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STABILITY AND BIFURCATION OF A FOUR-NEURON NETWORK WITH MULTIPLE DISCRETE DELAYS

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Abstract. *In this paper, a four-neuron network with multiple discrete delays is investigated. Applying suitable variable transformation, we obtain the equivalent form of the four-neuron network. Regarding the different time delays as parameters and analyzing the corresponding characteristic equations, we derive some conditions which ensure the local stability and the existence of Hopf bifurcation at the zero equilibrium of the system. It is shown that different time delays have different effects on the stability and Hopf bifurcation behavior of the system. Some numerical simulations supporting the theoretical analysis are carried out. Finally, main conclusions are given. Our results are new and complement previously known results.*

Keywords: Four-neuron network, Stability, Hopf bifurcation, Discrete delay, Periodic solution

1. **Introduction.** During the past decade, the dynamics of neural network models with delays or without delays has become a subject of intense research activity of mathematical fields due to their theoretical and practical significance. It is well known that the Hopf bifurcation phenomenon is a widespread phenomenon in the nature. Recently, many excellent and interesting results on Hopf bifurcation of the neural network models have been reported. For example, Xiao et al. [1] investigated the Hopf bifurcation of an $(n+1)$-neuron bidirectional associative memory neural network model with delays. Song et al. [2] focused on the stability and Hopf bifurcation in an unidirectional ring of n neurons with distributed delays. Guo and Huang [3] addressed the Hopf bifurcating periodic orbits in a ring of neurons with delays. Guo [4] made a detailed discussion on the equivariant Hopf bifurcation for functional differential equations of mixed type. For more related work, one can see [5-16]. Here we would like to point out that all the work mentioned above investigated the stability and Hopf bifurcation by choosing the single time delay or the sum of multiple time delays as bifurcation parameter. Further, they considered the stability, direction and period of Hopf bifurcation by applying normal form theory and center manifold argument. A natural problem is that in many cases, different time delays exist in networks and what different time delays have effect on the dynamical behavior of neural networks. This plays a key role in the design of neural networks. We feel regret that all the work mentioned above does not consider the effect of different time delay on the stability and local Hopf bifurcation behavior of neural network. We think that it is important to analyze the effect of different time delay on the dynamics of neural networks.

Motivated by the analysis above, in this paper, we will deal with the stability and local Hopf bifurcation of the following four-neuron network with multiple discrete delays

$$\begin{cases} \dot{u}_1(t) = -a_1u_1(t) + c_{14}f_4[u_4(t-\tau_1)] + c_{13}f_5[u_3(t-\tau_1)], \\ \dot{u}_2(t) = -a_2u_2(t) + c_{21}f_1[u_1(t-\tau_2)] + c_{24}f_6[u_4(t-\tau_2)], \\ \dot{u}_3(t) = -a_3u_3(t) + c_{32}f_2[u_2(t-\tau_3)], \\ \dot{u}_4(t) = -a_4u_4(t) + c_{43}f_3[u_3(t-\tau_4)], \end{cases} \tag{1}$$

where $\dot{u}(t) = \frac{du(t)}{dt}$, $u_i(t)$ represents the state of the ith neuron at time t, $a_i > 0$ is the internal decay rate, c_{ij} is the connection weight, f_k is the neuron transfer function, $f_k(0) = 0$, $f_k \in C^1$, τ_k is the non-negative transmission time dealy, and $i, j = 1, 2, 3, 4$; $k = 1, 2, 3, 4, 5, 6$. We believe that the investigation on the effect of time delay on the dynamical behavior will be an important supplement to the previous publications.

This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are studied. In Section 3, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 4.

2. Stability of Equilibrium and Local Hopf Bifurcation.

2.1. The analysis of the characteristic equation.

In this subsection, we will discuss the roots of the characteristic equation. It is easy to see that system (1) has a unique equilibrium $E(0, 0, 0, 0)$. Since system (1) contains four different time delays, it is difficult to study the dynamical behavior of the model. Hence, we employ a set of four new state variable so that system (1) can be expressed as a set of nonlinear differential equations with three distinct time delays. For simplicity, we introduce the following set of new state variables

$$\begin{cases} x_1(t) = u_1(t-\tau_2-\tau_3-\tau_4), \\ x_2(t) = u_2(t-\tau_3-\tau_4), \\ x_3(t) = u_3(t-\tau_4), \\ x_4(t) = u_4(t) \end{cases} \tag{2}$$

and let

$$\begin{cases} \tau_1+\tau_2+\tau_3+\tau_4 = s_1, \\ \tau_1+\tau_2+\tau_3 = s_2, \\ \tau_2+\tau_3+\tau_4 = \tau. \end{cases} \tag{3}$$

Then system (1) can be rewritten as the equivalent form

$$\begin{cases} \dot{x}_1(t) = -a_1x_1(t) + c_{14}f_4[x_4(t-\tau)] + c_{13}f_5[x_3(t-s_1)], \\ \dot{x}_2(t) = -a_2x_2(t) + c_{21}f_1[x_1(t)] + c_{24}f_6[x_4(t-s_2)], \\ \dot{x}_3(t) = -a_3x_3(t) + c_{32}f_2[x_2(t)], \\ \dot{x}_4(t) = -a_4x_4(t) + c_{43}f_3[x_3(t)]. \end{cases} \tag{4}$$

The linear system of (4) at $E(0, 0, 0, 0)$ takes the form:

$$\begin{cases} \dot{x}_1(t) = -a_1x_1(t) + a_{14}[x_4(t-\tau)] + a_{13}[x_3(t-s_1)], \\ \dot{x}_2(t) = -a_2x_2(t) + a_{21}[x_1(t)] + a_{24}[x_4(t-s_2)], \\ \dot{x}_3(t) = -a_3x_3(t) + a_{32}[x_2(t)], \\ \dot{x}_4(t) = -a_4x_4(t) + a_{43}[x_3(t)], \end{cases} \tag{5}$$

where $a_{14} = c_{14}f_4'(0)$, $a_{13} = c_{13}f_5'(0)$, $a_{21} = c_{21}f_1'(0)$, $a_{24} = c_{24}f_6'(0)$, $a_{32} = c_{32}f_2'(0)$, $a_{43} = c_{43}f_3'(0)$.

The associated characteristic equation of (5) is given by

$$\det\begin{bmatrix} \lambda + a_1 & 0 & -a_{13}e^{-\lambda s_1} & -a_{14}e^{-\lambda\tau} \\ -a_{21} & \lambda + a_2 & 0 & -a_{24}e^{-\lambda s_2} \\ 0 & -a_{32} & \lambda + a_3 & 0 \\ 0 & 0 & -a_{43} & \lambda + a_2 \end{bmatrix} = 0,$$

which leads to

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 + b_5e^{-\lambda\tau} + (b_6\lambda + b_7)e^{-\lambda s_2} + (b_8\lambda + b_9)e^{-\lambda s_1} = 0, \tag{6}$$

where

$$\begin{aligned} &b_1 = a_1 + a_2 + a_3 + a_4,\ b_2 = a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4, \\ &b_3 = a_1a_2a_3 + a_1a_3a_4 + a_1a_2a_4 + a_2a_3a_4,\ b_4 = a_1a_2a_3a_4,\ b_5 = a_{21}a_{32}a_{14}a_{43}, \\ &b_6 = -a_{24}a_{32}a_{43},\ b_7 = -a_1a_{24}a_{32}a_{43},\ b_8 = a_{21}a_{13}a_{32},\ b_9 = a_2a_{21}a_{13}a_{32}. \end{aligned}$$

Let

$$\Delta(\lambda, s_1, s_2, \tau) = \lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4+b_5e^{-\lambda\tau}+(b_6\lambda+b_7)e^{-\lambda s_2}+(b_8\lambda+b_9)e^{-\lambda s_1} = 0. \tag{7}$$

In order to investigate the roots of (7) with three time delays, we will begin with the analysis when there is no delay. When $s_1 = 0, s_2 = 0$ and $\tau = 0$ in system (7), we get

$$\Delta(\lambda, 0, 0, 0) = \lambda^4 + b_1\lambda^3 + b_2\lambda^2 + (b_3 + b_6 + b_8)\lambda + b_4 + b_5 + b_7 + b_9 = 0. \tag{8}$$

In view of the Rorth-Hurwitz criterion, all the roots of (8) have negative real parts if the following conditions hold.

$$\begin{cases} B_1 = b_1 > 0,\ B_2 = b_1b_2 - b_3 - b_6 - b_8 > 0, \\ B_3 = (b_3 + b_6 + b_8)B_2 - b_1^2(b_4 + b_5 + b_7 + b_9) > 0, \\ B_4 = (b_4 + b_5 + b_7 + b_9)B_3 > 0. \end{cases} \tag{9}$$

In the sequel, we will find the conditions for non-existence of stability switching. The following lemma (see [21]) is helpful.

Lemma 2.1. [21] *A set necessary and sufficient condition for the trivial solution* $(0, 0, 0, 0)$ *of system (5) to be asymptotically stable for* $s_1 \geq 0$, $s_2 \geq 0$ *and* $\tau \geq 0$ *is the following:*
(i) The real parts of all the roots of $\Delta(\lambda, 0, 0, 0) = 0$ *are negative;*
(ii) For any real $s_1 \geq 0$, $s_2 \geq 0$ *and* $\tau \geq 0$*, the following holds:* $\Delta(\lambda, s_1, s_2, \tau) \neq 0$*, where* $i = \sqrt{-1}$.

Let $\lambda = \pm i\omega$ be a pair of purely imaginary roots of (7). For $\omega = 0$, then

$$\Delta(0, s_1, s_2, \tau) = b_4 + b_5 + b_7 + b_9 \neq 0. \tag{10}$$

For $\omega \neq 0$, then

$$\begin{aligned} \Delta(i\omega, s_1, s_2, \tau) = &\,\omega^4 + b_1\omega^3 i - b_2\omega^2 + b_3\omega i + b_4 + b_5e^{-i\omega\tau} \\ &+ (b_6\omega i + b_7)e^{-i\omega s_2} + (b_8\omega i + b_9)e^{-i\omega s_1} = 0. \end{aligned} \tag{11}$$

Separating the real and imaginary parts of (11), we have

$$\begin{aligned}(\omega^2 - b_2\omega^2 + b_4) &= -b_5\cos(\omega\tau) + \omega b_6\cos\left(\omega s_2 + \frac{\pi}{2}\right) - d_7\cos(\omega s_2)\\ &\quad +\omega b_8\cos\left(\omega s_1 + \frac{\pi}{2}\right) - b_9\cos(\omega s_1), \end{aligned} \tag{12}$$

$$\begin{aligned}(b_1\omega^2 - b_3\omega) &= -b_5\sin(\omega\tau) + \omega b_6\sin\left(\omega s_2 + \frac{\pi}{2}\right) - b_7\sin(\omega s_2)\\ &\quad +\omega b_8\sin\left(\omega s_1 + \frac{\pi}{2}\right) - b_9\sin(\omega s_1). \end{aligned} \tag{13}$$

It follows from (12) and (13) that

$$\begin{aligned}&\omega^8 + (b_1^2 - 2b_2)\omega^6 + (b_2^2 - 2b_1b_3 + 2b_4)\omega^4\\ &+(b_3^2 - 2b_2b_4 - b_6^2 - b_8^2)\omega^2 + (b_4^2 - b_5^2 - b_7^2 - b_9^2)\\ =\ & 2\Big[-\omega b_5b_6\cos\left(\omega s_2 + \frac{\pi}{2} - \omega\tau\right) + b_5b_7\cos(\omega s_2 - \omega\tau)\\ &-\omega b_5b_8\cos\left(\omega s_1 + \frac{\pi}{2} - \omega\tau\right) + b_5b_9\cos(\omega s_1 - \omega\tau)\\ &+\omega^2 b_6b_8\cos(\omega s_1 - \omega s_2) - \omega b_6b_8\cos\left(\omega s_2 + \frac{\pi}{2} - \omega s_1\right)\\ &- \omega b_7b_8\cos\left(\omega s_1 + \frac{\pi}{2} - \omega s_2\right) + b_7b_9\cos(\omega s_1 - \omega s_2)\Big]. \end{aligned} \tag{14}$$

It follows from (14) that

$$\begin{aligned}&\omega^8 + (b_1^2 - 2b_2)\omega^6 + (b_2^2 - 2b_1b_3 + 2b_4)\omega^4\\ &+(b_3^2 - 2b_2b_4 - b_6^2 - b_8^2)\omega^2 + (b_4^2 - b_5^2 - b_7^2 - b_9^2)\\ \le\ & 2[|b_5b_7| + |b_5b_9| + |b_7b_9| + \omega^2|b_6b_8|\\ &-|\omega|(|b_5b_6| + |b_5b_8| + |b_6b_9| + |b_7b_8|)]. \end{aligned} \tag{15}$$

Then from (14) and (15), we have

$$\begin{aligned}&\omega^8 + \left(b_1^2 - 2b_2\right)\omega^6 + \left(b_2^2 - 2b_1b_3 + 2b_4\right)\omega^4 + \left[b_3^2 - 2b_2b_4 - (b_6 + b_8)^2\right]\omega^2\\ &+2\omega\left(b_4b_6 + b_5b_8 + b_6b_9 + b_7b_8\right) + \left[b_4^2 - (b_5 + b_7 + b_9)^2\right] > 0. \end{aligned} \tag{16}$$

Let

$$\begin{cases}\theta_1 = b_1^2 - 2b_2 > 0,\\ \theta_2 = b_2^2 - 2b_1b_3 + 2b_4 > 0,\\ \theta_3 = b_3^2 - 2b_2b_4 - (b_6 + b_8)^2,\\ \theta_4 = b_4b_6 + b_5b_8 + b_6b_9 + b_7b_8,\\ \theta_5 = b_4^2 - (b_5 + b_7 + b_9)^2. \end{cases} \tag{17}$$

Then (16) takes the form

$$\omega^8 + \theta_1\omega^6 + \theta_2\omega^4 + \theta_3\left(\omega + \frac{\theta_4}{\theta_3}\right)^2 + \left(\theta_5 - \frac{\theta_4^2}{\theta_3}\right) > 0. \tag{18}$$

Thus, we can conclude that a set of sufficient conditions for the non-existence of a real number ω satisfying $\Delta(i\omega, s_1, s_2, \tau) \neq 0$ are given by

$$\theta_3 > 0, \quad \theta_5 > \frac{\theta_4^2}{\theta_3}. \tag{19}$$

According to the analysis above, we have the following theorem.

Theorem 2.1. *If the following inequalities*

$$b_3^2 > 2b_2b_4 + (b_6 + b_8)^2, \tag{20}$$

$$\left[b_3^2 - 2b_2b_4 - (b_6 + b_8)^2\right]\left[b_4^2 - (b_5 + b_7 + b_9)^2\right] > (b_5b_6 + b_5b_8 + b_6b_9 + b_7b_8)^2, \tag{21}$$

then the stability switching of system (5) does not exist.

2.2. **Stability and Hopf bifurcation for three cases.** In this subsection, we discuss the local asymptotic stability of the zero equilibrium of system (1) and the existence of local Hopf bifurcation near the zero equilibrium.

Case a. When $s_1 = 0$, $s_2 = 0$ in (7), then $\tau = 0$. Thus, (7) becomes (8). It is easy to obtain the following result.

Lemma 2.2. *Assuming that* $B_i > 0$ $(i = 2, 3, 4)$, *then all roots of (8) have negative real parts. That is, the trivial equilibrium of (4) is asymptotically stable for* $s_1 \geq 0$, $s_2 \geq 0$ *and* $\tau \geq 0$.

Case b. When $s_2 = 0$, then $s_1 = \tau = \tau_4$, and then we get

$$\Delta(\lambda, s_1, 0, \tau) = \lambda^4 + b_1\lambda^3 + b_2\lambda^2 + (b_3 + b_6)\lambda + (b_4 + b_7) + [b_8\lambda + (b_5 + b_9)]e^{-\lambda\tau} = 0. \tag{22}$$

In this case, τ can be regarded as a parameter.

Let $\lambda = \pm i\upsilon$ $(\upsilon > 0)$ be a pair of purely imaginary roots of (22), then

$$\Delta(i\upsilon, s_1, 0, \tau) = \upsilon^4 - b_1\upsilon^3 i - b_2\upsilon^2 + (b_3 + b_6)\upsilon i + (b_4 + b_7) + [b_8\upsilon i + (b_5 + b_9)]e^{-i\upsilon\tau} = 0. \tag{23}$$

Separating the real and imaginary parts of (23), we get

$$\begin{cases} (b_5 + b_9)\cos\upsilon\tau + b_8\upsilon\sin\upsilon\tau = -\upsilon^4 + b_2\upsilon^2 - b_4 - b_7, \\ b_8\upsilon\cos\upsilon\tau - (b_5 + b_9)\upsilon\sin\upsilon\tau = (b_3 + b_6)\upsilon - b_1\upsilon^3. \end{cases} \tag{24}$$

Eliminating τ from (24) leads to

$$\upsilon^8 + \mu_1\upsilon^6 + \mu_2\upsilon^4 + \mu_3\upsilon^2 + \mu_4 = 0, \tag{25}$$

where

$$\begin{cases} \mu_1 = b_1^2 - 2b_2, \\ \mu_2 = 2(b_4 + b_7) + b_2^2 - 2b_1(b_3 + b_6), \\ \mu_3 = (b - 3 + b_6)^2 - 2b_2(b_4 + b - 7) - b_8^2, \\ \mu_4 = b_4 + b_7^2 - (b_5 + b_9)^2. \end{cases} \tag{26}$$

Differentiating λ with respect to τ in (22), we have

$$\lambda'(\tau) = \frac{\lambda e^{-\lambda\tau}(b_8\lambda + b_5 + b_9)}{4\lambda^3 + 3b_1\lambda^2 + 2b_2\lambda + b_5 + b_6 + [(b_8\lambda + b_5 + b_9)\tau + b_8]e^{-\lambda\tau}}. \tag{27}$$

From the analysis above, we have the following results.

Theorem 2.2. *The following assertions hold true if all the roots of (8) have negative real parts.*

(i) If (25) has no positive root, then the trivial equilibrium of (4) is asymptotically stable for an arbitrary delay τ.

(ii) If (25) has at least one positive and simple root and $\mathrm{Re}[\lambda'(\tau_c)] > 0$, *then the trivial equilibrium of (4) is asymptotically stable for* $\tau \in [0, \tau_c)$ *and (4) undergoes a branch of periodic solutions bifurcating from the trivial equilibrium at* τ_c.

Case c. When $s_1 > 0$, $s_2 > 0$ and $\tau > 0$ in system (4). Let $\lambda = \pm i\sigma$ ($\sigma > 0$) be the pair of purely imaginary roots of (6). Then

$$\begin{aligned}\Delta(i\sigma, s_1, s_2, \tau) = &\sigma^4 - b_1\sigma^3 - b_2\sigma^2 + b_3\sigma i + b_4 + b_5 e^{-i\sigma\tau} \\ &+ (b_6\sigma i + b_7)e^{-i\sigma s_2} + (b_8\sigma i + b_9)e^{-i\sigma s_1} = 0,\end{aligned} \tag{28}$$

Separating the real and imaginary parts of (28), we get

$$\begin{cases} \sigma^4 - b_2\sigma^2 + b_4 + b_5\cos(\sigma\tau) + b_6\sigma\sin(\sigma s_2) \\ \quad +b_7\cos(\sigma s_2) + b_8\sigma\sin(\sigma s_1) + b_9\cos(\sigma s_1) = 0, \\ b_3\sigma - b_1\sigma^3 - b_5\cos(\sigma\tau) + b_6\sigma\cos(\sigma s_2) \\ \quad -b_7\sin(\sigma s_2) + b_8\sigma\cos(\sigma s_1) - b_9\cos(\sigma s_1) = 0. \end{cases} \tag{29}$$

Then

$$\begin{cases} \cos\sigma s_1 = \dfrac{(b_1b_8 - b_9)\sigma^4 + (b_2b_9 - b_3b_8)\sigma^2 - b_4b_9}{b_9^2 + b_8^2\sigma^2} \\ \qquad + \dfrac{(b_7b_8 - b_6b_9)\sigma\sin(\sigma s_2) - (b_7b_9 + b_6b_8\sigma^2)\cos(\sigma s_2)}{b_9^2 + b_8^2\sigma^2} \\ \qquad - \dfrac{b_5b_9\cos\sigma\tau - b_5b_8\sigma\sin(\sigma\tau)}{b_9^2 + b_8^2\sigma^2}, \\ \sin\sigma s_1 = \dfrac{(b_1b_8 - b_9)\sigma^4 + (b_2b_9 - b_3b_8)\sigma^2 - b_4b_9}{b_9^2 + b_8^2\sigma^2} \\ \qquad + \dfrac{(b_7b_8 - b_6b_9)\sigma\sin(\sigma s_2) - (b_7b_9 + b_6b_8\sigma^2)\cos(\sigma s_2)}{b_9^2 + b_8^2\sigma^2} \\ \qquad - \dfrac{b_5b_9\cos\sigma\tau - b_5b_8\sigma\sin(\sigma\tau)}{b_9^2 + b_8^2\sigma^2}. \end{cases} \tag{30}$$

Eliminating s_1 from (30), together with (29), we have

$$\begin{aligned} G(\sigma, s_2, \tau) = &\sigma^8 + (b_1 - 2b_2)\sigma^6 + 2b_2\sin(\sigma s_2)\sigma^5 + [b_2^2 + 2b_4 - 2b_1b_3 + 2b_7\cos(\sigma s_2) \\ &+2b_5\cos(\sigma\tau) - 2b_1b_6\cos(\sigma s_2)]\sigma^4 + [2b_1b_7\sin(\sigma s_2) - 2d_2d_6\sin(\sigma s_2) \\ &+2b_1b_5\sin(\sigma s_2)]\sigma^3 + [b_3^2 + b_6^2 - b_8^2 - 2b_2b_4 + 2b_3b_6\cos(\sigma s_2) \\ &-2b_2b_7\cos(\sigma s_2) - 2b_2b_5\cos(\sigma\tau)]\sigma^2 + [2b_4b_6\sin(\sigma s_2) - 2b_3b_7\sin(\sigma s_2) \\ &-2b_3b_5\sin(\sigma\tau) + 2b_5b_6\sin(\sigma s_2)\cos(\sigma\tau) - 2b_5b_6\sin(\sigma\tau)\cos(\sigma s_2)]\sigma \\ &+[b_4^2 - b_5^2 + b_6^2 - b_9^2 + 2b_4b_7\cos(\sigma s_2) + 2b_4b_5\cos(\sigma\tau) \\ &+2b_5b_7\sin(\sigma s_2)\sin(\sigma\tau) + 2b_5b_7\cos(\sigma s_2)\cos(\sigma\tau)] = 0. \end{aligned} \tag{31}$$

If (31) has a number of positive and simple root σ_i, then (7) has the following set of critical time delays determined from (30)

$$s_{1ij} = \frac{\varrho_i + 2j\pi}{\sigma_i}, \quad i = 1, 2, \ldots; \quad j = 0, 1, 2, \ldots, \tag{32}$$

where $\varrho_i \in [0, 2\pi)$ and ϱ_i satisfies

$$\begin{cases} \cos \varrho_i = \dfrac{(b_1b_8 - b_9)\sigma_i^4 + (b_2b_9 - b_3b_8)\sigma_i^2 - b_4b_9}{b_9^2 + b_8^2\sigma_i^2} \\ \qquad + \dfrac{(b_7b_8 - b_6b_9)\sigma_i \sin(\sigma_i s_2) - (b_7b_9 + b_6b_8\sigma_i^2)\cos(\sigma_i s_2)}{b_9^2 + b_8^2\sigma_i^2} \\ \qquad - \dfrac{b_5b_9 \cos \sigma_i\tau - b_5b_8\sigma_i \sin(\sigma_i\tau)}{b_9^2 + b_8^2\sigma_i^2}, \\ \sin \varrho_i = \dfrac{(b_1b_8 - b_9)\sigma_i^4 + (b_2b_9 - b_3b_8)\sigma_i^2 - b_4b_9}{b_9^2 + b_8^2\sigma_i^2} \\ \qquad + \dfrac{(b_7b_8 - b_6b_9)\sigma_i \sin(\sigma s_2) - (b_7b_9 + b_6b_8\sigma_i^2)\cos(\sigma_i s_2)}{b_9^2 + b_8^2\sigma_i^2} \\ \qquad - \dfrac{b_5b_9 \cos \sigma_i\tau - b_5b_8\sigma_i \sin(\sigma_i\tau)}{b_9^2 + b_8^2\sigma_i^2}. \end{cases} \tag{33}$$

Denote

$$s_{1c} = \min\{s_{1ij}\}, \quad i = 1, 2, \ldots; \quad j = 0, 1, 2, \ldots. \tag{34}$$

and let σ_c be the positive and simple root of (31) when $s_1 = s_{1c}$. Differentiating λ with respect to s_1 in (7), we have

$$\lambda'(s_1) = \frac{b_8\lambda^2 + b_9\lambda}{\Lambda e^{\lambda s_1} - (b_8\lambda + b_9)s_1 + b_8}, \tag{35}$$

where $\Lambda = 4\lambda^3 + 3b_1\lambda^2 + 2b_2\lambda + b_3 - \tau b_5 e^{-\lambda\tau} - s_2(b_6\lambda + b_7)e^{-\lambda s_2} + b_6 e^{-\lambda s_2}$. According to the analysis above and Hopf bifurcation theory for functional differential equations [22], we have the following results.

Theorem 2.3. *The following assertions hold true if all the roots of (8) have negative real parts.*

(i) If (31) has no positive root, then the trivial equilibrium of (4) is asymptotically stable for an arbitrary delay τ.

(ii) If (31) has at least one positive and simple root and $\mathrm{Re}[\lambda'(s_1)] > 0$ *is satisfied, then the trivial equilibrium of (4) is asymptotically stable for $s_1 \in [0, s_{1c})$ and (4) undergoes a Hopf bifurcation at $s_1 = s_{1c}$. In other word, a small amplitude periodic solution of (4) bifurcating from the trivial equilibrium near s_{1c} occurs.*

3. **Numerical Examples.** In this section, we present some numerical results of system (1) to verify the analytical predictions obtained in the previous section. Let us consider the following special case of system (1).

$$\begin{cases} \dot{u}_1(t) = -2u_1(t) + 2\tanh[u_4(t - \tau_1)] + 2\tanh[u_3(t - \tau_1)], \\ \dot{u}_2(t) = -2u_2(t) + 1.4\tanh[u_1(t - \tau_2)] + \tanh[u_4(t - \tau_2)], \\ \dot{u}_3(t) = -2u_3(t) - 1.2\tanh[u_2(t - \tau_3)], \\ \dot{u}_4(t) = -0.8u_4(t) + 2\tanh[u_3(t - \tau_4)]. \end{cases} \tag{36}$$

By (2) and (3), system (36) can be rewritten as the equivalent form

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) + 2\tanh[x_4(t - \tau)] + 2\tanh[x_3(t - s_1)], \\ \dot{x}_2(t) = -2x_2(t) + 1.4\tanh[x_1(t)] + \tanh[x_4(t - s_2)], \\ \dot{x}_3(t) = -2x_3(t) - 1.2\tanh[x_2(t)], \\ \dot{x}_4(t) = -0.8x_4(t) + 2\tanh[x_3(t)]. \end{cases} \tag{37}$$

Obviously, system (37) has a unique positive equilibrium $E(0,0,0,0)$. It is easy to see that $a_1 = 2$, $a_2 = 2$, $a_3 = 2$, $a_4 = 0.8$, $c_{14} = 2$, $c_{13} = 2$, $c_{21} = 1.4$, $c_{24} = 1$, $c_{32} = -1.2$, $c_{43} = 2$. Then $a_{14} = 2$, $a_{13} = 2$, $a_{21} = 1.4$, $a_{24} = 1$, $a_{32} = -1.2$, $a_{43} = 2$, $b_1 = 6.8$, $b_2 = 16.8$, $b_3 = 17.6$, $b_4 = 6.4$, $b_5 = -6.72$, $b_6 = 2.2$, $b_7 = -4.8$, $b_8 = -3.36$, $b_9 = -6.72$. Thus, we can easily check that (20), (21) and $B_i > 0$ $(i = 1,2,3,4)$. Thus, we can conclude that the trivial equilibrium of (37) is asymptotically stable for $s_1 \geq 0$, $s_2 \geq 0$, $\tau \geq 0$ and the stability switching of system (37) does not exist which is shown in Figure 1.

Let $s_2 = 0$ and by Matlab 7.0 software, we can know that (25) has two positive and simple roots and $\tau_c \approx 2.018$ and $\mathrm{Re}[\lambda'(\tau_c)] \approx 0.7045 > 0$, then the trivial equilibrium of (37) is asymptotically stable for $\tau \in [0, 2.018)$ and (37) undergoes a branch of periodic solutions bifurcating from the trivial equilibrium at $\tau_c \approx 2.018$ which can be shown in Figure 2 and Figure 3.

Let $s_2 = 0.92$, $\tau = 1.02$ and by Matlab 7.0 software, we can know that (25) has two positive and simple roots and $\tau_c \approx 2.018$ and $\mathrm{Re}[\lambda'(s_{1c})] \approx 1.0921 > 0$, then the trivial equilibrium of (37) is asymptotically stable for $\tau \in [0, 1.12)$ and (37) undergoes a branch of periodic solutions bifurcating from the trivial equilibrium at $s_{1c} \approx 1.12$ which can be shown in Figure 4 and Figure 5.

Remark 3.1. *All the numerical simulations are carried out by means of MATLAB software 7.0. With the aid of the Hopf bifurcation theory, we show that the different time delays have different effects on the dynamical behavior. It is shown that when we choose a certain time delay as bifurcation parameter, the system is asymptotically stable for some range of time delays and when the time delay crosses some critical values, then the system becomes unstable and a Hopf bifurcation will appear. This plays an important role in the design of neural networks. To the best of our knowledge, there are few papers that focus on the effect of time delay on stability and Hopf bifurcation of neural networks with multiple delays. From the viewpoint, we can conclude that our results are new and complement some previously known results such as [1-16].*

4. **Conclusions.** In this paper, we have investigated local stability of the zero equilibrium $E(0,0,0,0)$ and local Hopf bifurcation in a four-neuron network with multiple discrete delays. We have showed that under some suitable conditions, the zero equilibrium $E(0,0,0,0)$ of system (1) is asymptotically stable for $s_1 \geq 0$, $s_2 \geq 0$ and $\tau \geq 0$. If some conditions hold true, then the stability switching of system (1) does not exist. Fixing $s_2 = 0$ and regarding the time delay τ as bifurcation parameter, we find that the zero equilibrium $E(0,0,0,0)$ of system (1) is asymptotically stable for $\tau \in [0, \tau_c)$, as the delay τ increases, the equilibrium loses its stability and a sequence of Hopf bifurcations occur at the zero equilibrium $E(0,0,0,0)$; in other words, a family of periodic orbits bifurcate from the zero equilibrium $E(0,0,0,0)$. We have also shown that if $s_2 > 0$, $\tau > 0$, then the zero equilibrium $E(0,0,0,0)$ of system (1) is asymptotically stable for $s_1 \in [0, s_{1c})$, as the delay s_1 increases, the equilibrium loses its stability and a sequence of Hopf bifurcations occur at the zero equilibrium $E(0,0,0,0)$. Studies show that the different time delays have different effects on the stability and Hopf bifurcation behavior of four-neuron network with multiple discrete delays.

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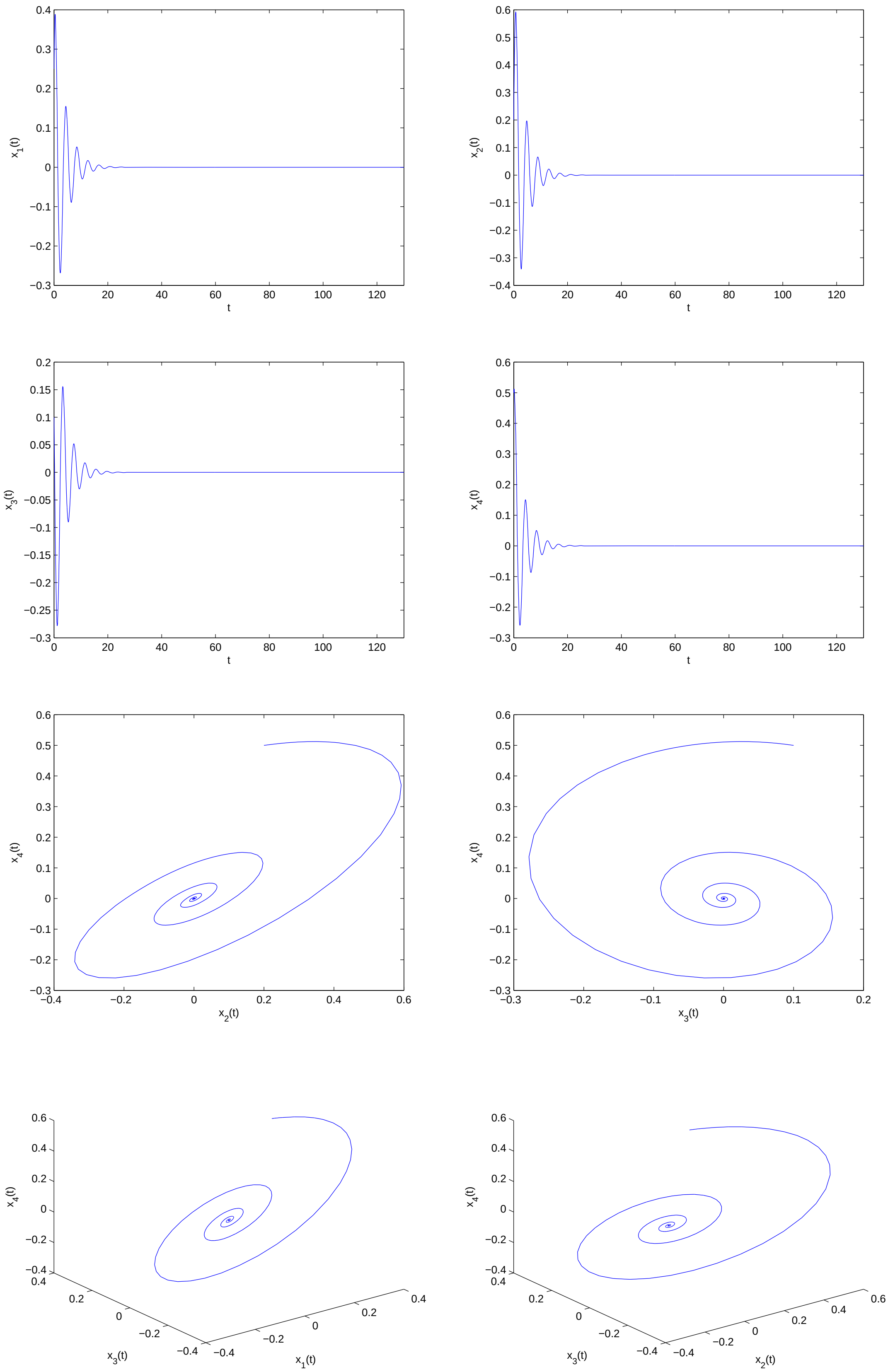

FIGURE 1. Dynamic behavior of system (37). Computer simulations of the asymptotically stable zero equilibrium to system (37) with $s_1 = 3.7$, $s_2 = 3.4$ and $\tau = 3.1$ and the initial value $(0.3; 0.3; 0.3; 0.3)$.

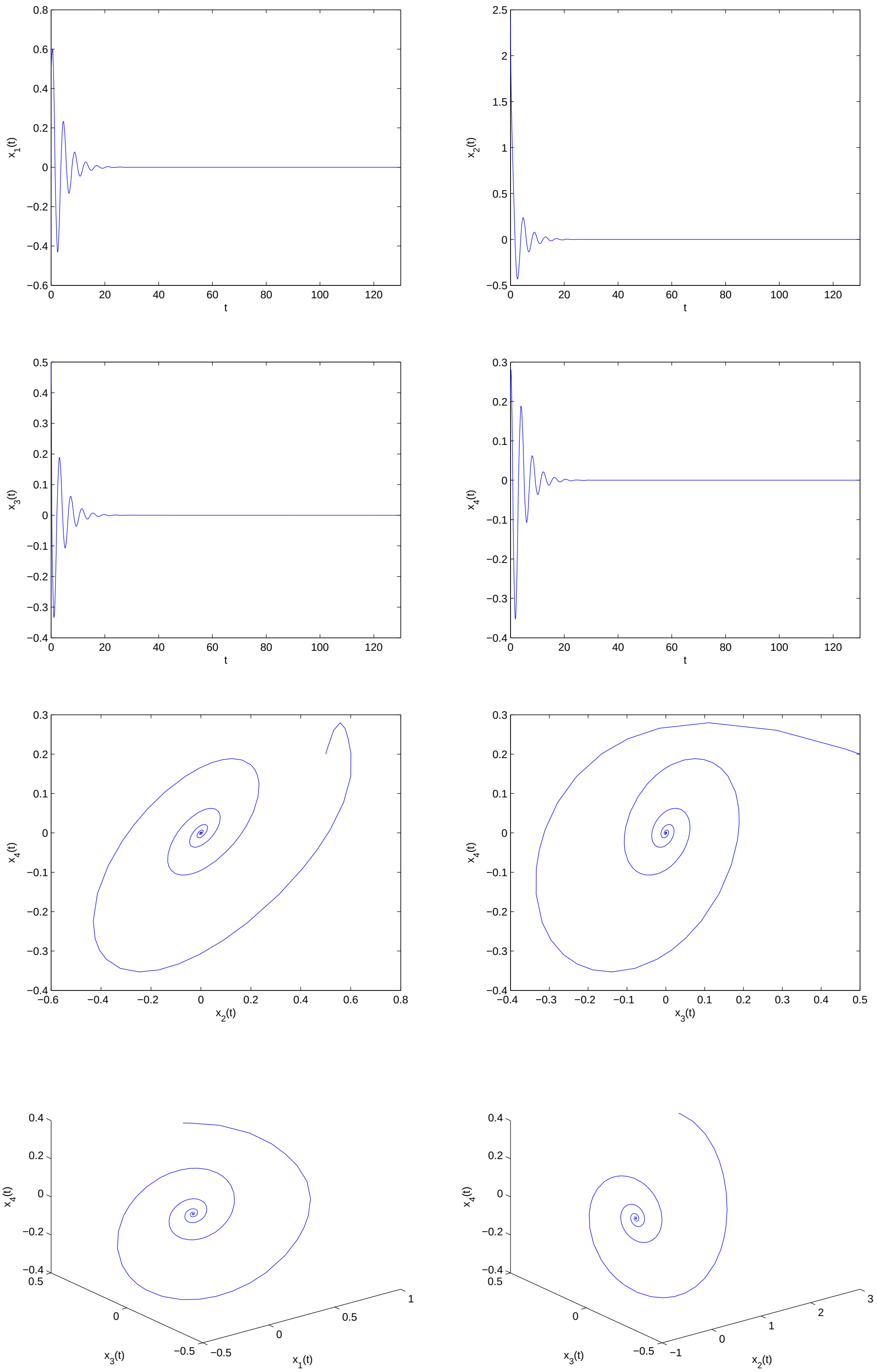

FIGURE 2. Dynamic behavior of system (37). Computer simulations of the asymptotically stablezero equilibrium to system (37) with $s_2 = 0$ and $\tau = 2.8 < \tau_c \approx 3.1$ and the initial value is $(0.5, 2.4, 0.5, 0.2)$.

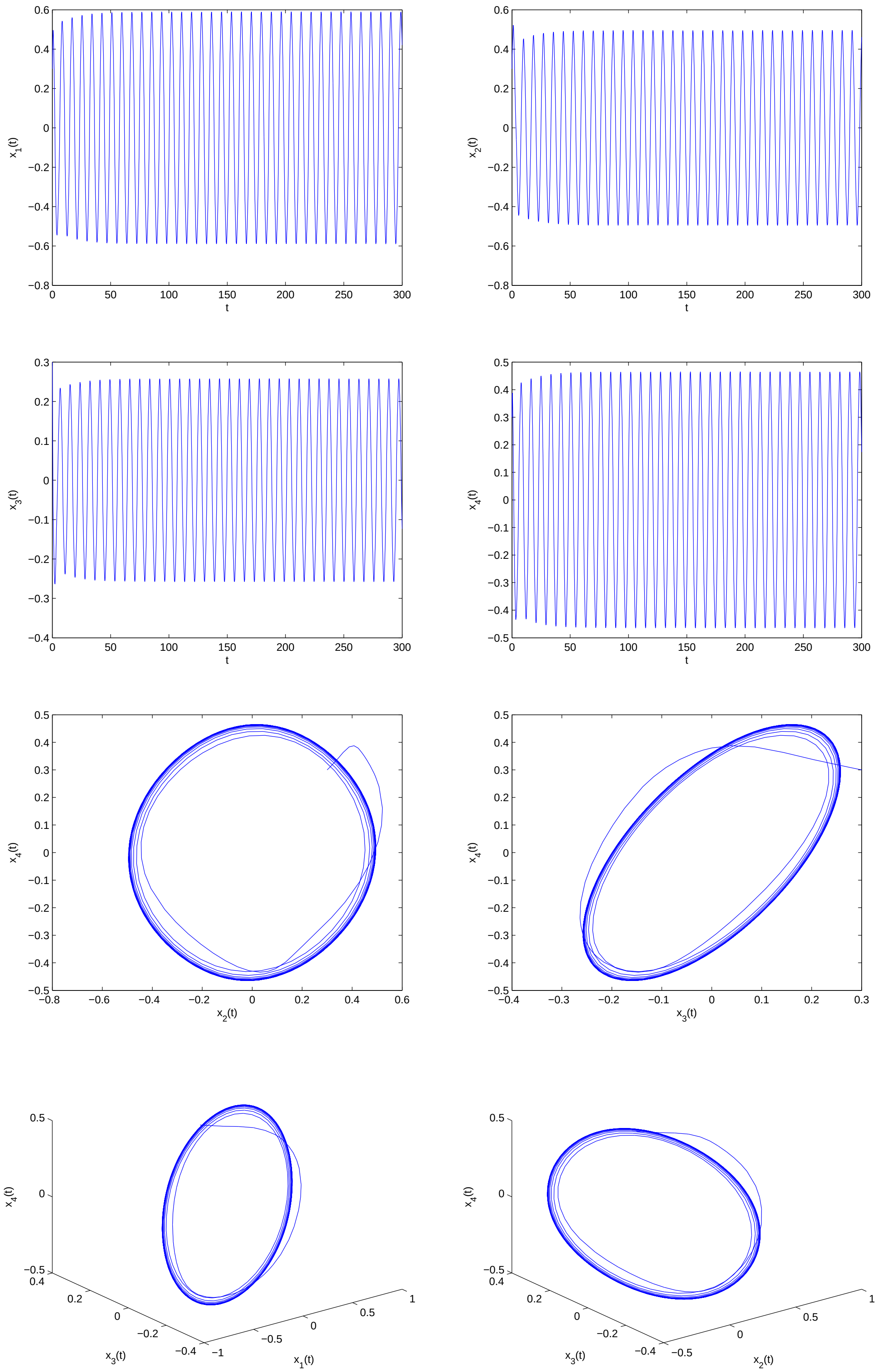

FIGURE 3. Dynamic behavior of system (37). Computer simulations of periodic solutions to system (37) with $s_2 = 0$ and $\tau = 3.4 > \tau_c \approx 3.1$ and the initial value $(0.5, 2.4, 0.5, 0.2)$.

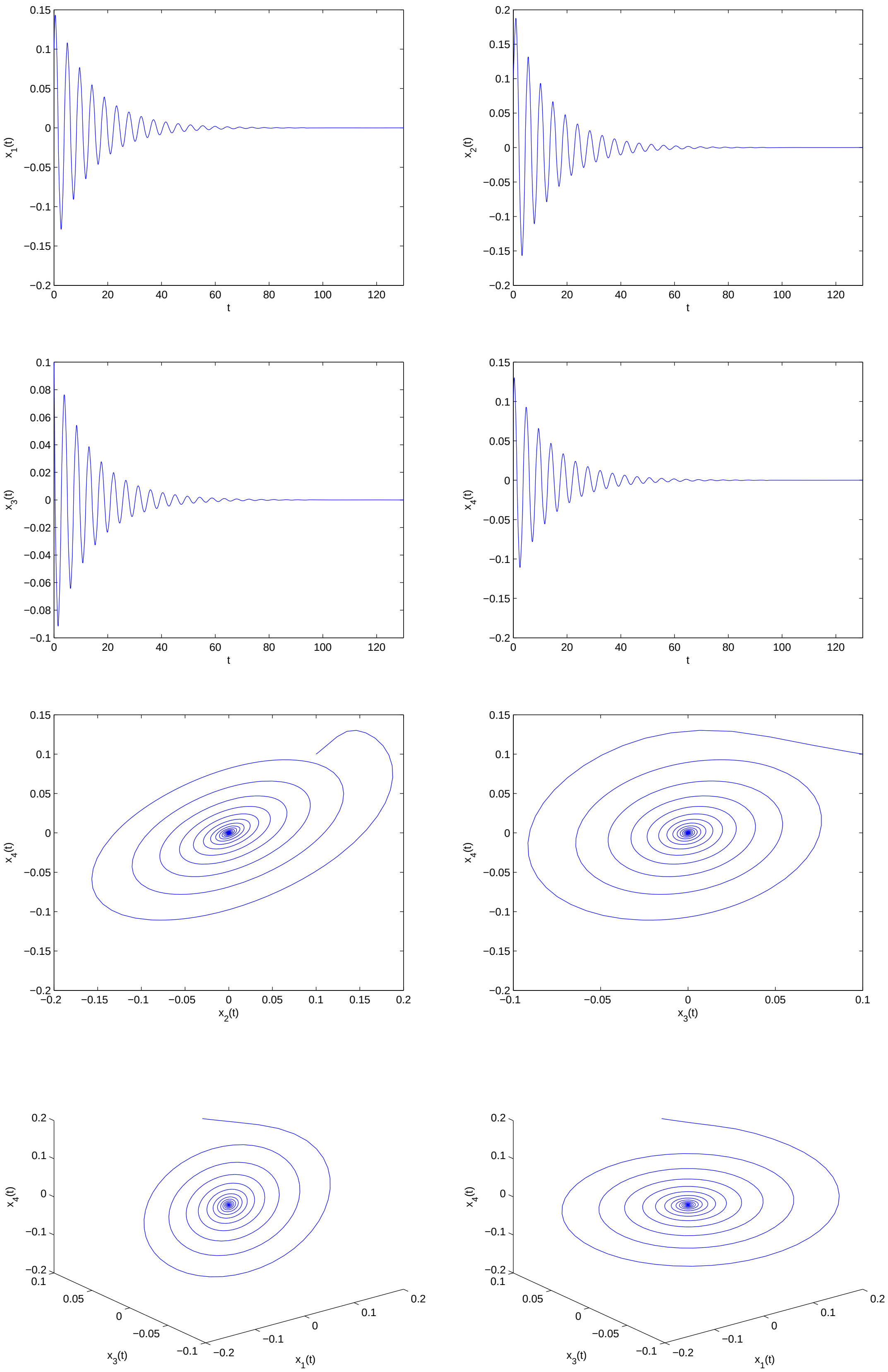

FIGURE 4. Dynamic behavior of system (37). Computer simulations of the asymptotically stable positive equilibrium to system (37) with $s_2 = 0.92$, $\tau = 1.02$ and $s_1 = 0.7 < s_{1c} \approx 1.12$ and the initial value $(0.5, 2.4, 0.5, 0.2)$.

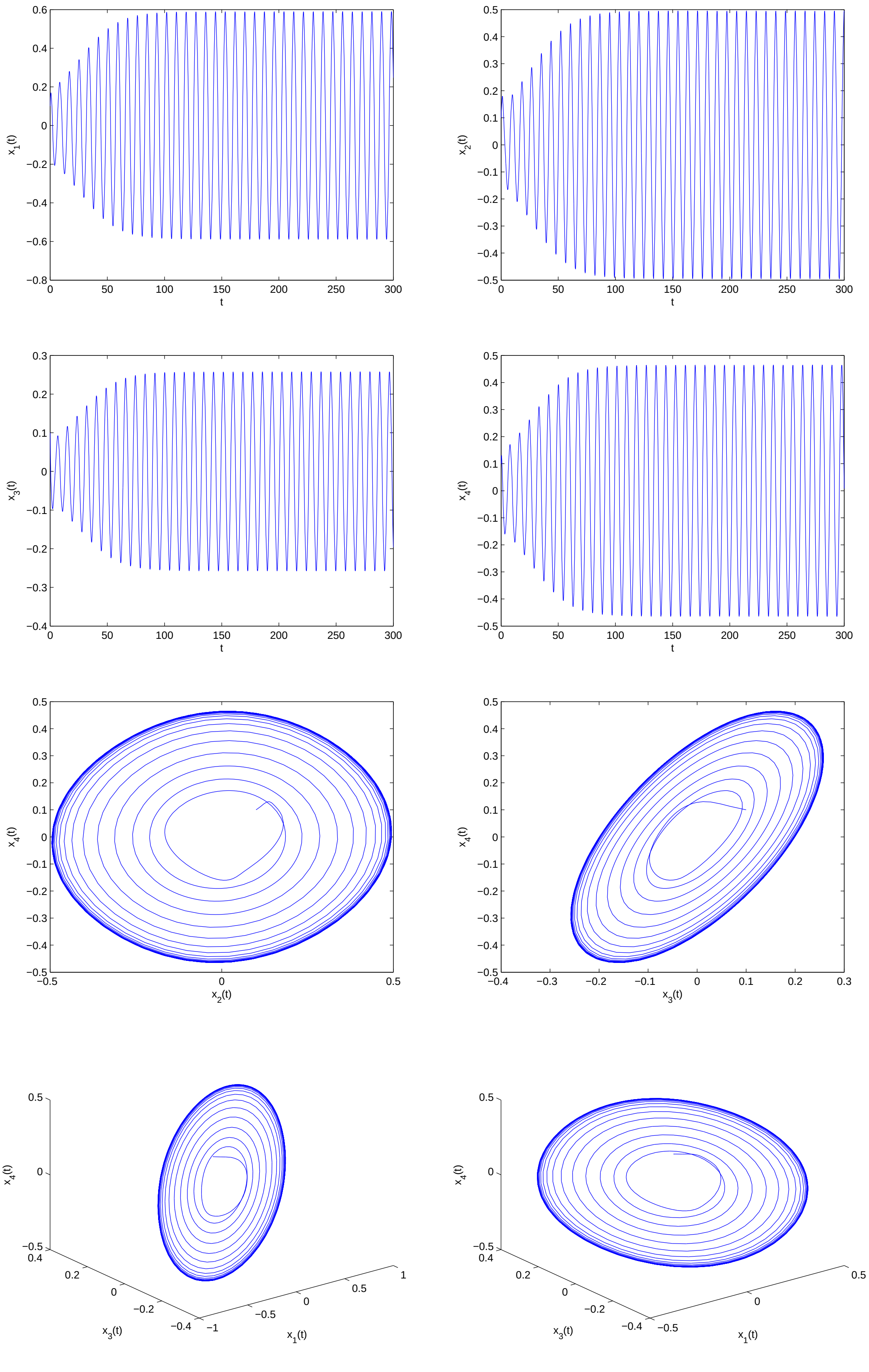

FIGURE 5. Dynamic behavior of system (37). Computer simulations of periodic solutions to system (37) with $s_2 = 0.92$, $\tau = 1.02$ and $s_1 = 3.7 > s_{1c} \approx 1.12$ and the initial value $(0.5, 2.4, 0.5, 0.2)$.

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