International Journal of Innovative Computing, Information and Control
Volume **11**, Number **6**, December **2015**
ICIC International ©2015 ISSN 1349-4198
pp. 1881–1895

OPTIMAL ESTIMATION PROBLEM FOR DISCRETE-TIME SYSTEMS WITH MULTI-CHANNEL MULTIPLICATIVE NOISE

Xin Wu[1], Xinmin Song[1] and Xuehua Yan[2]

[1]School of Information Science and Engineering
Shandong Normal University
No. 88, East Wenhua Road, Lixia District, Jinan 250014, P. R. China
wuxintengzhou@126.com; xinminsong@sina.com

[2]School of Electrical Engineering
University of Jinan
No. 336, West Road of Nan Xinzhuang, Jinan 250022, P. R. China
yanxh@mail.sdu.edu.cn

Received April 2015; revised September 2015

Abstract. *Multiplicative noise is usually assumed to be a scalar in existing literature works. Motivated by recent applications of communication technology, we consider multi-channel multiplicative noise represented by a diagonal matrix. Firstly we consider the estimation problem for multi-channel multiplicative noise systems, where the multiplicative noise appears both in state equation and measurement equation respectively. Based on the projection theory, the estimator is derived in terms of a difference Riccati equation with Hadamard product and a difference Lyapunov equation. Then as an extension, we consider the estimation problem for multi-channel multiplicative noise systems with time delay. The solution to the estimator can be obtained by calculating a partial difference Riccati equation and a Lyapunov equation. Compared with the conventional augmentation approach, the presented approach lessens the computational demand when the delay is large. Finally we present a numerical example to demonstrate the efficiency of the proposed method in this paper.*

Keywords: Multi-channel, Riccati equation, Estimation, Multiplicative noise, Projection

1. **Introduction.** Recently multiplicative noise has been a mainstream research topic, due to the fact that the signals contaminated by multiplicative noise are common in engineering and society. Such examples can be found in image processing [1], communication systems [2], etc. Different from the additive noise, the second order statistics of the multiplicative noise is usually unknown as it depends on the control solution, which leads to additional difficulties. Hence, the control and estimation problems for systems with multiplicative noise have received much attention [3-8].

The early work [9] considered the linear minimum mean squared error (LMMSE) estimation. By modeling the uncertainty via a sequence of i.i.d. binary random variables, the author derived a recursion similar to the Kalman filter by utilizing the statistics of the unobserved binary uncertainty sequence. [10] gave conditions for obtaining recursive filtering when the uncertainty sequence was not necessarily i.i.d. In [11], Tugnait defined the observability and controllability of the discrete system with multiplicative noise and introduced the classical equivalent filter system in a sense of linear minimum mean squared error. Apart from the above work, the author also discussed the stability of Rajasekaran's state filter algorithm. In [12], the authors proposed the block component search algorithm based on the principle of maximum likelihood. The advantages of this algorithm are that statistical parameters of multiplicative noise need not to be known in advance; moreover, the optimal estimation of state and multiplicative noise sequence are still able to be obtained at the same time. With the development of system with multiplicative noise rapidly, [13] proposed the optimal estimation algorithm based on the idea of Kalman filter, giving the recursive optimal filtering algorithm of system which contained entries and non-entries respectively. [14] investigated the robust state estimation problem with missing measurements. Though introducing a monotonic function and using the so-called squeeze rule, the new robust estimator is proved to converge to a stable system. [15] presented the mean-square optimal data-based quadratic-Gaussian controller for stochastic nonlinear polynomial systems with a polynomial multiplicative noise, a linear control input, and a quadratic criterion over linear observations. [16] discussed the problem of estimation of the remote signal generated by a class of discrete dynamical systems with periodic coefficients subject to multiplicative noise and additive noise. It should be noted that all the multiplicative noises in [3-16] are in scalar form. In other words, the multiplicative noise of each channel is assumed to be same, which is restricted and unrealistic. In [17], the authors studied the optimal fixed domain smoothing algorithm of systems with multi-channel multiplicative noise. However, the multiplicative noise only occurs in measurement equation.

State estimation problems for stochastic systems with time delay have caused the extensive concern in the past decades [18-23]. Generally speaking, for the discrete-time systems with time delay, the LMMSE problem is firstly considered to be dealt with by using the state augmentation method of [25] and standard Kalman filtering formula of [26]. However, the state augmentation may bring higher state dimension and higher computational cost. To prevent the high computational cost, [18] proposed a new method which was called reorganized innovation analysis. This new approach is effective to solve the state estimation problems for a class of discrete-time systems with measurement delay. Furthermore, [19] discussed the problems of filtering and fixed-lag smoothing for linear multiplicative noise systems with single delayed measurement and derived the estimator by using the reorganized innovation approach. It is, however, noted that the LMMSE estimator problem for systems with single measurement channel but multiple measurement delays which contain the systems studied in [18, 19] as a special case is much more challenging and cannot be directly dealt with by using the reorganized innovation approach. [20] considered the problem of optimal linear estimation for multiplicative noise systems with time delay, where the delay appeared both in state equation and measurement equation. The estimator was obtained in terms of a forward partial difference Riccati equation and a forward Lyapunov equation. The LMMSE filtering problem was investigated for uncertain stochastic systems with time-invariant state delay d_0, bounded random observation delays and missing measurements in [21]. In addition, for discrete-time stochastic linear systems with random measurement delays and packet dropouts, [22] got the optimal estimators including filter, predictor and smoother in a sense of linear minimum variance. The authors in [23] considered the optimal linear estimation problem for networked stochastic uncertain systems with multiple packet dropouts and delays. The random uncertainties of system parameters were described by white multiplicative noise. [24] concerned with the Kalman filtering for discrete stochastic systems with multiplicative noises and random two-step sensor delays.

Motivated by the work mentioned above, we consider the optimal estimation problem for discrete-time systems with multiplicative noise represented by a diagonal matrix, where the multiplicative noise of one channel is allowed to be different from another channel. To the best of our knowledge, there is no other work dealing with this problem in the literature. The main contributions of this paper are highlighted as follows: (1) We extend the scalar multiplicative case to the diagonal matrix multiplicative case. The multiplicative noise represented by a diagonal matrix makes the problem more challenging. In order to overcome this difficulty, we introduce the Hadamard product, which makes the results simple and beautiful; (2) We extend the optimal estimation problem to the systems with time delay. Considering the disadvantages of traditional approach of state augmentation, we adopt the method of calculating a partial difference Riccati equation which has the same dimension as the original system. When the delay is very large, the proposed approach is efficient to solve optimal estimation problem for systems with time delay; (3) [17] investigated the estimation problem for multi-channel multiplicative noise systems; however, the multiplicative noise only appeared in measurement equation. [20] considered the estimation problem for time-delay systems with multiplicative noise, but the multiplicative noises are in scalar form; moreover, the multiplicative noises in state equation and the noises in measurement equation are assumed to be uncorrelated. However, the multiplicative noises that appeared in our paper are assumed to be diagonal matrix case. Also, the multiplicative noises in state equation and the noises in measurement equation are assumed to be correlated. Hence, compared with the existing works [17, 20], our work is more general.

The arrangement of this paper is as follows. Section 2 gives problem statement and proposes the assumption system model. Section 3 derives the main results on estimator for multi-channel multiplicative noise systems. In Section 4, we extend the estimation problem in Section 3 to time-delay case. A numerical example is showed in Section 5 to demonstrate the efficiency of the proposed approach. Section 6 is the conclusion, containing the main work and significance of this paper.

Notation: Throughout this paper, a real symmetric matrix $P > 0$ (≥ 0) denotes P being a positive definite (or positive semi-definite) matrix. I denotes an identity matrix of appropriate dimension. The superscripts "-1" and "T" represent the inverse and transpose of a matrix. $\mathcal{R}^n$ stands for the n-dimensional Euclidean space. $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices. $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. The Hadamard product by $\odot$. $diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ stand for the diagonal matrix having $\lambda_1, \lambda_2, \cdots, \lambda_n$ as its diagonal elements. Furthermore, the mathematical expectation operator is denoted by E. We can define $\langle x, y\rangle = E\left\{xy^T\right\} = R_{xy}$ and $\langle x, x\rangle = E\left\{xx^T\right\} = R_x$. Matrices, if the dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. **Problems Statement.**

Consider the following discrete-time systems with multi-channel multiplicative noise

$$x(k+1) = [A + \xi(k)A_0]x(k) + n(k), \tag{1}$$

$$y(k) = [B + \eta(k)B_0]x(k) + v(k), \tag{2}$$

where $x(k) \in \mathcal{R}^n$, and $y(k) \in \mathcal{R}^m$ are respectively the system state and measurement. $n(k)$ and $v(k)$ are white noises with zero mean and covariances $E\left\{n(k)n^T(j)\right\} = Q\delta_{kj}$, $E\left\{v(k)v^T(j)\right\} = R\delta_{kj}$ respectively. Here $\xi(k) = diag\{\xi_1(k), \ldots, \xi_n(k)\}$, whose elements are random processes with mean $E\{\xi_i(k)\} = 0$ and covariances $E\left\{\xi_i(k)\xi_j^T(s)\right\} = \sigma_{ij}\delta_{ks}$, $\eta(k) = diag\{\eta_1(k), \cdots, \eta_m(k)\}$, whose elements are random processes with mean $E\{\eta_i(k)\} = 0$ and covariances $E\left\{\eta_i(k)\eta_j^T(s)\right\} = \rho_{ij}\delta_{ks}$. Also, for $\forall i, j$, $E\left\{\xi_i(k)\eta_j^T(s)\right\} = \gamma_{ij}\delta_{ks}$. For convenience, we further denote $\Pi_1 = [\sigma_{ij}]_{i,j=1,2,\ldots,n}$, $\Pi_2 = [\rho_{ij}]_{i,j=1,2,\ldots,m}$, $\Pi_3 = [\gamma_{ij}]_{i=1,2,\ldots,n,j=1,2\ldots,m}$. The initial state $x(0)$ is a random vector with mean μ_0 and covariance matrix $D(0)$. The random processes $n(k)$, $v(k)$, $\xi(k)$ and $\eta(k)$ for all k and the initial state $x(0)$ are mutually independent.

Problem I: Based on systems (1) and (2), our aim is to find the LMMSE estimate $\hat{x}(k|k-1)$ of the system state $x(k)$ based on the measurements sequence $\{y(k-1), \cdots, y(0)\}$, i.e., minimize the Euclidean 2-norm

$$\begin{aligned} J &= E\left\{[x(k)-\hat{x}(k|k-1)]^T[x(k)-\hat{x}(k|k-1)]\right\} \\ &= E\left\{\tilde{x}^T(k|k-1)\tilde{x}(k|k-1)\right\} \end{aligned}$$

at every time moment k.

Based on the above research, we will consider the linear system with multi-channel multiplicative noise and time delay

$$x(k+1) = Ax(k) + A_d x(k-d) + \xi(k)A_\xi x(k) + n(k), \tag{3}$$

$$y(k) = Bx(k) + B_d x(k-d) + \eta(k)B_\eta x(k) + v(k), \tag{4}$$

where $x(k)$, $y(k)$, $\xi(k)$, $\eta(k)$, $n(k)$ and $v(k)$ have the same properties with the above problem; we do not go into details here. The initial states satisfy $E\{x(-i)\} = \mu_{-i}$ $(i = 0, \ldots, d)$, $E\left\{x(-i)x^T(-j)\right\} = P_0(-i,-j)$. It is assumed that $n(k)$, $v(k)$, $\xi(k)$, $\eta(k)$ for all k and the initial states $x(-i)$ are mutually independent. For convenience, we suppose that A, A_d, A_ξ, B, B_d, B_η are the constant and the compatible dimension matrices. Also,

$$A_\xi = \left[\begin{array}{ccc} A_{\xi 1}^T & \cdots & A_{\xi n}^T \end{array}\right]^T, \; B_\eta = \left[\begin{array}{ccc} B_{\eta 1}^T & \cdots & B_{\eta m}^T \end{array}\right]^T.$$

Problem II: Based on systems (3) and (4), the aim is to find the LMMSE estimate $\hat{x}(k-l|k)$ $(l = -1, 0, \cdots, d)$ of the system state $x(k-l)$ based on the measurements sequence $\{y(k), \cdots, y(0)\}$, i.e., minimize the Euclidean 2-norm

$$\begin{aligned} J &= E\left\{[x(k-l)-\hat{x}(k-l|k)]^T[x(k-l)-\hat{x}(k-l|k)]\right\} \\ &= E\left\{\tilde{x}^T(k-l|k)\tilde{x}(k-l|k)\right\} \end{aligned}$$

at every time moment k.

Remark 2.1. *Most studies on multiplicative noise systems focus on the scalar multiplicative noise case* [3-16, 19-23]. [17] *considered the optimal estimation problem for multi-channel multiplicative noise systems, but the multiplicative noise only occurs in measurement equation. In order to get more general and more accurate result, the multi-channel multiplicative noise occurs both in state equation and measurement equation in this paper. Also, we will consider the estimation problem for multiplicative noise systems with time delay. Hence the model considered in our paper is more general compared with* [17, 20].

3. Estimator Design.

In this section we will deduce the LMMSE estimator for multi-channel multiplicative noise systems.

For convenience, we define

$$A_0 = \left[\begin{array}{ccc} A_{01}^T & \cdots & A_{0n}^T \end{array}\right]^T, \; B_0 = \left[\begin{array}{ccc} B_{01}^T & \cdots & B_{0m}^T \end{array}\right]^T,$$

$$I_{n\times n} = \left[\begin{array}{ccc} e_1 & \cdots & e_n \end{array}\right], \; I_{m\times m} = \left[\begin{array}{ccc} \alpha_1 & \cdots & \alpha_m \end{array}\right],$$

$$P(k) = E\left\{\tilde{x}(k|k-1)\tilde{x}^T(k|k-1)\right\}, \; \tilde{x}(k|k-1) = x(k) - \hat{x}(k|k-1),$$

$$D(k) = E\left\{x(k)x^T(k)\right\}.$$

Theorem 3.1. *Consider systems (1) and (2). The LMMSE predictor is computed by*

$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + K_p(k)\varepsilon(k), \tag{5}$$

$$\varepsilon(k) = B\tilde{x}(k|k-1) + \eta(k)B_0 x(k) + v(k), \tag{6}$$

where $K_p(k)$ is given by

$$K_p(k) = \left[AP(k)B^T + \Pi_3 \odot \left(A_0 D(k) B_0^T\right)\right] R_{\varepsilon(k)}^{-1}, \tag{7}$$

$$R_{\varepsilon(k)} = BP(k)B^T + \Pi_2 \odot \left(B_0 D(k) B_0^T\right) + R, \tag{8}$$

while $D(k)$ and $P(k)$ can be calculated by

$$D(k+1) = AD(k)A^T + \Pi_1 \odot \left(A_0 D(k) A_0^T\right) + Q, \tag{9}$$

$$P(k+1) = AP(k)A^T + \Pi_1 \odot \left(A_0 D(k) A_0^T\right) + Q - K_p(k) R_{\varepsilon(k)} K_p^T(k), \tag{10}$$

$$P(0) = D(0).$$

Proof: According to the definition of innovation, we can get

$$\begin{aligned} \varepsilon(k) &= y(k) - \hat{y}(k|k-1) \\ &= y(k) - B\hat{x}(k|k-1) \\ &= B\tilde{x}(k|k-1) + \eta(k)B_0 x(k) + v(k). \end{aligned} \tag{11}$$

Further, the covariance of innovation is obtained as follows

$$\begin{aligned} R_{\varepsilon(k)} &= E\left\{\varepsilon(k)\varepsilon^T(k)\right\} \\ &= E\left\{[B\tilde{x}(k|k-1) + \eta(k)B_0 x(k) + v(k)]\,[B\tilde{x}(k|k-1) + \eta(k)B_0 x(k) + v(k)]^T\right\} \\ &= \left[BP(k)B^T + E\left\{\left(\sum_{i=1}^{m} \alpha_i \eta_i(k) B_{0i}\right) x(k)x^T(k) \left(\sum_{j=1}^{m} B_{0j}^T \eta_j^T(k) \alpha_j^T\right)\right\} + R\right] \\ &= \left[BP(k)B^T + \sum_{i=1}^{m}\sum_{j=1}^{m} \rho_{ij}\alpha_i B_{0i} D(k) B_{0j}^T \alpha_j^T + R\right] \\ &= BP(k)B^T + \Pi_2 \odot \left(B_0 D(k) B_0^T\right) + R. \end{aligned}$$

By employing the projection theory, we have

$$\begin{aligned} \hat{x}(k+1|k) &= proj\{x(k+1)|\varepsilon(0), \cdots, \varepsilon(k)\} \\ &= proj\{[A + \xi(k)A_0]x(k) + n(k)|\varepsilon(0), \cdots, \varepsilon(k)\} \\ &= proj\{[A + \xi(k)A_0]x(k)|\varepsilon(0), \cdots, \varepsilon(k-1)\} \\ &\quad + E\left\{[A + \xi(k)A_0]x(k)\varepsilon^T(k)\right\} R_{\varepsilon(k)}^{-1}\varepsilon(k) \\ &= A\hat{x}(k|k-1) + K_p(k)\varepsilon(k), \end{aligned} \tag{12}$$

where $K_p(k)$ is calculated by

$$\begin{aligned} K_p(k) &= E\left\{[A + \xi(k)A_0]x(k)\varepsilon^T(k)\right\} R_{\varepsilon(k)}^{-1} \\ &= E\left\{[A + \xi(k)A_0]x(k)[B\tilde{x}(k|k-1) + \eta(k)B_0 x(k) + v(k)]^T\right\} R_{\varepsilon(k)}^{-1} \\ &= \left[AP(k)B^T + E\left\{\xi(k)A_0 x(k)x^T(k)B_0^T \eta(k)\right\}\right] R_{\varepsilon(k)}^{-1} \\ &= \left[AP(k)B^T + E\left\{\left(\sum_{i=1}^{n} e_i \xi_i(k) A_{0i}\right) x(k)x^T(k) \left(\sum_{j=1}^{m} B_{0j}^T \eta_j^T(k)\alpha_j^T\right)\right\}\right] R_{\varepsilon(k)}^{-1} \\ &= \left[AP(k)B^T + \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_{ij} e_i A_{0i} D(k) B_{0j}^T \alpha_j^T\right] R_{\varepsilon(k)}^{-1} \\ &= \left[AP(k)B^T + \Pi_3 \odot \left(A_0 D(k) B_0^T\right)\right] R_{\varepsilon(k)}^{-1}. \end{aligned} \tag{13}$$

Thus, (5)-(8) is proved. Meanwhile, it follows from (1) one has

$$\begin{aligned} D(k+1) &= E\left\{x(k+1)x^T(k+1)\right\} \\ &= E\left\{([A + \xi(k)A_0]x(k) + n(k))([A + \xi(k)A_0]x(k) + n(k))^T\right\} \end{aligned}$$

$$
\begin{aligned}
&= \left[AD(k)A^T + E\left\{\left(\sum_{i=1}^{n} e_i\xi_i(k)A_{0i}\right)x(k)x^T(k)\left(\sum_{j=1}^{n} A_{0j}^T\xi_j^T(k)e_j^T\right)\right\} + Q\right] \\
&= \left[AD(k)A^T + \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}e_iA_{0i}D(k)A_{0j}^Te_j^T + Q\right] \\
&= AD(k)A^T + \Pi_1 \odot \left(A_0D(k)A_0^T\right) + Q,
\end{aligned}
$$

hence (9) is obtained. After straightforward calculation based on (1) and (12) yield

$$
\begin{aligned}
\tilde{x}(k+1|k) = {} & [A - K_p(k)B]\tilde{x}(k|k-1) + n(k) + [\xi(k)A_0 - K_p(k)\eta(k)B_0]x(k) \\
& -K_p(k)v(k),
\end{aligned} \tag{14}
$$

thus

$$
\begin{aligned}
P(k+1) &= E\left\{\tilde{x}(k+1|k)\tilde{x}^T(k+1|k)\right\} \\
&= [A - K_p(k)B]P(k)[A - K_p(k)B]^T + Q + K_p(k)RK_p^T(k) \\
&\quad + E\left\{[\xi(k)A_0 - K_p(k)\eta(k)B_0]x(k)x^T(k)[\xi(k)A_0 - K_p(k)\eta(k)B_0]^T\right\} \\
&= [A - K_p(k)B]P(k)[A - K_p(k)B]^T + Q + K_p(k)RK_p^T(k) \\
&\quad + E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} e_i\xi_i(k)A_{0i}x(k)x^T(k)A_{0j}^T\xi_j^T(k)e_j^T\right\} \\
&\quad - E\left\{\sum_{i=1}^{n}\sum_{j=1}^{m} e_i\xi_i(k)A_{0i}x(k)x^T(k)B_{0j}^T\eta_j^T(k)\alpha_j^T\right\}K_p^T(k) \\
&\quad - K_p(k)E\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_i\eta_i(k)B_{0i}x(k)x^T(k)A_{0j}^T\xi_j^T(k)e_j^T\right\} \\
&\quad + K_p(k)E\left\{\sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i\eta_i(k)B_{0i}x(k)x^T(k)B_{0j}^T\eta_j^T(k)\alpha_j^T\right\}K_p^T(k) \\
&= [A - K_p(k)B]P(k)[A - K_p(k)B]^T + Q + K_p(k)RK_p^T(k) + \Pi_1 \odot \left(A_0D(k)A_0^T\right) \\
&\quad + K_p(k)\left[\Pi_2 \odot \left(B_0D(k)B_0^T\right)\right]K_p^T(k) - K_p(k)\left[\Pi_3^T \odot \left(B_0D(k)A_0^T\right)\right] \\
&\quad - \left[\Pi_3 \odot \left(A_0D(k)B_0^T\right)\right]K_p^T(k) \\
&= AP(k)A^T + \Pi_1 \odot \left(A_0D(k)A_0^T\right) + Q - K_p(k)R_{\varepsilon(k)}K_p^T(k).
\end{aligned} \tag{15}
$$

Therefore, (10) is obtained. The proof is ended here. ∇

Remark 3.1. *By employing the projection theory, we have derived the LMMSE predictor for multi-channel multiplicative noise systems. The multiplicative noise represented by a diagonal matrix makes the problem more challenging. In order to overcome this difficulty, we introduce the Hadamard product, which renders it possible to obtain a similar result to the scalar multiplicative noise case. It will play an important role for us to deal with the estimation problem for multi-channel multiplicative noise systems with time delay, which will be presented later.*

4. **Extension to Time-Delay Systems.** This problem can be settled directly by employing the augmentation approach; here we give a simple explanation.

Define

$$\overline{X}(k)=\begin{bmatrix} x(k)\\ x(k-1)\\ \vdots\\ x(k-d)\end{bmatrix},\ \overline{A}=\begin{bmatrix} A & 0 & \cdots & 0 & A_d\\ I & 0 & \cdots & 0 & 0\\ 0 & I & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & 0 & I & 0\end{bmatrix},\ \overline{n}(k)=\begin{bmatrix} n(k)\\ 0\\ \vdots\\ 0\end{bmatrix},$$

$$\overline{\xi}(k)=\begin{bmatrix} \xi(k) & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & 0 & 0 & 0\end{bmatrix},\ \overline{A}_\xi=\begin{bmatrix} A_\xi & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix},$$

$\overline{B}=\begin{bmatrix} B & 0 & \cdots & 0 & B_d\end{bmatrix}$, $\overline{B}_\eta=\begin{bmatrix} B_\eta & 0 & \cdots & 0\end{bmatrix}$, $\overline{\eta}(k)=diag\{\eta(k),\ 0,\ \cdots,\ 0\}$, thus systems (3) and (4) can be changed as an equivalent system:

$$\overline{X}(k+1) = \left[\overline{A}+\overline{\xi}(k)\overline{A}_\xi\right]\overline{X}(k)+\overline{n}(k), \tag{16}$$

$$y(k) = \left[\overline{B}+\overline{\eta}(k)\overline{B}_\eta\right]\overline{X}(k)+v(k). \tag{17}$$

Hence the estimator can be designed directly for time-delay system by employing the result of Theorem 3.1. Apart from the augmentation approach, we consider another method by calculating a partial difference Riccati equation and a Lyapunov equation, which has the same dimension as the original system (3). Before solving the optimal estimator $\hat{x}(k-j|k)$ of systems (3) and (4), we introduce an important lemma as follows.

Lemma 4.1. *For system (3), let* $D_{i,j}(k)=E\left\{x(k-i)x^T(k-j)\right\}$, *and then we can obtain the correlated functions as follows*

$$D_{i,j}(k) = D_{j,i}^T(k) = \begin{cases} D_{i-j,0}(k-j), & i>j,\\ D_{0,j-i}(k-i), & i\le j,\end{cases} \tag{18}$$

where

$$\begin{aligned} D_{0,0}(k) &= AD_{0,0}(k-1)A^T + A_dD_{0,0}(k-1-d)A_d^T + \Pi_1\odot\left(A_\xi D_{0,0}(k-1)A_\xi^T\right)\\ &\quad + AD_{0,d}(k-1)A_d^T + A_dD_{0,d}^T(k-1)A^T + Q, \qquad (19)\\ D_{0,s}(k) &= AD_{0,s-1}(k-1) + A_dD_{0,d-s+1}^T(k-s), \qquad (20)\\ D_{i,j}(0) &= P_0(-i,-j). \end{aligned}$$

Proof: According to the definition of $D_{i,j}(k)$, it is obvious that (18) holds. It follows from (18), and one has

$$\begin{aligned} D_{0,0}(k) &= E\left\{x(k-0)x^T(k-0)\right\}\\ &= E\big\{[Ax(k-1)+A_dx(k-d-1)+\xi(k-1)A_\xi x(k-1)+n(k-1)][Ax(k-1)\\ &\quad + A_dx(k-d-1)+\xi(k-1)A_\xi x(k-1)+n(k-1)]^T\big\}\\ &= AE\left\{x(k-1)x^T(k-1)\right\}A^T + A_dE\left\{x(k-d-1)x^T(k-d-1)\right\}A_d^T\\ &\quad + E\left\{\left(\sum_{i=1}^n e_i\xi_i(k-1)A_{\xi i}\right)x(k-1)x^T(k-1)\left(\sum_{j=1}^n A_{\xi j}^T\xi_j^T(k-1)e_j^T\right)\right\}\\ &\quad + AE\left\{x(k-1)x^T(k-1-d)\right\}A_d^T + A_dE\left\{x(k-1-d)x^T(k-1)\right\}A^T\\ &\quad + E\left\{n(k-1)n^T(k-1)\right\}\\ &= AD_{0,0}(k-1)A^T + A_dD_{0,0}(k-1-d)A_d^T \end{aligned}$$

$$+\Pi_1 \odot \left[A_\xi D_{0,0}(k-1)A_\xi^T\right] + AD_{0,d}(k-1)A_d^T + A_d D_{0,d}^T(k-1)A^T + Q,$$

which is (19). Following the same line we can derive

$$\begin{aligned}
D_{0,s}(k) &= E\left\{x(k-0)x^T(k-s)\right\} \\
&= E\left\{[Ax(k-1) + A_d x(k-d-1) + \xi(k-1)A_\xi x(k-1) + n(k-1)]x^T(k-s)\right\} \\
&= AE\left\{x(k-1)x^T(k-s)\right\} + A_d E\left\{x(k-1-d)x^T(k-s)\right\} \\
&= AE\left\{x(k-1)x^T(k-1-(s-1))\right\} + A_d E\left\{x(k-s-(d-s+1))x^T(k-s)\right\} \\
&= AD_{0,s-1}(k-1) + A_d D_{d-s+1,0}(k-s) \\
&= AD_{0,s-1}(k-1) + A_d D_{0,d-s+1}^T(k-s),
\end{aligned}$$

thus (20) is proved. The proof of this lemma is completed here. ∇

Under the condition of the standard Kalman filtering, based on systems (3) and (4), we define the innovation sequence $\varepsilon(k)$ as follows:

$$\begin{aligned}
\varepsilon(k) &= y(k) - \hat{y}(k|k-1) \\
&= B\tilde{x}(k|k-1) + B_d\tilde{x}(k-d|k-1) + \eta(k)B_\eta x(k) + v(k). \qquad (21)
\end{aligned}$$

Meanwhile, we let

$$P(i,j,k) = E\left\{[x(i) - \hat{x}(i|k)]\,[x(j) - \hat{x}(j|k)]^T\right\}. \qquad (22)$$

We are devoted to deriving the LMMSE filter and smoother by innovation analysis method in next step.

Theorem 4.1. *For systems (3) and (4), the LMMSE filter and smoother are given as follows*

$$\begin{aligned}
\hat{x}(k+1|k) &= A\hat{x}(k|k) + A_d\hat{x}(k-d|k) + N_0(k)\varepsilon(k), \qquad (23) \\
\hat{x}(k-j|k) &= \hat{x}(k-j|k-1) + K_j(k)\varepsilon(k),\ j = 0,1,\ldots,d, \qquad (24) \\
\hat{x}(-k|0) &= \mu_{-k},
\end{aligned}$$

where $N_0(k)$ and $K_j(k)$ can be calculated by

$$\begin{aligned}
N_0(k) &= \Pi_3 \odot \left(A_\xi D_{0,0}(k)B_\eta^T\right) R_{\varepsilon(k)}^{-1}, \qquad (25) \\
K_j(k) &= \left[P(k-j,k,k-1)B^T + P(k-j,k-d,k-1)B_d^T\right] R_{\varepsilon(k)}^{-1}, \qquad (26) \\
R_{\varepsilon(k)} &= BP(k,k,k-1)B^T + B_dP(k-d,k-d,k-1)B_d^T \\
&\quad + BP(k,k-d,k-1)B_d^T + B_dP(k-d,k,k-1)B^T \\
&\quad + \Pi_2 \odot \left(B_\eta D_{0,0}(k)B_\eta^T\right) + R, \qquad (27)
\end{aligned}$$

while $D_{0,0}(k)$ is calculated from Lemma 4.1, and $P(\cdot,\cdot,\cdot)$ can be calculated as follows

$$\begin{aligned}
P(k-i,k-j,k) &= P(k-i,k-j,k-1) - K_i(k)R_{\varepsilon(k)}K_j^T(k),\ 0 \le i \le j \le d, \qquad (28) \\
P(k+1,k-j,k) &= AP(k,k-j,k) + A_dP(k-d,k-j,k) \\
&\quad - N_0(k)[BP(k,k-j,k-1) \\
&\quad + B_dP(k-d,k-j,k-1)],\ 0 \le j \le d, \qquad (29) \\
P(k+1,k+1,k) &= AP(k,k,k)A^T + A_dP(k-d,k-d,k)A_d^T \\
&\quad + AP(k,k-d,k)A_d^T + A_dP(k-d,k,k)A^T \\
&\quad + \Pi_1 \odot \left(A_\xi D_{0,0}(k)A_\xi^T\right) + Q \\
&\quad - N_0(k)R_{\varepsilon(k)}\left[K_0^T(k)A^T + K_d^T(k)A_d^T\right] \\
&\quad - [N_0(k) + AK_0(k) + A_dK_d(k)]R_{\varepsilon(k)}N_0^T(k), \qquad (30)
\end{aligned}$$

$$\begin{aligned} P(k-i,k-j,k) &= P^T(k-j,k-i,k), \\ P(-i,-j,0) &= P_0(-i,-j),\ 0\le i\le j,\ 0\le j\le d. \end{aligned} \tag{31}$$

Proof: By employing the projection theory, we have

$$\begin{aligned} \hat{x}(k+1|k) &= proj\{x(k+1)|\varepsilon(0),\cdots,\varepsilon(k)\} \\ &= proj\{[Ax(k)+A_dx(k-d)+\xi(k)A_\xi x(k)+n(k)]|\varepsilon(0),\cdots,\varepsilon(k)\} \\ &= A\hat{x}(k|k)+A_d\hat{x}(k-d|k)+E\left\{\xi(k)A_\xi x(k)\varepsilon^T(k)\right\}R^{-1}_{\varepsilon(k)}\varepsilon(k) \\ &= A\hat{x}(k|k)+A_d\hat{x}(k-d|k)+N_0(k)\varepsilon(k), \end{aligned} \tag{32}$$

where $N_0(k)$ is calculated by

$$\begin{aligned} N_0(k) &= E\left\{\xi(k)A_\xi x(k)\varepsilon^T(k)\right\}R^{-1}_{\varepsilon(k)} \\ &= E\left\{\xi(k)A_\xi x(k)[B\tilde{x}(k|k-1)+B_d\tilde{x}(k-d|k-1)+\eta(k)B_\eta x(k)+v(k)]^T\right\}R^{-1}_{\varepsilon(k)} \\ &= E\left\{\xi(k)A_\xi x(k)x^T(k)B_\eta^T\eta^T(k)\right\}R^{-1}_{\varepsilon(k)} \\ &= E\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}e_i\xi_i(k)A_{\xi i}x(k)x^T(k)B_{\eta j}^T\eta_j^T(k)\alpha_j^T\right\}R^{-1}_{\varepsilon(k)} \\ &= \Pi_3\odot\left(A_\xi D_{0,0}(k)B_\eta^T\right)R^{-1}_{\varepsilon(k)}, \end{aligned} \tag{33}$$

hence (23) and (25) are proved. On the other hand, based on the projection theory one has

$$\begin{aligned} \hat{x}(k-j|k) &= proj\{x(k-j)|\varepsilon(0),\cdots,\varepsilon(k)\} \\ &= proj\{x(k-j)|\varepsilon(0),\cdots,\varepsilon(k-1)\}+proj\{x(k-j|\varepsilon(k)\} \\ &= \hat{x}(k-j|k-1)+E\left\{x(k-j)\varepsilon^T(k)\right\}R^{-1}_{\varepsilon(k)}\varepsilon(k) \\ &= \hat{x}(k-j|k-1)+K_j(k)\varepsilon(k), \end{aligned}$$

where $K_j(k)$ can be calculated by

$$\begin{aligned} K_j(k) &= E\left\{x(k-j)\varepsilon^T(k)\right\}R^{-1}_{\varepsilon(k)} \\ &= E\left\{[\hat{x}(k-j|k-1)+\tilde{x}(k-j|k-1)]\varepsilon^T(k)\right\}R^{-1}_{\varepsilon(k)} \\ &= E\left\{\tilde{x}(k-j|k-1)\varepsilon^T(k)\right\}R^{-1}_{\varepsilon(k)} \\ &= E\big\{\tilde{x}(k-j|k-1)[B\tilde{x}(k|k-1)+B_d\tilde{x}(k-d|k-1) \\ &\quad +\eta(k)B_\eta x(k)+v(k)]^T\big\}R^{-1}_{\varepsilon(k)} \\ &= \left[P(k-j,k,k-1)B^T+P(k-j,k-d,k-1)B_d^T\right]R^{-1}_{\varepsilon(k)}, \end{aligned}$$

therefore (24) and (26) are obtained. Next, we will derive the covariance matrices of the innovation $\varepsilon(k)$.

$$\begin{aligned} R_{\varepsilon(k)} &= E\big\{[B\tilde{x}(k|k-1)+B_d\tilde{x}(k-d|k-1)+\eta(k)B_\eta x(k)+v(k)][B\tilde{x}(k|k-1) \\ &\quad +B_d\tilde{x}(k-d|k-1)+\eta(k)B_\eta x(k)+v(k)]^T\big\} \\ &= BP(k,k,k-1)B^T+B_dP(k-d,k-d,k-1)B_d^T+BP(k,k-d,k-1)B_d^T \\ &\quad +B_dP(k-d,k,k-1)B^T+\Pi_2\odot\left(B_\eta D_{0,0}(k)B_\eta^T\right)+R, \end{aligned}$$

thus (27) is proved. Meanwhile, it follows from (24) that

$$\tilde{x}(k-i|k) = \tilde{x}(k-i|k-1)-K_i(k)\varepsilon(k),\ 0\le i\le d, \tag{34}$$

$$\tilde{x}(k-j|k) = \tilde{x}(k-j|k-1)-K_j(k)\varepsilon(k),\ 0\le j\le d. \tag{35}$$

Based on (34) and (35), we can derive the estimation error covariance matrices as follows

$$\begin{aligned}
P(k-i,k-j,k) &= E\left\{\left[\tilde{x}(k-i|k)\tilde{x}^T(k-j|k)\right]\right\} \\
&= E\left\{[\tilde{x}(k-i|k-1)-K_i(k)\varepsilon(k)][\tilde{x}(k-j|k-1)-K_j(k)\varepsilon(k)]^T\right\} \\
&= E\left\{\tilde{x}(k-i|k-1)\tilde{x}^T(k-j|k-1)\right\}-E\left\{\tilde{x}(k-i|k-1)\varepsilon^T(k)\right\}K_j^T(k) \\
&\quad -K_i(k)E\left\{\varepsilon(k)\tilde{x}^T(k-j|k-1)\right\}+K_i(k)E\left\{\varepsilon(k)\varepsilon^T(k)\right\}K_j^T(k) \\
&= P(k-i,k-j,k-1)-K_i(k)R_{\varepsilon(k)}K_j^T(k),
\end{aligned}$$

which is (28). Combining (3) and (32), we will obtain

$$\begin{aligned}
\tilde{x}(k+1|k) &= x(k+1)-\hat{x}(k+1|k) \\
&= A\tilde{x}(k|k)+A_d\tilde{x}(k-d|k)+\xi(k)A_\xi x(k)+n(k)-N_0(k)\varepsilon(k), \qquad (36)
\end{aligned}$$

then making use of the results of (35) and (36), one can show that

$$\begin{aligned}
P(k+1,k-j,k) &= E\left\{\tilde{x}(k+1|k)\tilde{x}^T(k-j|k)\right\} \\
&= E\big\{[A\tilde{x}(k|k)+A_d\tilde{x}(k-d|k)+\xi(k)A_\xi x(k)+n(k) \\
&\quad -N_0(k)\varepsilon(k)]\tilde{x}^T(k-j|k)\big\} \\
&= AP(k,k-j,k)+A_dP(k-d,k-j,k)-N_0(k)E\left\{\varepsilon(k)\tilde{x}^T(k-j|k)\right\} \\
&\quad +E\left\{\xi(k)A_\xi x(k)\tilde{x}^T(k-j|k)\right\} \\
&= AP(k,k-j,k)+A_dP(k-d,k-j,k)-N_0(k)E\left\{\varepsilon(k)\tilde{x}^T(k-j|k)\right\} \\
&\quad -E\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}e_i\xi_i(k)A_{\xi i}x(k)x^T(k)B_{\eta j}^T\eta_j^T(k)\alpha_j^T\right\}K_j^T(k) \\
&= AP(k,k-j,k)+A_dP(k-d,k-j,k) \\
&\quad -\Pi_3\odot\left(A_\xi D_{0,0}(k)B_\eta^T\right)K_j^T(k)-N_0(k)E\left\{\varepsilon(k)\tilde{x}^T(k-j|k)\right\} \\
&= AP(k,k-j,k)+A_dP(k-d,k-j,k)-N_0(k)R_{\varepsilon(k)}K_j^T(k) \\
&\quad -N_0(k)E\left\{\varepsilon(k)\tilde{x}^T(k-j|k)\right\}, \qquad (37)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
E\left\{\varepsilon(k)\tilde{x}^T(k-j|k)\right\} &= E\big\{[B\tilde{x}(k|k-1)+B_d\tilde{x}(k-d|k-1)+\eta(k)B_\eta x(k) \\
&\quad +v(k)][\tilde{x}(k-j|k-1)-K_j(k)\varepsilon(k)]^T\big\} \\
&= BP(k,k-j,k-1)+B_dP(k-d,k-j,k-1) \\
&\quad -R_{\varepsilon(k)}K_j^T(k), \qquad (38)
\end{aligned}$$

by combining (37) and (38), the proof of (29) can be completed. Noting that

$$\tilde{x}(k|k) = \tilde{x}(k|k-1)-K_0(k)\varepsilon(k), \qquad (39)$$

$$\tilde{x}(k-d|k) = \tilde{x}(k-d|k-1)-K_d(k)\varepsilon(k), \qquad (40)$$

meanwhile, with the help of (21), one has

$$\begin{aligned}
E\left\{\tilde{x}(k|k)\varepsilon^T(k)\right\} &= E\left\{[\tilde{x}(k|k-1)-K_0(k)\varepsilon(k)]\varepsilon^T(k)\right\} \\
&= E\left\{\tilde{x}(k|k-1)\varepsilon^T(k)\right\}-K_0(k)R_{\varepsilon(k)} \\
&= P(k,k,k-1)B^T+P(k,k-d,k-1)B_d^T-K_0(k)R_{\varepsilon(k)}, \quad (41) \\
E\left\{\tilde{x}(k-d|k)\varepsilon^T(k)\right\} &= E\left\{[\tilde{x}(k-d|k-1)-K_d(k)\varepsilon(k)]\varepsilon^T(k)\right\} \\
&= E\left\{\tilde{x}(k-d|k-1)\varepsilon^T(k)\right\}-K_d(k)R_{\varepsilon(k)} \\
&= P(k-d,k,k-1)B^T+P(k-d,k-d,k-1)B_d^T
\end{aligned}$$

$$- K_d(k)R_{\varepsilon(k)}. \tag{42}$$

Then according to (39) and (40), we have

$$\begin{aligned} E\left\{A\tilde{x}(k|k)[\xi(k)A_\xi x(k)]^T\right\} &= E\left\{A[\tilde{x}(k|k-1) - K_0(k)\varepsilon(k)][\xi(k)A_\xi x(k)]^T\right\} \\ &= -AK_0(k)\left(\Pi_3^T \odot \left(B_\eta D_{0,0}(k)A_\xi^T\right)\right), \end{aligned} \tag{43}$$

$$\begin{aligned} E\left\{A_d\tilde{x}(k-d|k)[\xi(k)A_\xi x(k)]^T\right\} &= E\left\{A_d[\tilde{x}(k-d|k-1) - K_d(k)\varepsilon(k)][\xi(k)A_\xi x(k)]^T\right\} \\ &= -A_dK_d(k)\left(\Pi_3^T \odot \left(B_\eta D_{0,0}(k)A_\xi^T\right)\right). \end{aligned} \tag{44}$$

Therefore, it follows from (41)-(44) that

$$\begin{aligned}
& P(k+1,k+1,k) \\
=\ & E\left\{\tilde{x}(k+1|k)\tilde{x}^T(k+1|k)\right\} \\
=\ & E\big\{[A\tilde{x}(k|k) + A_d\tilde{x}(k-d|k) + \xi(k)A_\xi x(k) + n(k) - N_0(k)\varepsilon(k)][A\tilde{x}(k|k) \\
& + A_d\tilde{x}(k-d|k) + \xi(k)A_\xi x(k) + n(k) - N_0(k)\varepsilon(k)]^T\big\} \\
=\ & AP(k,k,k)A^T + A_dP(k-d,k-d,k)A_d^T + AP(k,k-d,k)A_d^T + A_dP(k-d,k,k)A^T \\
& - AE\left\{\tilde{x}(k|k)\varepsilon^T(k)\right\}N_0^T(k) - A_dE\left\{\tilde{x}(k-d|k)\varepsilon^T(k)\right\}N_0^T(k) + \Pi_1 \odot \left(A_\xi D_{0,0}(k)A_\xi^T\right) \\
& - N_0(k)E\left\{\varepsilon(k)\tilde{x}^T(k|k)\right\}A^T - N_0(k)E\left\{\varepsilon(k)\tilde{x}^T(k-d|k)\right\}A_d^T + N_0(k)R_{\varepsilon(k)}N_0^T(k) \\
& - E\left\{\xi(k)A_\xi x(k)\varepsilon^T(k)\right\}N_0^T(k) - N_0(k)E\left\{\varepsilon(k)x^T(k)A_\xi^T\xi^T(k)\right\} \\
& + E\left\{A\tilde{x}(k|k)[\xi(k)A_\xi x(k)]^T\right\} + E\left\{[\xi(k)A_\xi x(k)]\tilde{x}^T(k|k)A^T\right\} \\
& + E\left\{A_d\tilde{x}(k-d|k)[\xi(k)A_\xi x(k)]^T\right\} + E\left\{[\xi(k)A_\xi x(k)]\tilde{x}^T(k-d|k)A_d^T\right\} + Q \\
=\ & AP(k,k,k)A^T + A_dP(k-d,k-d,k)A_d^T + AP(k,k-d,k)A_d^T + A_dP(k-d,k,k)A^T \\
& - \left[AP(k,k,k-1)B^T + AP(k,k-d,k-1)B_d^T - AK_0(k)R_{\varepsilon(k)}\right]N_0^T(k) \\
& - \left[A_dP(k-d,k,k-1)B^T + A_dP(k-d,k-d,k-1)B_d^T - A_dK_d(k)R_{\varepsilon(k)}\right]N_0^T(k) \\
& - N_0(k)\left[AP(k,k,k-1)B^T + AP(k,k-d,k-1)B_d^T - AK_0(k)R_{\varepsilon(k)}\right]^T \\
& - N_0(k)\left[A_dP(k-d,k,k-1)B^T + A_dP(k-d,k-d,k-1)B_d^T - A_dK_d(k)R_{\varepsilon(k)}\right]^T \\
& + \Pi_1 \odot \left(A_\xi D_{0,0}(k)A_\xi^T\right) + Q + N_0(k)R_{\varepsilon(k)}N_0^T(k) - N_0(k)R_{\varepsilon(k)}N_0^T(k) \\
& - N_0(k)R_{\varepsilon(k)}N_0^T(k) - AK_0(k)\left(\Pi_3^T \odot \left(B_\eta D_{0,0}(k)A_\xi^T\right)\right) \\
& - \left(\Pi_3 \odot \left(A_\xi D_{0,0}(k)B_\eta^T\right)\right)K_0^T(k)A^T - A_dK_d(k)\left(\Pi_3^T \odot \left(B_\eta D_{0,0}(k)A_\xi^T\right)\right) \\
& - \left(\Pi_3 \odot \left(A_\xi D_{0,0}(k)B_\eta^T\right)\right)K_d^T(k)A_d^T \\
=\ & AP(k,k,k)A^T + A_dP(k-d,k-d,k)A_d^T + AP(k,k-d,k)A_d^T + A_dP(k-d,k,k)A^T \\
& + \Pi_1 \odot \left(A_\xi D_{0,0}(k)A_\xi^T\right) + Q - N_0(k)R_{\varepsilon(k)}\left[K_0^T(k)A^T + K_d^T(k)A_d^T\right] \\
& - [N_0(k) + AK_0(k) + A_dK_d(k)]R_{\varepsilon(k)}N_0^T(k).
\end{aligned}$$

which is (30). At last (31) can be obtained by the definition of (22). So far, the proof is completed. ∇

Remark 4.1. *From Theorem 4.1 one can see the estimator gain can be calculated by solving a partial difference Riccati Equation (28)-(30) and a Lyapunov equation, where the equations have the same dimension as the original system. Denote C_{aug} and C_{new} the numbers of system with multi-channel multiplicative noise and time delay for augmentation method and our proposed approach in one step, respectively. According to* [27], *one can see that the order of d in C_{aug} is 3, while the order of d in C_{new} is 2. Therefore, if d is large enough, $C_{aug} \gg C_{new}$. Therefore, compared with the conventional augmented* *approach [25], the presented approach greatly lessens the computational demand when the delay is very large.*

Remark 4.2. *When the delay is free, that is $A_d = B_d = 0$, the result of Theorem 4.1 can be reduced to the result of Theorem 3.1. Here we only give a simple explanation on how to deduce the Riccati Equation (10) from partial Riccati Equations (28)-(30).*

It follows from (28)-(30) one has

$$\begin{aligned}
P(k+1,k+1,k) &= AP(k,k,k)A^T + \Pi_1 \odot \left(A_\xi D_{0,0}(k)A_\xi^T\right) + Q - N_0(k)R_{\varepsilon(k)}K_0^T(k)A^T \\
&\quad - AK_0(k)R_{\varepsilon(k)}N_0^T(k) - N_0(k)R_{\varepsilon(k)}N_0^T(k) \\
&= AP(k,k,k-1)A^T - AK_0(k)R_{\varepsilon(k)}K_0^T(k)A^T + \Pi_1 \odot \left(A_\xi D_{0,0}(k)A_\xi^T\right) \\
&\quad + Q - N_0(k)R_{\varepsilon(k)}K_0^T(k)A^T - AK_0(k)R_{\varepsilon(k)}N_0^T(k) \\
&\quad - N_0(k)R_{\varepsilon(k)}N_0^T(k) \\
&= AP(k,k,k-1)A^T + \Pi_1 \odot \left(A_\xi D_{0,0}(k)A_\xi^T\right) + Q \\
&\quad - K_{p^*}(k)R_{\varepsilon(k)}K_{p^*}^T(k),
\end{aligned} \tag{45}$$

where $K_{p^*}(k)$ is

$$K_{p^*}(k) = AK_0(k) + N_0(k) = \left[AP(k,k,k-1)B^T + \Pi_3 \odot \left(A_\xi D_{0,0}(k)B_\eta^T\right)\right] R_{\varepsilon(k)}^{-1}.$$

From the definitions of $P(k+1,k+1,k)$, $P(k)$, $D_{0,0}(k)$, $D(k)$, one can see that $P(k+1,k+1,k) = P(k)$, $D_{0,0}(k) = D(k)$. Hence we draw the conclusion that (45) is consistent with (10).

5. Numerical Example.

Here we consider the linear discrete-time systems with multi-channel multiplicative noises

$$\begin{aligned}
x(k+1) &= \left(\begin{bmatrix} 0.48 & 0.1 \\ 0.3 & 0.4 \end{bmatrix} + \xi(k)\begin{bmatrix} 0.6 & 0.4 \\ 0.1 & -0.5 \end{bmatrix}\right) x(k) + n(k), \\
y(k) &= \left(\begin{bmatrix} 0.95 & 0.2 \\ -0.3 & 0.4 \end{bmatrix} + \eta(k)\begin{bmatrix} 0.85 & 0.5 \\ 0.6 & -0.5 \end{bmatrix}\right) x(k) + v(k),
\end{aligned}$$

with

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \hat{x}(0|-1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ P(0) = D(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ R = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

When the multiplicative noise $\xi(k)$ and $\eta(k)$ are correlated, we set $\Pi_1 = \Pi_2 = \Pi_3 = \begin{bmatrix} 1 & 0.75 \\ 0.75 & 1 \end{bmatrix}$. On the contrary, when the multiplicative noise $\xi(k)$ and $\eta(k)$ are uncorrelated, Π_1, Π_2, Π_3 are given as $\Pi_1 = \Pi_2 = \Pi_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Based on Theorem 3.1, we obtain the result of one-step predictor $\hat{x}(k+1|k)$. When the multiplicative noise $\xi(k)$ and $\eta(k)$ are correlated, the simulation results are given in Figures 1 and 2, respectively. At the same time, when the multiplicative noises are uncorrelated, we also provide the simulation results in Figures 3 and 4. It can be seen from the Figures 1-4 that the predictor $\hat{x}(k|k-1)$ can track the original state $x(k)$, which implies the presented approach in this paper is effective.

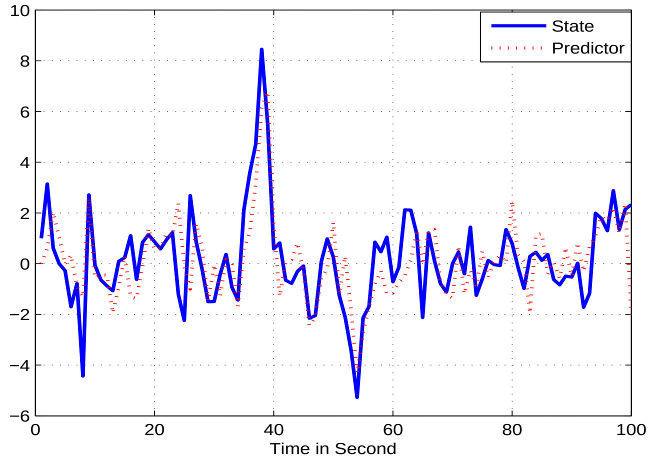

FIGURE 1. The first state component $x_1(k)$ and the predictor $\hat{x}_1(k|k-1)$

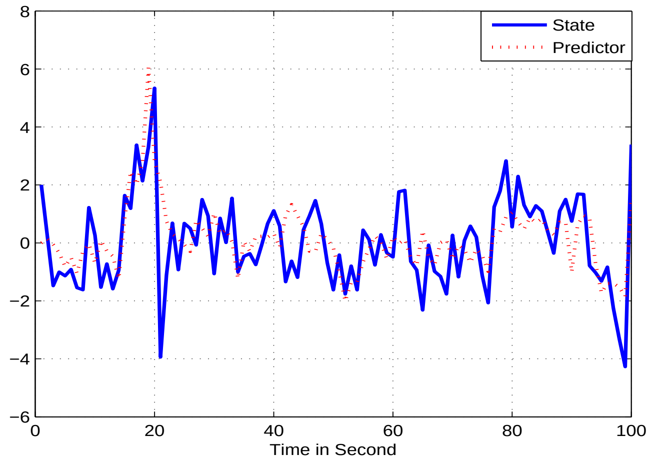

FIGURE 2. The second state component $x_2(k)$ and the predictor $\hat{x}_2(k|k-1)$

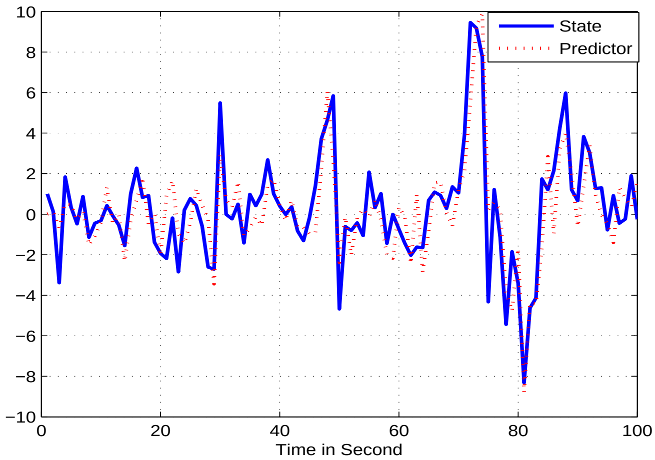

FIGURE 3. The first state component $x_1(k)$ and the predictor $\hat{x}_1(k|k-1)$

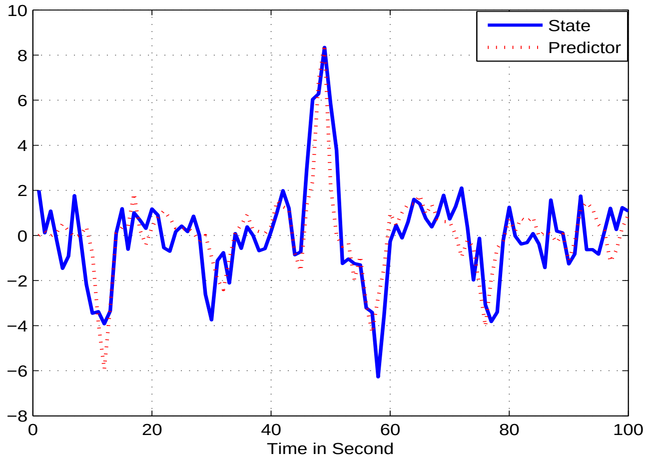

FIGURE 4. The second state component $x_2(k)$ and the predictor $\hat{x}_2(k|k-1)$

6. **Conclusions.** In this paper, we have discussed the optimal estimation problem for systems with diagonal multiplicative noise, where the multiplicative noise occurs both in state equation and measurement equation respectively. By using projection theory and introducing the Hadamard product, the estimator has been designed in terms of a difference Riccati equation and a difference Lyapunov equation. Moreover, we have extended the estimation problem to the case of time delay. The solution to the estimator has been given by calculating a partial difference Riccati equation which has the same dimension as the original system and a difference Lyapunov equation. Hence the traditional approach of the state augmentation has been abandoned and the computation demand will be lessened when the delay is very large. The infinite horizon estimation problems for systems with multi-channel multiplicative noise and the estimation problem for systems with packet dropping merit further study.

Acknowledgment. This work was supported in part by the National Natural Science Foundation of China (61202225, 61304013), the Excellent Young Scholars Research Fund of Shandong Normal University, the Shandong Provincial Key Laboratory for Novel Distributed Computer Software Technology. The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

REFERENCES

[1] Y. Huang, M. K. Ng and Y. Wen, A new total variation method for multiplicative noise removal, *SIAM Journal on Imaging Sciences*, vol.2, no.1, pp.20-40, 2010.
[2] O. Besson, F. P. Stoica and A. B. Gershman, Approximate maximum likelihood estimators for array processing in multiplicative noise environments, *IEEE Trans. Signal Processing*, vol.48, no.9, pp.2506-2518, 2000.
[3] I. Matei and J. S. Baras, Optimal state estimation for discrete-time Markovian jump linear systems, in the presence of delayed output observations, *IEEE Trans. Automatic Control*, vol.56, no.9, pp.2235-2240, 2011.
[4] W. Liu and G. Hu, State estimation for discrete-time Markov jump linear systems with multiplicative noises and delayed mode measurements, *Digital Signal Processing*, vol.30, no.4, pp.27-36, 2014.
[5] L. Zhang, Y. Zhu, P. Shi and Y. Zhao, Resilient asynchronous H_∞ filtering for Markov jump neural networks with unideal measurements and multiplicative noises, *IEEE Trans. Cybernetics*, 2015.

[6] V. Dragan, H. Mukaidani and P. Shi, Near optimal linear quadratic regulator for a class of stochastic systems modeled by singularly perturbed Ito differential equations with state and control multiplicative white noise, *ICIC Express Letters*, vol.6, no.3, pp.595-602, 2012.
[7] M. Basin and P. Shi, Central suboptimal H_∞ filtering for nonlinear polynomial systems with multiplicative noise, *Journal of Franklin Institute*, vol.347, no.9, pp.1740-1754, 2010.
[8] F. Peng and S. Sun, Distributed fusion estimation for multisensor multirate systems with stochastic observation multiplicative noises, *Mathematical Problems in Engineering*, 2014.
[9] N. E. Nahi, Optimal recursive estimation with uncertain observation, *IEEE Trans. Information Theory*, vol.15, no.4, pp.457-462, 1969.
[10] M. T. Haditi and S. C. Schwartz, Linear recursive state estimators under uncertain observations, *IEEE Trans. Automatic Control*, vol.24, no.6, pp.944-948, 1979.
[11] J. K. Tugnait, Stability of optimal linear estimations of stochastic signals in white multiplicative noise, *IEEE Trans. Automatic Control*, vol.26, no.3, pp.757-761, 1981.
[12] W. Zhang, D. Chu and Y. Hou, Optimal estimation algorithm of maximum likelihood for the systems with multiplicative noises, *Periodical of Ocean University of China*, vol.29, no.2, pp.332-338, 1999.
[13] D. Xu and M. Zhao, Optimal filtering algorithm of a class of stochastic system with multiplicative noise, *Computing Technology and Automation*, vol.25, no.1, pp.17-19, 2006.
[14] H. Liang and T. Zhou, Robust state estimation for uncertain discrete-time stochastic systems with missing measurements, *Automatica*, vol.47, no.7, pp.1520-1524, 2011.
[15] M. Basin, P. Shi and P. Soto, Mean-square data-based controller for nonlinear polynomial systems with multiplicative noise, *Information Sciences*, vol.195, pp.256-265, 2012.
[16] V. Dragan, Optimal filtering for discrete-time linear systems with multiplicative white noise perturbations and periodic coefficients, *IEEE Trans. Automatic Control*, vol.58, no.4, pp.1029-1034, 2013.
[17] B. Liu, D. Chu, H. Han and M. Liang, Optimal fixed domain smoothing algorithm of system with multi-channel multiplicative noise, *Journal of Ocean University of Qingdao*, vol.32, no.5, pp.843-847, 2002.
[18] H. Zhang, L. Xie, D. Zhang and Y. C. Soh, A reorganized innovation approach to linear estimation, *IEEE Trans. Automatic Control*, vol.49, no.10, pp.1810-1814, 2004.
[19] H. Zhao, H. Zhang, C. Zhang and X. Song, Robust filtering and fixed-lag smoothing for linear uncertain systems with single delayed measurement, *Proc. of the 26th Chinese Control Conference*, pp.23-27, 2007.
[20] X. Song, X. Yan and D. Yuan, Optimal linear estimation for multiplicative noise systems with time delay and its duality, *Proc. of the 31st Chinese Control Conference*, pp.1614-1619, 2012.
[21] B. Chen, L. Yu and W. A. Zhang, Robust Kalman filtering for uncertain state delay systems with random observation delays and missing measurements, *IET Control Theory and Applications*, vol.17, no.5, pp.1945-1954, 2011.
[22] S. Sun, Linear minimum variance estimators for systems with bounded random measurement delays and packet dropouts, *Signal Processing*, vol.89, no.7, pp.1457-1466, 2009.
[23] N. Li, J. Ma and S. Sun, Optimal linear estimation for stochastic uncertain systems with multiple packet dropouts and delays, *Acta Automatica Sinica*, vol.41, no.3, pp.611-619, 2015.
[24] D. Chen, Y. Yu, L. Xu and X. Liu, Kalman filtering for discrete stochastic systems with multiplicative noises and random two-step sensor delays, *Discrete Dynamics in Nature and Society*, 2015.
[25] L. Xiao, A. Hassibi and J. P. How, Control with random communication delays via a discrete-time jump system approach, *Proc. of the American Control Conference*, pp.2199-2204, 2000.
[26] B. Anderson and J. Moore, *Optimal Filtering*, Prentice Hall, 1979.
[27] S. Liu, L, Xie and H. Zhang, Infinite horizon LQR for systems with multiple delays in a single input channel, *Journal of Control Theory and Applications*, vol.8, no.3, pp.368-374, 2010.