

STABILITY ANALYSIS AND BIFURCATION OF A PREDATOR-PREY MODEL WITH TIME DELAY IN PREY AND DISEASES IN PREDATOR

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ABSTRACT. *In this paper, we present a predator-prey model with time delay, in which predator can be infected. The epidemics cannot be transmitted between prey and predator by predation. The predation ability of susceptible predators is stronger than infected ones. Based on the assumptions above, we study the stability and bifurcation of some equilibrium points, where the time delay is regarded as a parameter. It is found that there are stability switches, and Hopf bifurcation occurs when the delay passes through a sequence of critical values. We provide numerical results to illustrate our conclusion about stability and obtain the properties of Hopf bifurcation. In addition, we also discuss some interesting biological cases that our model exhibits.*

Keywords: Predator-prey model, Time delay, Hopf bifurcation

1. Introduction. The dynamical behavior of a class of the predator-prey models with disease has been considered in recent years, such as [1, 2, 3]. While the models of more species, or the models with spatial structure, delay effect, non local effect are mathematically much more challenging. Considering the biological maturation cycle, the model with time delays deserves more attention, such as [4, 5, 6, 7, 8, 9, 10, 11, 12]. For example, P. J. Pal and P. K. Mandal studied a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response in [13].

$$\begin{cases} \frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) (x - m) - \frac{qxy}{a + bx + cy}, \\ \frac{dy}{dt} = y \left(h - \frac{ny(t - \tau)}{d + x(t - \tau)}\right) \end{cases} \quad (1)$$

X. Meng and J. Wei studied the following model in [4]

$$\begin{cases} \frac{dx}{dt} = r_1x(t) \left(1 - \frac{x(t - \tau)}{k_1}\right) + ax(t)y(t), \\ \frac{dy}{dt} = r_2y(t) \left(1 - \frac{y(t - \tau)}{k_2}\right) + bx(t)y(t), \end{cases} \quad (2)$$

where r_i, k_i, a, b ($i = 1, 2$) are positive.

It is well known that the delayed logistic differential equation

$$\frac{dx}{dt} = rx \left(1 - \frac{x(t - \tau)}{k}\right) \quad (3)$$

is used to model the evolution of a single species $x(t)$ (see [14]).

In this paper, we established a predator-prey model with time delay in which the predator could be infected.

$$\begin{cases} \frac{dx}{dt} = ax \left(1 - \frac{x(t-\tau)}{k} \right) - px(y + hz), \\ \frac{dy}{dt} = -my - qyz + rx(y + hz), \\ \frac{dz}{dt} = -nz + qyz \end{cases} \quad (4)$$

In our model:

- $x(t)$ is the population size of prey at time t
- $y(t)$ is the population size of susceptible predator at time t
- $z(t)$ is the population size of infected predator at time t

where a, k, p, h, m, n, q, r are positive. a is the intrinsic growth rate of the prey, and k is the carrying capacity of the prey in the absence of predator in the environment. The prey will be preyed on with rate p . The infectious rates in predator is denoted as q . The parameters m and n ($m < n$) are the death rate of susceptible predator and infected predator, respectively. r is the conversion factor for the predator due to consumption prey. The parameter h stands for the impact of disease on predation rate, where $0 < h < 1$. The time delay $\tau \geq 0$ is considered as parameter to investigate the local stability and Hopf bifurcation of model (4).

In this paper, we discuss a predator-prey model in which not only predator could be infected, but it also exists delay for prey. We find that lots of diseases are not transmitted vertically, so it is also assumed that the disease in predator is not genetic.

The paper is organized as follows. In Section 2, the existence of the positive equilibrium is given. The stability of some of equilibriums and the existence of local Hopf bifurcation is also considered. In Section 3, we provide some numerical examples to illustrate our results and give its biological interpretation. Finally, the conclusion is shown in the last section.

2. Stability of the Equilibrium and Existence of the Hopf Bifurcation. Obviously system (4) has five equilibrium points as below,

$$E_1(0, 0, 0), \quad E_2(k, 0, 0), \quad E_3\left(\frac{m}{r}, \frac{a(rk - m)}{rpk}, 0\right), \quad E_4\left(0, \frac{n}{q}, \frac{-m}{q}\right), \quad E_5(x^*, y^*, z^*)$$

Considering the biological meaning of the system, we do not discuss the negative equilibrium point E_4 . It is obvious that E_1, E_2 are always non-negative, and E_3 is also non-negative when $rk > m$. For the last one E_5 , we have the following conclusion.

Theorem 2.1. *For system (4) under the assumptions in Section 1, we have*

1. *If $aq + phm < np$, then E_5 is not non-negative.*
2. *If $aq(rk - m) - rknp > 0$, then E_5 is positive.*

Proof: Since $n - hm > 0$, we may define

$$y^* = \frac{n}{q}, \quad \Delta = \sqrt{(karh - an)^2 + 4arhkpy^*(n - hm)} > 0 \quad (5)$$

which leads to

$$x_{\pm}^* = \frac{1}{2arh}(karh + an \pm \Delta), \quad z_{\pm}^* = \frac{(m - rx_{\pm}^*)y^*}{rx_{\pm}^*h - qy^*} \quad (6)$$

Then $E_5 = (x_-^*, y^*, z_-^*)$ or $E_5 = (x_+^*, y^*, z_+^*)$. If $aq + phm < np$, then $x_-^* < 0$ and $z_+^* < 0$. If $aq(rk - m) - rkn p > 0$, then $krh + n > 2hm$ and $aq + phm > np$, which leads to $x_-^* > 0$ and $z_-^* > 0$. The proof is complete.

From the above discussion, we note that if and only if $aq(rkh - m) - rkn p > 0$, then system (4) has unique positive equilibrium $E_5(x^*, y^*, z^*)$, where

$$\begin{aligned} x^* &= \frac{1}{2arh} \left(karh + an - \sqrt{(karh - an)^2 + 4arhkpy^*(n - hm)} \right) \\ y^* &= \frac{n}{q} \\ z^* &= \frac{(m - rx^*)y^*}{rx^*h - qy^*} \text{ or } z^* = \frac{ak - ax^* - pky^*}{kph}. \end{aligned}$$

We will discuss the stability and bifurcation of the non-negative equilibrium points of (4) in the following.

2.1. Stability analysis of the equilibrium E_1 . The linearization of (4) at E_1 is

$$\begin{cases} \frac{dx}{dt} = ax \left(1 - \frac{x(t - \tau)}{k} \right), \\ \frac{dy}{dt} = -my, \\ \frac{dz}{dt} = -ny \end{cases} \quad (7)$$

Thus, the characteristic equation of (7) is

$$D(\lambda) = (\lambda + m)(\lambda + n) \left(\lambda - a + \frac{a}{k} e^{-\lambda\tau} \right) \quad (8)$$

we can obtain the following results about the stability of equilibrium E_1 .

Theorem 2.2. *If $k > 1$, then the equilibrium E_1 is unstable for all $\tau > 0$.*

Proof: By hypothesis, the distribution of the roots of $D(\lambda) = 0$ is depended on the equation $\lambda - a - \frac{a}{k} e^{-\lambda\tau} = 0$. Note that, k is the carrying capacity of the prey in the absence of predator in the environment. Therefore, $k > 1$ conforms to the actual situation. Denote $g(\lambda) = \lambda - a - \frac{a}{k} e^{-\lambda\tau} = 0$, we find that $g(0) = \frac{a}{k} - a < 0$ and $\lim_{\lambda \rightarrow +\infty} g(\lambda) = +\infty$. Hence, there exists a $\lambda^* > 0$ such that $g(\lambda^*) = 0$. Therefore, the equilibrium E_1 is unstable for all $\tau > 0$. This completes the proof.

2.2. Stability analysis of the equilibrium E_2 . By the translation $u_1(t) = x(t) - k$, $u_2(t) = y(t)$, $u_3(t) = z(t)$, system (4) is equivalent to

$$\begin{cases} \frac{du_1}{dt} = -a(u_1(t) + k) \frac{u_1(t - \tau)}{k} - p(u_1(t) + k)(u_2(t) + hu_3(t)), \\ \frac{du_2}{dt} = -mu_2(t) - qu_2(t)u_3(t) + r(u_1(t) + k)(u_2(t) + hu_3(t)), \\ \frac{du_3}{dt} = -nu_3(t) + qu_2(t)u_3(t) \end{cases} \quad (9)$$

The linearization of (9) at $(u_1, u_2, u_3) = (0, 0, 0)$ is

$$\begin{cases} \frac{du_1}{dt} = -pk u_2(t) - pk h u_3(t) - a u_1(t - \tau), \\ \frac{du_2}{dt} = (rk - m) u_2(t) + rk h u_3(t), \\ \frac{du_3}{dt} = -n u_3(t) \end{cases} \quad (10)$$

Thus, the characteristic equation of (10) is

$$D(\lambda) = (\lambda + n)(\lambda + m - rk)(\lambda + ae^{-\lambda\tau}) = 0 \quad (11)$$

The E_2 is asymptotically stable if all roots of (11) have negative real parts and are unstable and if (11) has at least one root with positive real part. In order to discuss the distribution of the roots of (11), we state a result due to Ruan and Wei [15].

Lemma 2.1. *For the transcendental equation*

$$\begin{aligned} &\lambda^n + p_1^0 \lambda^{n-1} + \cdots + p_{n-1}^0 \lambda + p_n^0 + [p_1^1 \lambda^{n-1} + \cdots + p_{n-1}^1 \lambda + p_n^1] e^{-\lambda\tau_1} \\ &+ \cdots + [p_1^m \lambda^{n-1} + \cdots + p_{n-1}^m \lambda + p_n^m] e^{-\lambda\tau_m} = 0 \end{aligned}$$

as $(p_1^0, \dots, p_n^0, \dots, p_n^m; \tau_1, \dots, \tau_m)$ varies, the sum of the order of the zeros of above equation in the open right half-plane can change only if a zero appears on or crosses the imaginary axis. Here $\tau_j \geq 0$ ($j = 1, \dots, m$) and $p_j^i \in C$ ($0 \leq i \leq m, 1 \leq j \leq n$).

Let us consider the distribution of the roots of (11). It is clearly that if $kr - m > 0$, then (11) has a positive root $\lambda = kr - m$ for all $\tau \geq 0$.

In the following, we suppose that $kr - m < 0$. We can find that all roots of (11) are negative when $\tau = 0$. If $i\omega$ ($\omega > 0$) is a root of (11) for some $\tau > 0$, then

$$D(i\omega) = (i\omega + n)(i\omega + m - rk)(i\omega + ae^{-i\omega\tau}) = 0 \quad (12)$$

which is equivalent to

$$i\omega + ae^{-i\omega\tau} = 0 \quad (13)$$

Separating the real and imaginary parts, we have

$$\begin{cases} \omega = a \sin \omega\tau \\ a \cos \omega\tau = 0 \end{cases} \quad (14)$$

From (14), we find

$$\omega = a, \quad \tau_j = \frac{\pi}{2a} + \frac{2j\pi}{a}, \quad j = 0, 1, \dots \quad (15)$$

Denote that

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau) \quad (16)$$

is the root of

$$\lambda + ae^{-\lambda\tau} = 0 \quad (17)$$

such that $\alpha(\tau_j) = 0$, $\omega(\tau_j) = a$. Substituting $\lambda(\tau)$ into (17) and taking the derivative with respect to τ , we have

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{e^{\lambda\tau}}{\lambda a} - \frac{\tau}{\lambda} \quad (18)$$

which, together with (15), leads to

$$\operatorname{Re} \left[\frac{d\lambda}{d\tau} \right]_{\tau=\tau_j}^{-1} = \frac{\sin \omega\tau_j}{a\omega} = \frac{1}{a^2} > 0 \quad (19)$$

By Lemma 2.1 and the above computation, we can obtain the following results about the distribution of the characteristic roots of (11).

Lemma 2.2. *Let τ_j ($j = 0, 1, \dots$) be defined by (15).*

1. *If $kr - m > 0$, (11) has at least one positive root for all $\tau \geq 0$.*

2. If $kr - m < 0$, all roots of (11) have strictly negative real parts when $\tau \in [0, \tau_0)$, and (11) has a pair of imaginary roots $\pm ia$ and all other roots have strictly negative real parts when $\tau = \tau_0$, as well as when $\tau > \tau_0$, (11) has at least a pair of roots with positive real part.

Applying above Lemma 2.2 and Theorem 1.1 in Chapter 10 of Hale and Lunel [16], we have

Theorem 2.3. Consider system (4),

1. If $kr - m > 0$, then the equilibrium E_2 is unstable for all $\tau \geq 0$.
2. If $kr - m < 0$, then the equilibrium E_2 is asymptotically stable for $\tau \in [0, \tau_0)$, and unstable for $\tau > \tau_0$, and (11) undergoes a Hopf bifurcation at E_2 when $\tau = \tau_j$ ($j = 0, 1, \dots$).

Proof: The theorem is a direct application of Theorem 1.1 in Chapter 10 of Hale and Lunel [16]. The proof is easy, we omit it.

The formulae for determining the direction of Hopf bifurcation and the stability of bifurcating periodic solution of system (11) at τ_0 can be presented by employing the normal form method and center manifold theorem introduced by Hassard et al. [17]. This method is widely used by many authors, such as [4, 6]. Moreover, we have proved that the direction of numerical Hopf bifurcation and stability of bifurcating invariant curve are the same as that of original system for lots of numerical methods in [18, 19, 20]. Therefore, in this paper, we do not repeat the complex calculation. Instead, in Section 3, we apply some numerical methods to (11) and show the direction of Hopf bifurcation and the stability of bifurcating periodic solution of system (11).

2.3. Stability analysis of the equilibrium E_3 . Assume that $rk > m$ in this subsection. In this case $E_3 \left(\frac{m}{r}, \frac{a(rk-m)}{rkp}, 0 \right)$ is non-negative. Let $u_1(t) = x(t) - \frac{m}{r}$, $u_2(t) = y(t) - \frac{a(rk-m)}{rkp}$, $u_3(t) = z(t)$. Then system (4) is equivalent to

$$\begin{cases} \frac{du_1}{dt} = a \left(u_1(t) + \frac{m}{r} \right) \left(1 - \frac{u_1(t-\tau) + \frac{m}{r}}{k} \right) - p \left(u_1 + \frac{m}{r} \right) \left(u_2 + \frac{a(rk-m)}{rkp} + hu_3 \right), \\ \frac{du_2}{dt} = -m \left(u_2(t) + \frac{a(rk-m)}{rkp} \right) - q \left(u_2(t) + \frac{a(rk-m)}{rkp} \right) u_3(t) \\ \quad + r \left(u_1(t) + \frac{m}{r} \right) \left(u_2(t) + \frac{a(rk-m)}{rkp} + hu_3(t) \right), \\ \frac{du_3}{dt} = -nu_3(t) + q \left(u_2(t) + \frac{a(rk-m)}{rkp} \right) u_3(t) \end{cases} \quad (20)$$

The linearization of (20) at $(u_1, u_2, u_3) = (0, 0, 0)$ is

$$\begin{cases} \frac{du_1}{dt} = \frac{m}{r} \left(-pu_2(t) - phu_3(t) - \frac{a}{k}u_1(t-\tau) \right), \\ \frac{du_2}{dt} = \frac{a(rk-m)}{kp}u_1(t) + \left(mh - qa\frac{rk-m}{rkp} \right) u_3(t), \\ \frac{du_3}{dt} = \left(qa\frac{rk-m}{rkp} - n \right) u_3(t) \end{cases} \quad (21)$$

Denote

$$A = \frac{am}{rk}, \quad B = \frac{am(rk-m)}{rk}.$$

Thus, the characteristic equation of (21) is

$$D(\lambda) = \left(\lambda - \frac{qa(rk-m)}{rkp} + n \right) (\lambda + Ae^{-\lambda\tau} + B) = 0 \quad (22)$$

Obviously, if $\frac{qa(rk-m)}{rkp} - n > 0$, then the (22) has at least one root with positive real part.

If $\frac{qa(rk-m)}{rkp} - n < 0$, then the root $\lambda = \frac{qa(rk-m)}{rkp} - n < 0$, the stability of the system is determined by the roots of

$$\lambda + Ae^{-\lambda\tau} + B = 0 \quad (23)$$

Then, let us consider the distribution of the roots of (23).

Lemma 2.3. *Equation (23) has a pair of imaginary roots $\pm i\omega_{\pm}$ when $\tau = \tau_j^{\pm}$ ($j = 0, 1, \dots$), where*

$$\tau_j^+ = \frac{\pi}{2\omega_+} + \frac{2j\pi}{\omega_+}, \quad \tau_j^- = \frac{3\pi}{2\omega_-} + \frac{2j\pi}{\omega_-}, \quad \omega_{\pm} = \frac{\sqrt{A^2 + 4B} \pm A}{2} \quad (24)$$

Proof: It is obvious that $kr - m > 0$ provides that $A > 0, B > 0$. Let $i\omega$ ($\omega > 0$) be a root of (23), then

$$\begin{cases} \omega^2 - B = A\omega \sin \omega\tau, \\ A\omega \cos \omega\tau = 0 \end{cases} \quad (25)$$

From some simple calculation, we can obtain the conclusion. The proof is complete.

Denote that

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau) \quad (26)$$

is the root of (23) satisfying $\alpha(\tau_j^{\pm}) = 0, \omega(\tau_j^{\pm}) = \omega_{\pm}$.

Substituting $\lambda(\tau)$ into (23) and taking the derivative with respect to τ , it follows that

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{2\lambda + Ae^{-\lambda\tau} - A\lambda\tau e^{-\lambda\tau}}{A\lambda^2 e^{-\lambda\tau}} = \frac{2}{A\lambda} e^{\lambda\tau} + \lambda^{-2} - \frac{\tau}{\lambda} \quad (27)$$

which, together with (24), leads to

$$\operatorname{Re} \left[\frac{d\lambda}{d\tau} \right]_{\tau=\tau_j^+}^{-1} = \frac{2\omega_+ \sin \omega_+ \tau_j^+ - A}{A\omega_+^2} = \frac{2\omega_+ - A}{A\omega_+^2} > 0 \quad (28)$$

and

$$\operatorname{Re} \left[\frac{d\lambda}{d\tau} \right]_{\tau=\tau_j^-}^{-1} = \frac{2\omega_- \sin \omega_- \tau_j^- - A}{A\omega_-^2} = \frac{-2\omega_- - A}{A\omega_-^2} < 0 \quad (29)$$

Then, we can obtain the following lemma.

Lemma 2.4. *If $\frac{qa(rk-m)}{rkp} - n < 0$ and $rk - m > 0$, then all roots of (22) have strictly negative real parts when $\tau \in [0, \tau_0)$, and it has a pair of imaginary roots $\pm i\omega_+$ and all other roots have strictly negative real parts when $\tau = \tau_0$, as well as when $\tau > \tau_0$, it has at least a pair of roots with positive real part, where $\tau_0 = \tau_0^+$.*

Proof: Note that $\tau_0^+ = \frac{\pi}{2\omega_+} < \tau_0^- = \frac{3\pi}{2\omega_-}$. Under the assumption $\frac{qa(rk-m)}{rkp} - n < 0$ and $rk - m > 0$, all roots of (22) with $\tau = 0$ has strictly negative real parts. By above computation, the conclusion of this Lemma follows.

Therefore, we have the following results about stability of the equilibrium E_3 .

Theorem 2.4. *Consider system (4) under the assumption $rk - m > 0$.*

1. If $\frac{qa(rk-m)}{rkp} - n > 0$, then the equilibrium E_3 is unstable for all $\tau \geq 0$.
2. If $\frac{qa(rk-m)}{rkp} - n < 0$, then the equilibrium E_3 is asymptotically stable for $\tau \in [0, \tau_0)$, and unstable for $\tau > \tau_0$, and (11) undergoes a Hopf bifurcation at E_3 when $\tau = \tau_j^+$ ($j = 0, 1, \dots$).

Proof: The proof is similar to Theorem 2.3, we omit it.

2.4. Stability analysis of the equilibrium E_5 . In this subsection, we make the following hypotheses

$$(H_1): aq(rk - m) - rkn p > 0,$$

$$(H_2): A_3 + B_3 > 0,$$

$$(H_3): (A_1 + B_1)(A_2 + B_2) - (A_3 + B_3) > 0,$$

$$(H_4): A_3^2 - B_3^2 < 0,$$

$$(H_5): s_+ > 0, \phi(s_+) \leq 0,$$

where $A_i, B_i, i = 1, 2, 3; s_+, \phi(s_+)$ will be defined in the following parts.

Let $u_1(t) = x(t) - x^*, u_2(t) = y(t) - y^*, u_3(t) = z(t) - z^*$, then system (4) is equivalent to

$$\begin{cases} \frac{du_1}{dt} = -px^*u_2(t) - phx^*u_3(t) - \frac{a}{k}u_1(t - \tau) - \frac{a}{k}u_1(t - \tau)u_1(t) \\ \quad - pu_1(t)u_2(t) - phu_1(t)u_3(t), \\ \frac{du_2}{dt} = r \left(\frac{n}{q} + hz^* \right) u_1(t) - \frac{hrqx^*z^*}{n}u_2(t) + (rhx^* - n)u_3(t) \\ \quad + ru_1(t)u_2(t) + rhu_1(t)u_3(t) - qu_2(t)u_3(t), \\ \frac{du_3}{dt} = qz^*u_2(t) + qu_2(t)u_3(t). \end{cases} \quad (30)$$

The characteristic equation of the linear part of (30) is given by

$$D(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 + (B_1\lambda^2 + B_2\lambda + B_3)e^{-\lambda\tau} = 0 \quad (31)$$

where

$$A_1 = \frac{rghx^*z^*}{n}, \quad A_2 = rp(x^*)^2 + qnz^* - qrhx^*z^* + rphx^*z^*, \quad A_3 = rpqh x^*z^*(x^* + hz^*);$$

$$B_1 = \frac{ax^*}{k}, \quad B_2 = \frac{arhqz^*(x^*)^2}{kn}, \quad B_3 = (rx^*h - n)\frac{aqx^*z^*}{k}.$$

In the absence of delay, (31) reduces to

$$\lambda^3 + (A_1 + B_1)\lambda^2 + (A_2 + B_2)\lambda + (A_3 + B_3) = 0 \quad (32)$$

Note that $m < rx^* < \frac{n}{h}$, (H_2) , (H_3) , which provided that

$$A_1 + B_1 > 0, \quad A_2 + B_2 > 0, \quad A_3 + B_3 > 0, \quad (A_1 + B_1)(A_2 + B_2) - (A_3 + B_3) > 0.$$

Using the well-known Routh-Hurwitz criterion, we conclude that (32) is stable if and only if all of the above inequalities are satisfied. Thus, we have the following proposition.

Proposition 2.1. *If $H_1 - H_3$ are satisfied, then the equilibrium E_5 of (4) is asymptotically stable.*

Let $\lambda = i\omega$ ($\omega > 0$) be a root of (31) and separate it into real and imaginary parts to obtain

$$\begin{cases} -\omega^3 + A_2\omega = (B_3 - B_1\omega^2) \sin \omega\tau - B_2\omega \cos \omega\tau, \\ A_1\omega^2 - A_3 = (B_3 - B_1\omega^2) \cos \omega\tau + B_2\omega \sin \omega\tau, \end{cases} \quad (33)$$

which leads to

$$\varphi(\omega) = \omega^6 + (A_1^2 - B_1^2 - 2A_2)\omega^4 + (A_2^2 + 2B_1B_3 - B_2^2 - 2A_1A_3)\omega^2 + (A_3^2 - B_3^2) = 0 \quad (34)$$

Proposition 2.2. *If H_4 holds, then (34) has at least one positive root.*

Proof: By hypothesis (H_4): $A_3^2 - B_3^2 < 0$, we have $\varphi(0) = (A_3^2 - B_3^2) < 0$, and $\lim_{\omega \rightarrow +\infty} \varphi(\omega) = +\infty$. Hence, there exists an $\omega > 0$ such that $\varphi(\omega) = 0$. This completes the proof.

If we cannot be sure that the sign of $A_3^2 - B_3^2$, for this situation, we make the change of variable $s := \omega^2$ in (34). This gives the polynomial

$$\phi(s) = s^3 + Hs^2 + Ms + N = 0 \quad (35)$$

where $H = A_1^2 - B_1^2 - 2A_2$, $M = A_2^2 + 2B_1B_3 - B_2^2 - 2A_1A_3$, $N = A_3^2 - B_3^2$.

Then

$$\phi'(s) = 3s^2 + 2Hs + M.$$

The equation $3s^2 + 2Hs + M = 0$ has roots $s_{\pm} = \frac{-H \pm \sqrt{H^2 - 3M}}{3}$.

Proposition 2.3. *If H_5 holds, then (34) has at least one positive root.*

Proof: By H_5 : $s_+ > 0$, $\phi(s_+) \leq 0$ and $\phi(+\infty) > 0$, the (35) has at least one positive root $s_0 \geq s_+$. Therefore, the (34) has at least one positive root. This completes the proof.

Let $\omega_0 > 0$ be largest real root of (34) in Proposition 2.2 or 2.3, then solving $\sin(\omega_0\tau)$ and $\cos(\omega_0\tau)$ in (33) yields

$$\begin{aligned} \cos(\omega_0\tau) &= \frac{(B_3 - B_1\omega_0^2)(A_1\omega_0^2 - A_3) + B_2\omega_0(A_2\omega_0 - \omega_0^3)}{B_2^2\omega_0^2 + (B_3 - B_1\omega_0^2)^2} \\ \sin(\omega_0\tau) &= \frac{(A_1\omega_0^2 - A_3)B_2\omega_0 + (B_3 - B_1\omega_0^2)(A_2\omega_0 - \omega_0^3)}{B_2^2\omega_0^2 + (B_3 - B_1\omega_0^2)^2} \end{aligned}$$

We determine

$$\tau_j = \frac{1}{\omega_0} \cos^{-1} \left[\frac{(B_3 - B_1\omega_0^2)(A_1\omega_0^2 - A_3) + B_2\omega_0(A_2\omega_0 - \omega_0^3)}{B_2^2\omega_0^2 + (B_3 - B_1\omega_0^2)^2} \right] + \frac{2j\pi}{\omega_0}, \quad j = 0, 1, \dots$$

Now let $\lambda = \lambda(\tau)$ in (31), and differentiate it with respect to τ . For the simple root case, we have that $D(i\omega_0) = 0$. This leads to

Lemma 2.5.

$$\operatorname{Re} \left[\frac{d\lambda}{d\tau} \right]_{\tau=\tau_j} > 0, \quad j = 0, 1, \dots, \quad (36)$$

where τ_j , $j = 0, 1, \dots$, are defined above.

Proof: Differentiating both sides of (31) with respect to τ gives

$$\begin{aligned} \left[\frac{d\lambda}{d\tau} \right]^{-1} &= \frac{3\lambda^2 + 2A_1\lambda + A_2 + (2B_1\lambda + B_2)e^{-\lambda\tau} - \tau(B_1\lambda^2 + B_2\lambda + B_3)e^{-\lambda\tau}}{\lambda(B_1\lambda^2 + B_2\lambda + B_3)e^{-\lambda\tau}} \\ &= \frac{3\lambda^3 + 2A_1\lambda^2 + A_2\lambda}{-\lambda^2(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3)} + \frac{(2B_1\lambda^2 + B_2\lambda)e^{-\lambda\tau}}{\lambda^2(B_1\lambda^2 + B_2\lambda + B_3)e^{-\lambda\tau}} - \frac{\tau}{\lambda} \\ &= \frac{\lambda^3 - A_2\lambda - 2A_3}{-\lambda^2(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3)} + \frac{B_2\lambda + 2B_3}{-\lambda^2(B_1\lambda^2 + B_2\lambda + B_3)e^{-\lambda\tau}} - \frac{\tau}{\lambda} \end{aligned}$$

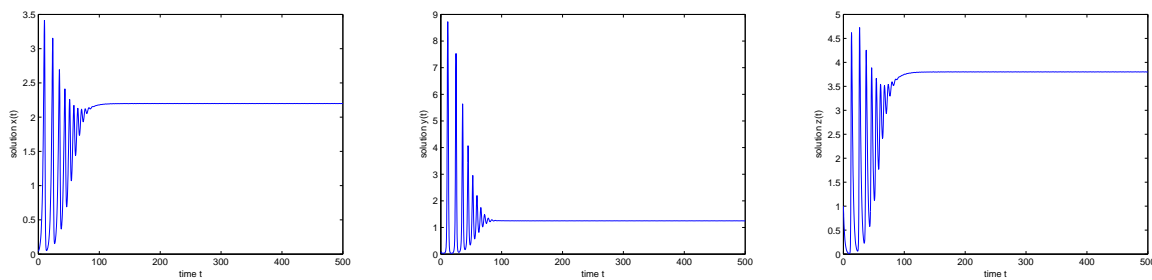


FIGURE 1. The equilibrium $E_1(0, 0, 0)$ is unstable when $\tau = 0.1$.

Hence,

$$\begin{aligned} \operatorname{Re} \left[\frac{d\lambda}{d\tau} \right]_{\tau=\tau_j}^{-1} &= \frac{\omega_0^6 + (2A_1A_3 - A_2^2 - 2B_1B_3 + B_2^2)\omega_0^2 + 2B_3^2 - 2A_3^2}{\omega_0^2[(B_3 - B_1\omega_0^2)^2 + B_2^2\omega_0^2]} \\ &= \frac{s^3 - Ms - 2N}{s[(B_3 - B_1s) + B_2^2s]} \end{aligned}$$

By Proposition 2.3 and (35), we can obtain the conclusion. This completes the proof.

Therefore, from Proposition 2.1 and Lemma 2.5, we have the following theorem.

Theorem 2.5. *Consider system (4) under the assumption H_1 .*

1. *If H_2 or H_3 does not hold, then the equilibrium E_5 is unstable for all $\tau \in [0, \tau^*)$, where τ^* is some small positive value.*
2. *If $H_2 - H_4$ or H_2, H_3, H_5 hold, then the equilibrium E_5 is asymptotically stable for $\tau \in [0, \tau_0)$, and unstable for $\tau > \tau_0$, and (4) undergoes a Hopf bifurcation at E_5 when $\tau = \tau_j^+$ ($j = 0, 1, \dots$).*

3. Numerical Simulation Examples and Biological Interpretation. From [7, 18, 19], we know that the properties of Hopf bifurcation of discrete schemes are the same as that of the corresponding delay differential equations, such as Runge-kutta or strictly stable linear multistep method. In this section, we use some numerical simulations to illustrate the analytical results we obtained in previous sections and to show the properties of Hopf bifurcation, such as the direction of bifurcation and stability of periodic solutions. Some numerical results of simulating system (4) are presented at different data of coefficients.

From the analysis in Section 2, if $k > 1$, then the E_1 is unstable for $\tau \geq 0$. We consider the system

$$\begin{cases} \frac{dx}{dt} = 0.5x \left(1 - \frac{x(t-0.1)}{100} \right) - 0.1x(y + 0.1z), \\ \frac{dy}{dt} = -0.2y - 0.4yz + 0.6x(y + 0.1z), \\ \frac{dz}{dt} = -0.5z + 0.4yz \end{cases} \quad (37)$$

Figure 1 shows that the situation matches with our conclusion. We can see that the solution in Figure 1 has a limit. Note that the $\lim_{t \rightarrow \infty} y(t) \approx 1.25 = \frac{0.5}{0.4} = \frac{n}{q}$, then we have reasons to believe that the limit of this solution is the equilibrium E_5 . In other words, susceptible populations and infected predator and prey can coexist in this situation.

For equilibrium E_2 , let $a = 0.5$, $k = 10$, $p = 0.1$, $h = 0.1$, $m = 0.2$, $q = 0.2$, $r = 0.01$, $n = 0.3$, then $\tau_0 \approx 3.1$. Here, rk is the natality of susceptible predator at the equilibrium

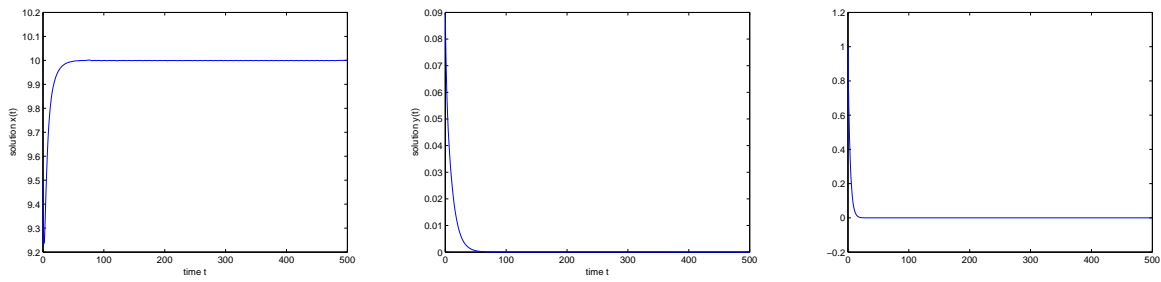


FIGURE 2. The equilibrium $E_2(10, 0, 0)$ is asymptotically stable when $kr - m > 0$ and $\tau = 2.5 < \tau_0$.

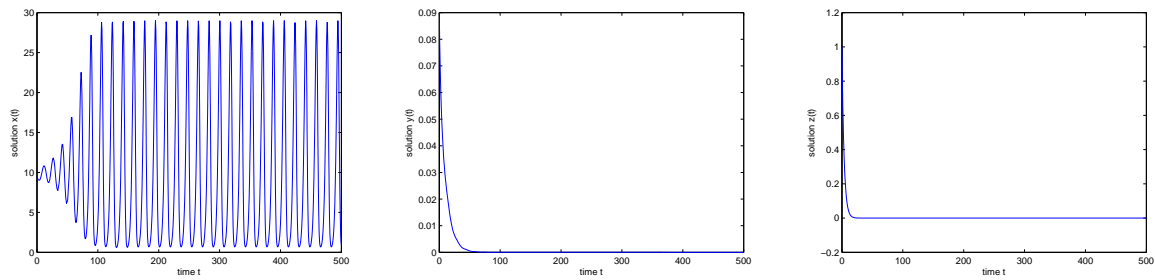


FIGURE 3. The equilibrium $E_2(10, 0, 0)$ is unstable when $kr - m < 0$ and $\tau = 3.5 > \tau_0$.

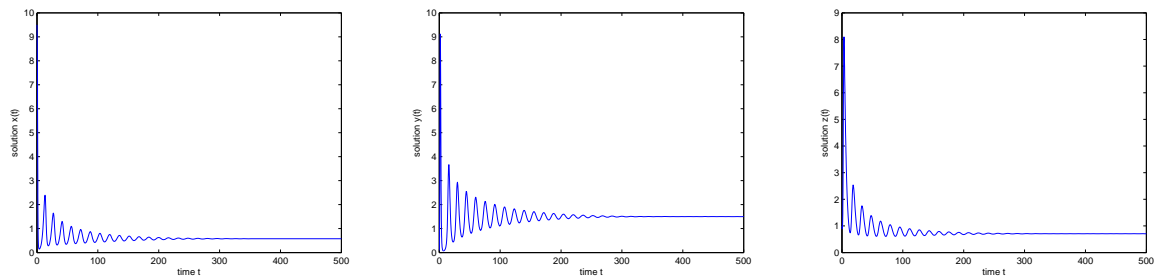


FIGURE 4. The equilibrium $E_2(10, 0, 0)$ is unstable when $kr - m > 0$ and $\tau = 0.5$.

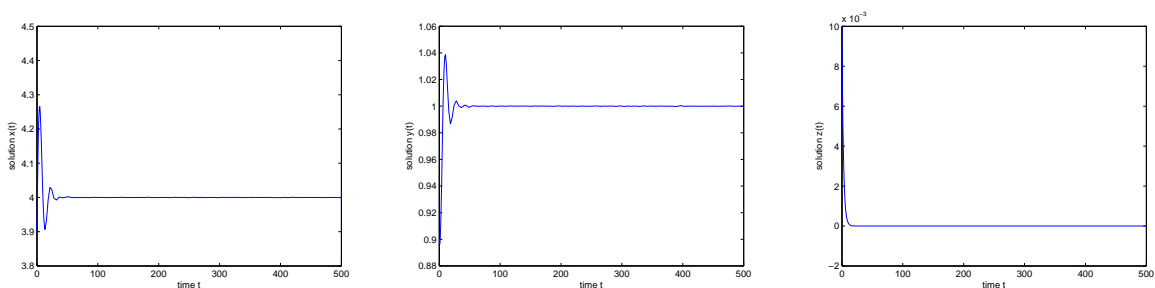


FIGURE 5. The equilibrium $E_3(4, 1, 0)$ is asymptotically stable when $\tau = 1.3 < \tau_0$.

point E_2 , and m is the death rate of the susceptible predator at E_2 . Therefore, $\frac{rk}{m}$

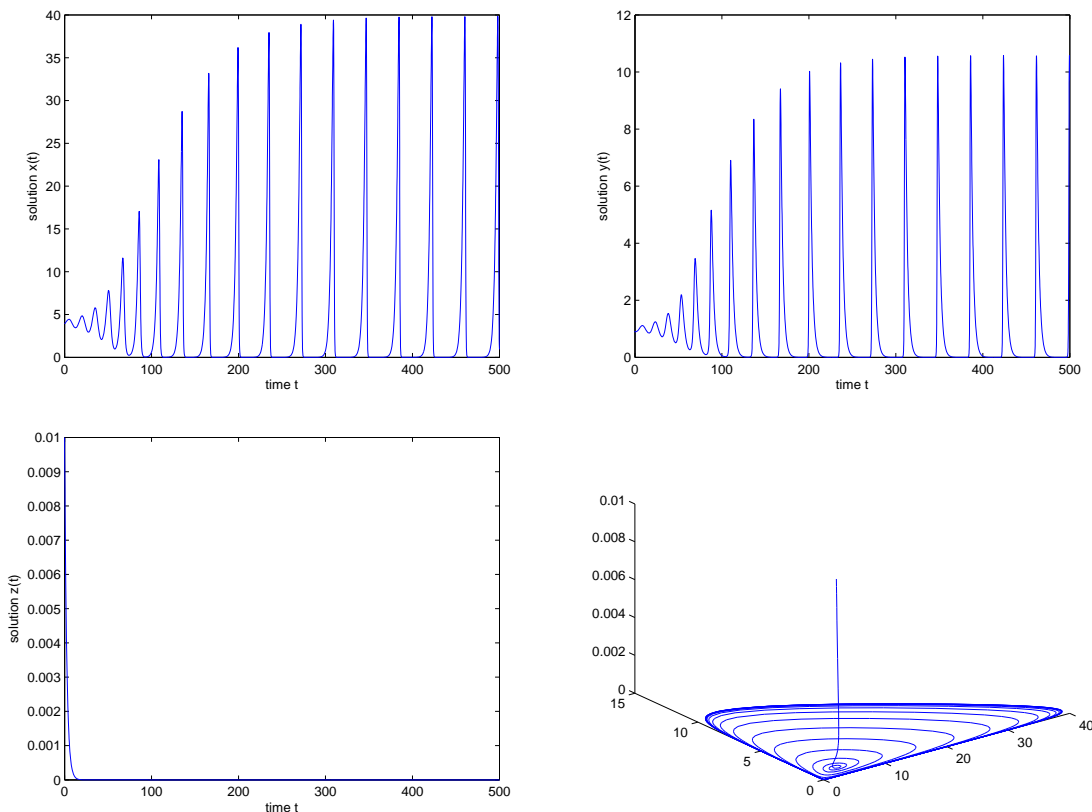


FIGURE 6. The equilibrium $E_3(4, 1, 0)$ is unstable when $\tau = 1.7 > \tau_0$.

is interpreted as ecological basic reproduction number. When $\frac{rk}{m} < 1$, the susceptible predator will become extinct, and so will the infected predator.

Figure 2 illustrates this point. In this case, the conversion factor r and delay τ are both too small, which implies that the predation ability of predator is too low and the mature period of prey is too short. So only preys exist at last, then we can call the system a stable SIS model.

For Figure 3, the delay increases to $3.5 > \tau_0$. Since the mature period of prey is long enough, the death rate $a\frac{x(t-\tau)}{k} + p(y+hz)$ has a large volatility around a , and the natality is fixed at a . This leads to the prey fluctuate around k .

In Figure 4, we change the values of r and m into $r = 0.4$, $m = 0.1$. The higher predation ability leads to the prey and predator coexisting.

In Figure 5, we choose $a = 0.5$, $k = 10$, $p = 0.3$, $h = 0.1$, $m = 0.4$, $q = 0.1$, $r = 0.1$, $n = 0.5$, then $\tau_0 \approx 1.3$. By Theorem 2.4, system (4) undergoes a Hopf bifurcation at $E_3(4, 1, 0)$ when $\tau = \tau_0$. We obtain the ecological basic reproduction number of infect predator $\frac{qa(rk-m)}{rkpn} < 1$ at the equilibrium point E_3 . The lower infectious rates q in predator such that the infect predator will become extinct. That is, the infectious disease in the predator will be cure. Therefore, we can call the system as a Lotka-Volterra model.

When the delay increases to $1.7 > \tau_0$ in Figure 6, the infect predator will also become extinct. The longer mature period leads the prey and susceptible predator both have fluctuations around $(4, 1)$.

For Figure 7, let $a = 0.5$, $k = 10$, $p = 0.3$, $h = 0.1$, $m = 0.4$, $q = 0.7$, $r = 0.1$, $n = 0.5$, such that $\frac{qa(rk-m)}{rkpn} > n$. The higher infectious rates q in predator such that the infect predator will always cannot be cured. However, the appropriate mature period is

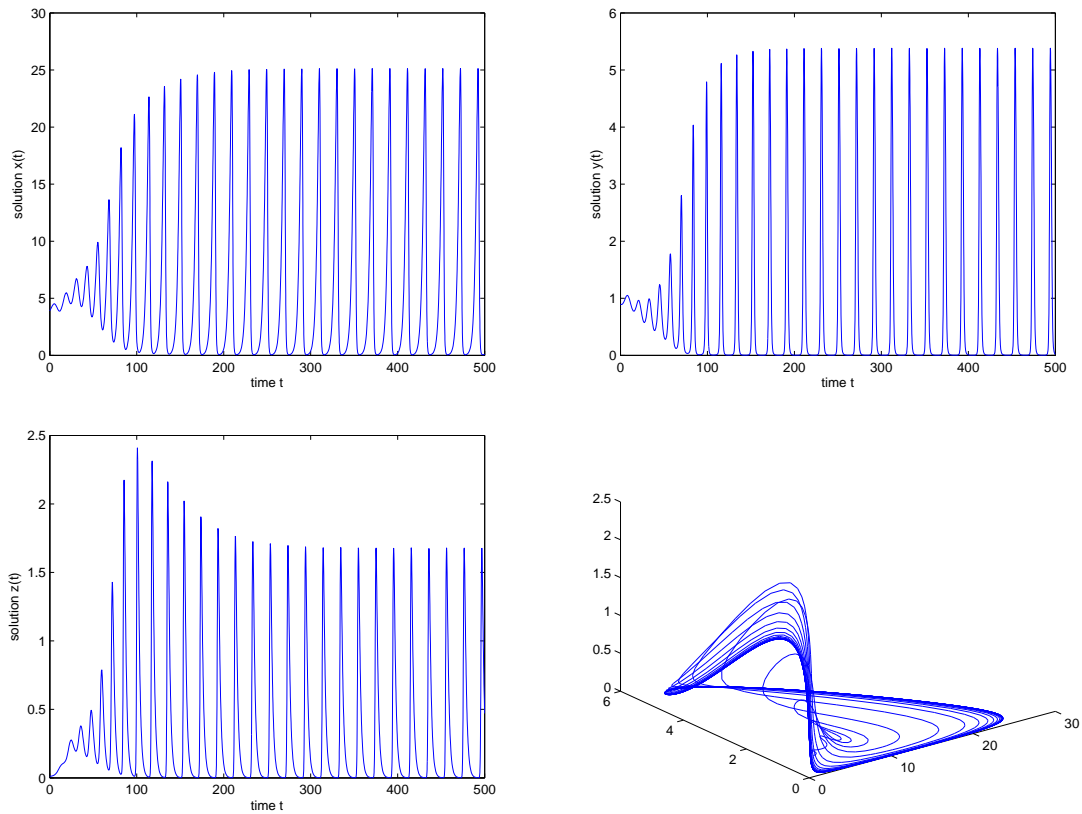


FIGURE 7. The equilibrium $E_3(4, 1, 0)$ is unstable when $\tau = 2$.

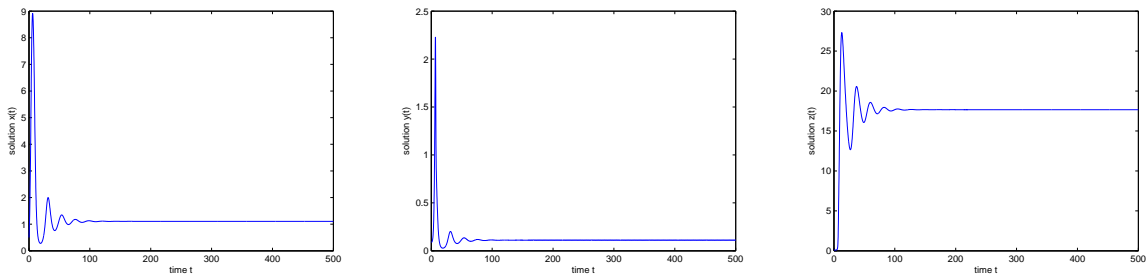


FIGURE 8. The equilibrium $E_5(0.1111, 1.1069, 17.6628)$ is asymptotically stable when $\tau = 4 < \tau_0$.

to ensure that predator has enough food. It is shown that all species coexist and have fluctuations.

At last, we discuss E_5 . In Figure 8 and Figure 9, we consider the large infectious rate on the system, the values of parameters as $a = 0.9$, $k = 10$, $p = 0.05$, $h = 0.9$, $m = 0.05$, $q = 0.9$, $r = 0.1$, $n = 0.1$, then $\tau_0 \approx 4.65$. These numbers satisfy $H_1 - H_4$, thus from Theorem 2.5 system (4) undergoes a Hopf bifurcation at $E_3(0.1111, 1.1069, 17.6628)$ when $\tau = \tau_0$. The ecological basic reproduction number of infect predator $\frac{qy^*}{n} = 1$ at the equilibrium point E_5 . Thus, the infect predator will exist forever. If $\tau = 4 < \tau_0$, all species will tend to E_5 as be showed in Figure 8, and when the delay increases to $5 > \tau_0$ in Figure 9. They will fluctuate.

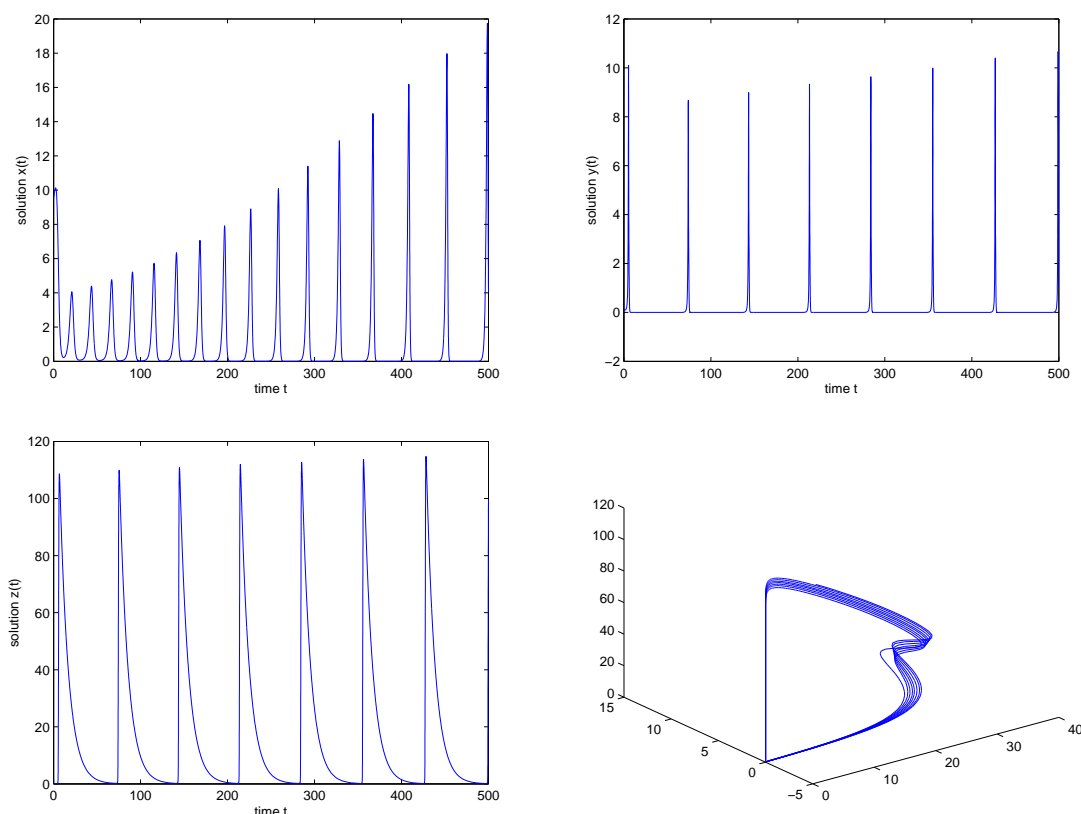


FIGURE 9. The equilibrium $E_5(0.1111, 1.1069, 17.6628)$ is unstable when $\tau = 5 > \tau_0$.

4. Conclusions. In this paper, we established a new predator-prey model with infectious disease which can transmit in predator and a time delay in prey. We focused our attention on the effects of time delay τ and the infectious disease on the model. By the analysis in Section 2 and numerical simulations in Section 3, we give some propositions about the properties of Hopf bifurcations at last.

Proposition 4.1.

1. If $kr - m < 0$, then the Hopf bifurcation at E_2 is supercritical and bifurcating periodic solution is orbitally asymptotically stable.
2. If $rk - m > 0$ and $\frac{qa(rk-m)}{r_{kp}} - n > 0$, then the Hopf bifurcation at E_3 is supercritical and bifurcating periodic solution is orbitally asymptotically stable.
3. If $H_2 - H_4$ or H_2, H_3, H_5 hold, then the Hopf bifurcation at E_5 is supercritical and bifurcating periodic solution is orbitally asymptotically stable.

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