their appropriate mathematical structure.

In this paper, we propose a new approximation in order to improve the approximation accuracy of the Gaussian Q-function at the cost of slightly increased calculation complexity. The differential evolution (DE) algorithm is employed to optimize the parameters of

the proposed approximation. It is found that the approximation presented in this study can predict the Gaussian Q-function with high accuracy.

The layout of this paper proceeds as follows. Section 2 provides a brief mathematical review of the approximations presented so far in the literature. In Section 3, the proposed model and the parameter identification process based on the DE algorithm are described. Section 4 gives the absolute relative error curves both for the proposed approximation and previously introduced approximations. Finally, based on the findings, some concluding remarks are made in Section 5.

2. **Background and Previous Work.**

A significant number of contributions have been dedicated in the literature to the development of approximations for the Gaussian Q-function. This section presents a brief review of some of the approximations existing in the literature.

In a preliminary work [6], some tight bounds and good approximations for the Gaussian Q-function were given where the emphasis was on the simplicity and practicality of the proposals. As a result of this work, the following tight approximation was presented [6, Equation (13)];

$$Q(x) \approx \frac{1}{a\sqrt{x^2+b}+(1-a)\,x} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad x \geq 0 \tag{1}$$

where a and b are the parameters of $Q(x)$, and the optimum values of these parameters are found as 0.339 and 5.510 in [6], respectively. Unfortunately, the approximation in Equation (1) is thought to be less useful in algebraic manipulations. Considering related works on approximations for the Gaussian Q-function, Chiani et al. [7, Equation (14)] presented an alternative expression which is quite simple but less accurate as

$$\operatorname{erfc}(x) \approx \frac{1}{6} e^{-x^2} + \frac{1}{2} e^{-4x^2/3} \qquad x \geq 0 \tag{2}$$

It is useful to note that $\operatorname{erfc}(\cdot)$ is related to the Q function by

$$Q(x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{3}$$

In 2009, Loskot et al. [8] revised Chiani et al.'s model by considering a finite sum of exponentials in the representation of the Q-function. The approximations that were formed as the sum of two and three terms can be given by [8, Equations (13c) and (13d)]

$$Q(x) \approx 0.208 e^{-0.971x^2} + 0.147 e^{-0.525x^2} \qquad x \geq 0 \tag{4}$$

$$Q(x) \approx 0.168 e^{-0.876x^2} + 0.144 e^{-0.525x^2} + 0.002 e^{-0.603x^2} \qquad x \geq 0 \tag{5}$$

Although the performance of Loskot et al.'s approximation is better than those previously proposed, it has poor approximation accuracy for high function arguments. Karagiannidis et al. [9, Equation (6)] then further presented a novel, simple and tight approximation for the Gaussian Q-function and its integer powers as

$$f(x, A, B) = \operatorname{erfc}(x) \approx \frac{\left(1 - e^{-Ax}\right) e^{-x^2}}{B\sqrt{\pi}x} \qquad x \geq 0 \tag{6}$$

For the argument region $x \in [0,\ 20]$, the values of A and B in the above equation are found numerically as 1.98 and 1.135 in [9], respectively. Isukapalli et al. extended the

work of [9] and derived a modified version of Equation (6) that can be easily integrated for any m of a Nakagami-m fading distribution as [11, Equation (3)]

$$Q(x) \approx \frac{\left(1-e^{-Ax/\sqrt{2}}\right)e^{-x^2/2}}{B\sqrt{2\pi}x} \approx e^{-x^2/2}\sum_{n=1}^{n_a}\frac{(-1)^{n+1}(A)^n}{B\sqrt{\pi}\left(\sqrt{2}\right)^{n+1}n!}\cdot x^{n-1} \qquad x \geq 0 \tag{7}$$

where n_a is the number of terms that is taken into account for the approximation. Instead of using exponential or rational based functions, a new polynomial approximation that employs only power functions was proposed by Chen et al. in [12, Equation (4)] as

$$\begin{aligned} Q(x) \approx Q_n(x) = 1 - \sum_{m=0}^{n}\sum_{p=0}^{n}\frac{(-1)^{m+p}\binom{n}{p}}{m!(n-m)!}\left(\tfrac{n}{12}\right)^{p/2}\cdot\left(\tfrac{n}{2}-m\right)^{n-p}x^p \\ \cdot U\left(x-\sqrt{\tfrac{12}{n}}\left(\tfrac{n}{2}-m\right)\right) \qquad |x| < \sqrt{3n} \end{aligned} \tag{8}$$

where $U(\cdot)$ denotes the unit step function. Very recently, inspired by the different types of approximations presented in previous works, a new mathematical model based on a second-order exponential function was considered by Lopez-Benitez et al. in [13, Equation (8)] as

$$Q(x) \approx e^{ax^2+bx+c} \qquad x \geq 0 \tag{9}$$

where $a, b, c \in \Re$ are fitting parameters. The optimum values of these parameters can be found in [13, Table I] for different argument ranges. Nevertheless, the second-order exponential function with a minimum sum of square errors (min-SSE) criterion has large approximation errors for small arguments. Therefore, this approximation is less appropriate for using in performance computation over fading channels.

3. **Proposed Approximation for the $Q(x)$.** The following model that is a combination of the Börjesson et al.'s and the Loskot et al.'s approximations (with 2 terms) was considered as

$$Q_a(x) = \frac{a_1}{a_2\sqrt{x^2+a_3}+(1-a_2)\,x}\cdot\frac{1}{\sqrt{2\pi}}e^{-x^2/a_4} + \left(a_5\cdot e^{a_6\cdot x^2} + a_7\cdot e^{a_8\cdot x^2}\right) \qquad x \geq 0 \tag{10}$$

where $\{a_1, a_2, \ldots, a_8\}$ represent the unknown parameters that will be determined by the differential evolution algorithm, and $Q_a(x)$ denotes the proposed approximation for the Gaussian Q-function. The expression in Equation (10) may alternatively be shown in closed form as

$$Q_a(x) = f\left(x, a_1, a_2, \ldots, a_8\right) \tag{11}$$

where $f(\cdot)$ represents the nonlinear relationship between x and $Q_a(x)$. The mean absolute model error may be expressed as follows:

$$E = \frac{1}{M}\sum_{k=1}^{M}\left|Q\left(k\right) - Q_a\left(k\right)\right|^2 \tag{12}$$

where k denotes the k^{th} sample and M is the number of all samples. $Q(k)$ and $Q_a(k)$ represent the original values and the results computed by the proposed approximation, respectively. By substituting Equation (11) into the above equation, we can rewrite Equation (12) as:

$$E = \frac{1}{M}\sum_{k=1}^{M}\left|Q\left(k\right) - f\left(k, a_1, a_2, \ldots, a_8\right)\right|^2 \tag{13}$$

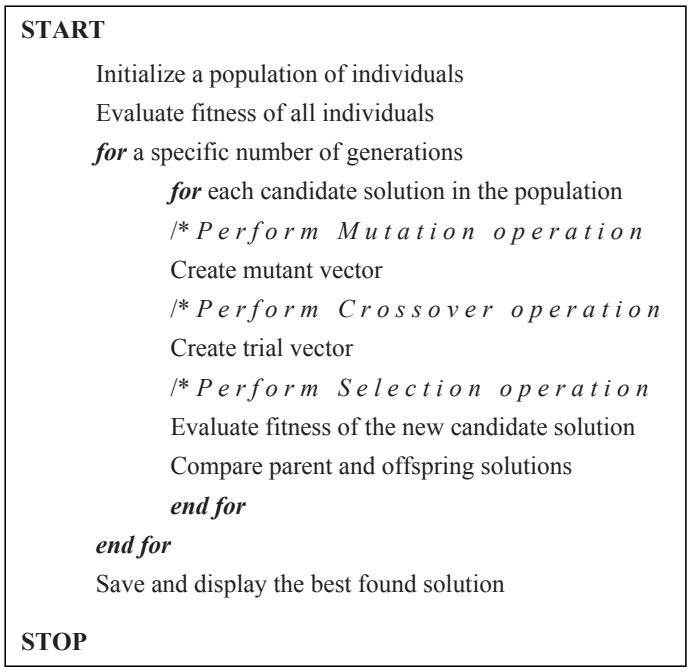

FIGURE 1. Main steps of the DE algorithm

The only unknowns in the above equation are the parameters $\{a_1, a_2, \ldots, a_8\}$ of the approximation. In this section, the mean absolute model error given in Equation (12) is used as the cost function, and the optimum approximation parameters are determined by using the DE algorithm.

Differential evolution (DE) developed by Storn and Price [14], [15] is a simple and powerful evolutionary algorithm for solving various difficult parameter optimization problems. Similar to many evolutionary algorithms, DE has three important operations: *mutation*, *crossover* and *selection* [16, 17]. The main steps of the DE algorithm are described in Figure 1.

The DE algorithm operates on a population with N_{pop} real-valued vectors $X_{i,G} = \left\{x^1_{i,G}, \ldots, x^D_{i,G}\right\}$, $i = 1, 2, \ldots, N_{pop}$, where i denotes the index of the individual and G represents the generation. Each individual belonging to the solution vector is composed of D optimization parameters. An initial population of N_{pop} solution vectors is generated randomly on a search space. Note that the initialized population is supposed to satisfy uniform probability distribution in the solving space. The initial population in the first generation ($G = 0$) is generated by the following equation:

$$x^p_{i,0} = x^p_{\min} + \text{rand}[p] \cdot (x^p_{\max} - x^p_{\min}), \quad p = 1, \ldots, D \tag{14}$$

where rand$[p]$ is a uniform random number produced within [0, 1]. Here $x^p_{\max}$ and $x^p_{\min}$ are the upper and lower bounds of the pth parameter, respectively. Finding a region that probably contains the optimum solution is the main motivation for choosing the values for $x^p_{\max}$ and $x^p_{\min}$. This region can be scaled by prior knowledge of the problem to improve search efficiency [18]. After the population is initialized, the algorithm evolves to the genetic evolution loop by applying the three basic genetic operations; *mutation*, *crossover* and *selection* in sequence.

The mutation process is the key procedure in the DE algorithm. The basic idea is to create a difference vector (*mutant vector*) by selecting three individuals (X_a, X_b and X_c) that are all randomly selected from the population and satisfy $a \neq b \neq c$ and $a, b, c \in$

$[1, N_{pop}]$. The mutation operator performs as follows:

$$V_{i,G+1}^{p} = X_{c,G}^{p} + F \times \left(X_{a,G}^{p} - X_{b,G}^{p}\right) \tag{15}$$

where F is the real-valued mutation factor which is commonly chosen from within the range [0.1, 1]. This factor weights the differential variations, and therefore controls the mutation operation. Crossover is the second operation that is carried out to increase the variety of the population. The crossover operation is applied between $V_{i,G}^{p}$ and $X_{i,G}^{p}$ according to the probability of a real-valued crossover factor P_{cross}, $P_{cross} \in [0, 1]$. The new vector called the trial vector $H_{i,G}^{p} = \left[h_{i,G}^{1}, h_{i,G}^{2}, \ldots, h_{i,G}^{D}\right]$ is generated by the following equation:

$$H_{i,G+1}^{p} = \begin{cases} V_{i,G+1}^{p} & \text{if rand } [p] \leq P_{cross} \\ \\ X_{i,G}^{p} & \text{otherwise} \end{cases} \tag{16}$$

After the crossover phase in which a parent chromosome vector generates its own offspring vector, the selection step is realized as a final operation in order to select better offspring. This operator compares the fitness of each parent vector $X_{i,G}^{p}$ and the corresponding trial vector $H_{i,G+1}^{p}$ based on the principle of greediness and decides whether the trial vector $H_{i,G+1}^{p}$ will have a place in the next generation. As a result, the trial vector competes with its parent vector to advance into the next generation. A greedy selection is performed as follows:

$$X_{i,G+1}^{p} = \begin{cases} H_{i,G+1}^{p} & \text{if } f\left(H_{i,G+1}^{p}\right) \leq f\left(X_{i,G}^{p}\right) \\ \\ X_{i,G}^{p} & \text{otherwise} \end{cases} \tag{17}$$

where $f(\cdot)$ is the fitness function that is equal to the cost function in Equation (12) and $X_{i,G+1}^{p}$ is the new generation vector. The processes of *mutation*, *crossover* and *selection* are repeated over and over again until a predetermined generation limit is reached or until a termination criterion is satisfied.

4. **Numerical Results.** In this section, we evaluate the accuracy of the approximation developed in the previous section and compare it with other existing approximations presented in the literature. For the simulations, a population of 40 individuals, $N_{pop} = 40$, was used while the real-valued mutation factor F was chosen as 0.7. Because the number of unknown parameters in Equation (10) is eight, $D = 8$. The DE algorithm performs very well in this optimization problem when the crossover probability P_{cross} is selected as 0.9. It is useful to note that the choice of these control parameters determines the ability of the DE to find an optimum solution. The parameters of the approximation were then optimally designated for the derivation of an accurate approximation for the $Q(x)$. This is achieved by tuning the parameters of the proposed approximation by the DE algorithm so as to minimize the cost function defined in Equation (12). Finally, the algorithm was defined to terminate when the maximum number of generations is equal to 250. The parameter values found by the DE algorithm are tabulated in Table 1.

In Figure 2, the proposed approximation is compared with the previously proposed approximations of Chiani et al., Loskot et al. and Börjesson et al. in terms of absolute relative error as a function of the argument x. It is evident from the figure that the proposed approximation outperforms these previously proposed approximations in terms of accuracy. As can be noted, while the argument x is low, the accuracy of the presented approximation turns out to be very powerful. Figure 3 presents the absolute relative error curve obtained by using the proposed approximation. The approximations of Chen et al., Isukapalli et al., Benitez et al. and Karagiannidis et al. are also included in this figure for

TABLE 1. Parameter values for the proposed approximation found by the DE algorithm

Parameters of the proposed approximation	Optimum values
a_1	0.9702
a_2	0.3987
a_3	3.6677
a_4	2.0055
a_5	–0.0013
a_6	–2.3690
a_7	–0.0054
a_8	–1.0436

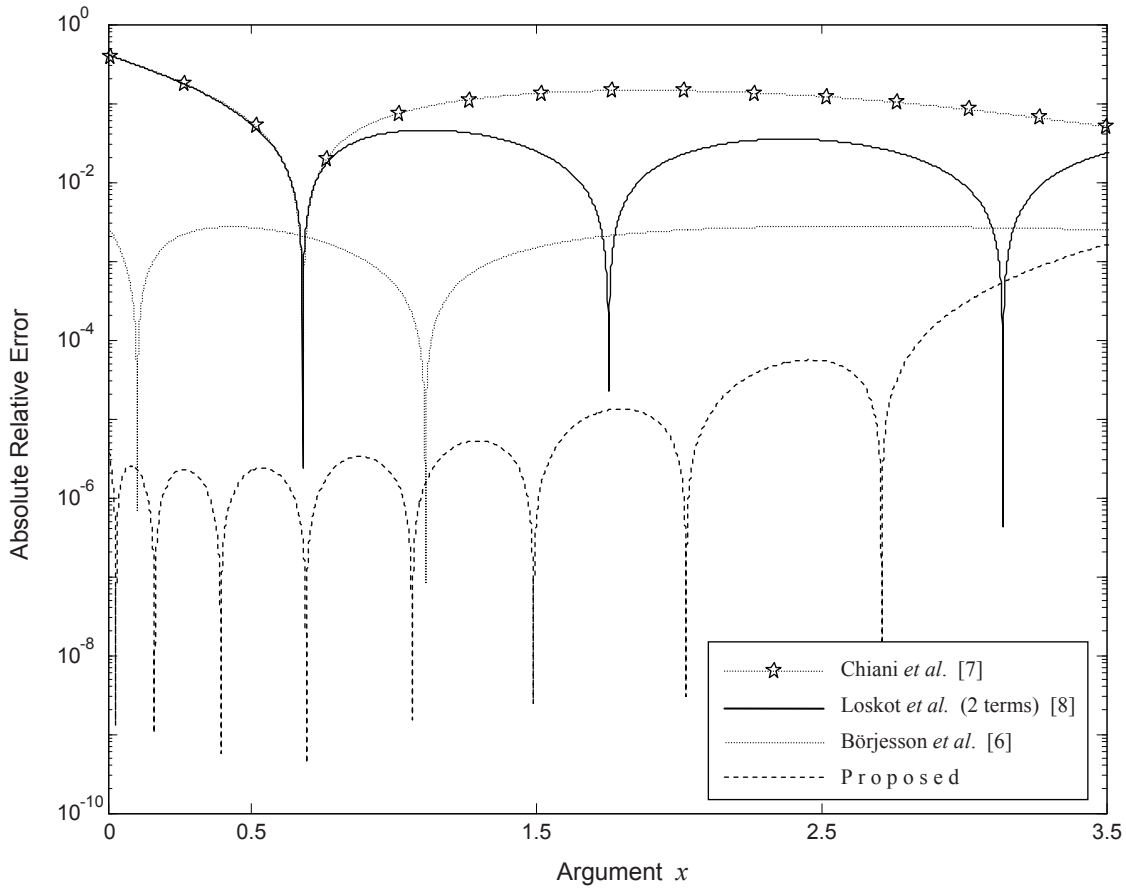

FIGURE 2. Comparison of the absolute relative error against the argument x between the proposed approximation, Chiani et al.'s approximation [7], Loskot et al.'s approximation [8], and Börjesson et al.'s approximation [6]

a better evaluation of accuracy. As can be easily seen from the figure, the accuracy of our approximation significantly outperforms the other four approximations. More specifically, the accuracy of our approach becomes more impressive as the argument x decreases.

5. **Concluding Remarks.** The Gaussian Q-function is of major importance in evaluating the performance of communication systems. This function is tabulated for a specific

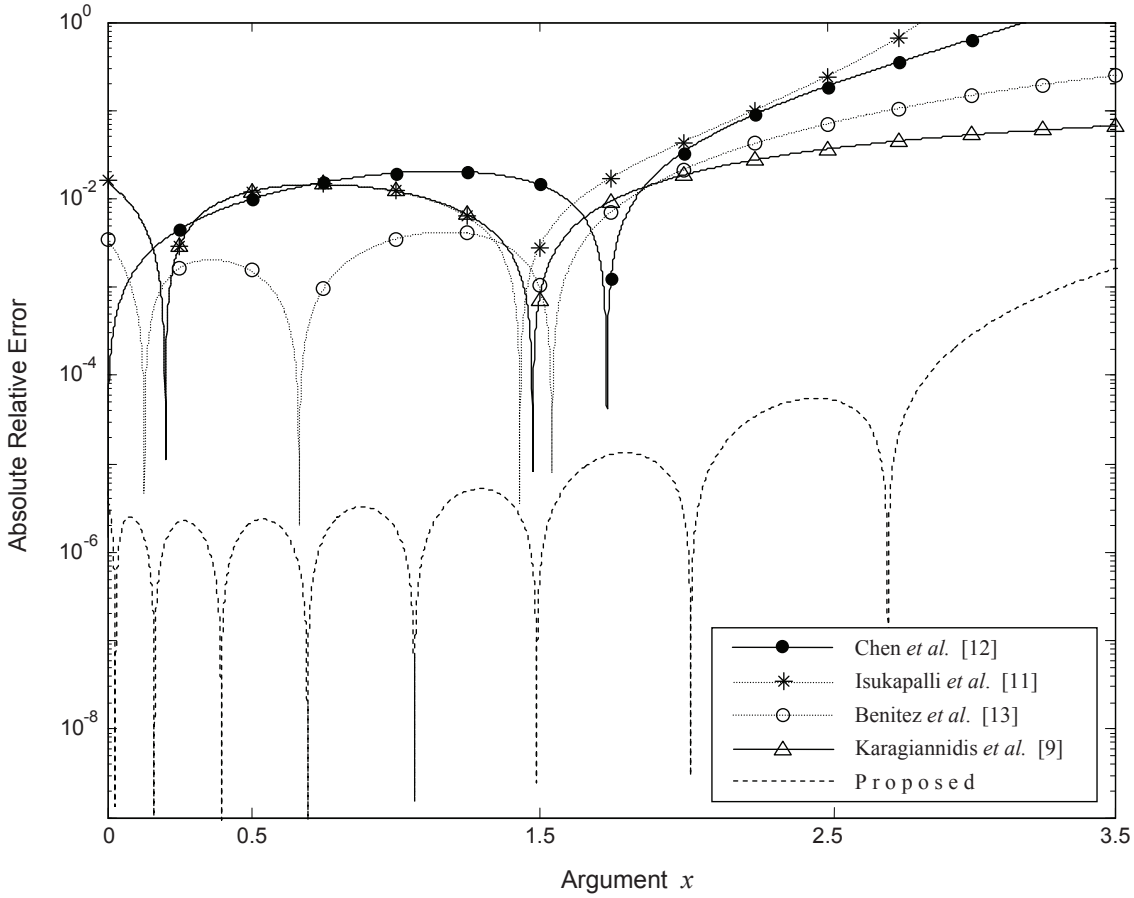

FIGURE 3. Comparison of the absolute relative error against the argument x between the proposed approximation, Chen et al.'s approximation [12], Isukapalli et al.'s approximation [11], Benitez et al.'s approximation [13], and Karagiannidis et al.'s approximation [9]

range of argument x, and frequently defined as a built-in function in mathematics software packages such as Matlab and Mathematica. However, in most states it is more suitable to hold approximations instead of the exact expression in order to simplify mathematical manipulations. In this paper, we have proposed an accurate expression that combines two well-known approximations to improve the approximation accuracy of the Gaussian Q-function for small arguments. The differential evolution algorithm, which is a computationally efficient global optimization method, is employed to estimate the parameters of the proposed approximation. The results indicate that the approximation derived in this paper can accurately predict the Gaussian Q-function especially in the low x region. However, as expected, its accuracy tends to decrease as the values of argument x increase. Although the proposed approximation is the most accurate it suffers from increased computational complexity as compared to both Börjesson et al.'s and Loskot et al.'s approximations. This is the main deficiency of our approach compared to two approximations mentioned. As a final remark, the theoretic results presented in this work can be practically used to the general problem of analysing the error probability of various communication systems in an additive white Gaussian noise channel.

Acknowledgment. The author thanks the associate editor and the anonymous reviewers who helped in the completion of the paper with their constructive comments and important suggestions.

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