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OPTIMAL SWITCHING TIME CONTROL OF DISCRETE-TIME SWITCHED AUTONOMOUS SYSTEMS

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Abstract. *Optimal switching time control is the study that investigates how best to switch between different modes. In this paper, we present an approach for solving the optimal switching time control problem of discrete-time switched linear systems, where the objective is to minimize a cost functional defined on the state. In particular, we assume that the sequence of active subsystems is pre-specified and the switching times are the only control variables. Firstly, using calculus of variations, for one switching time case, the difference of the cost with respect to the switching time is derived. Then, a method is provided to deal with the switched systems with multiple switching times. It is worth mentioning that the differences of the cost functional have an especially simple form and can be easily used to locate the optimal switching instants. Finally, a numerical example shows the viability of the proposed method.*
Keywords: Optimal control, Switched systems, Calculus of variation

1. **Introduction.** Many practical systems are best described by dynamics that comprise continuous state evolution within a mode of operation and discrete transitions from one mode to another, which are called switched systems. These systems differ from the conventional dynamical systems in that they have both continuous dynamics and discrete event dynamics. Such systems arise in a variety of applications, including power systems, industrial process control, automotive systems, and networked control systems [1-7], etc.

Recently, optimal control of switched systems has attracted wide attention, and many results have been obtained, due to the problems significant in theory and application [3, 8-21]. Many of them concern the problems whose control variables consist of a proper switching law and the input $u(t)$ [3, 14, 15]. However, a more special class considers autonomous systems, where the term $u(t)$ is absent, the modes sequence is fixed and the switching times are the sole control variables [16-20]. These problems are referred in the literature as optimal switching time control problem or time optimization problem. In particular, [16] considers general nonlinear continuous-time systems, derives the derivatives of the cost with respect to the switching instants and uses constrained nonlinear optimization techniques to locate the optimal switching instants. [19] considers the similar problem, and develops an especially simpler formula than the one in [16] for the gradient of the cost functional, which leads itself to be directly used in gradient descent algorithms.

The above existing results about the optimal switching time control problem focused on continuous-time systems. However, discrete-time switched systems also appear frequently in real applications [22-24]. For the optimal control of discrete-time switched systems, there is extensive literature in recent years [25-30]. Yet, these researches are basically based on the principle of dynamic programming, or the problems are first transformed into equivalent optimization problems. Meanwhile, the switching times are not regarded as control variables among the results in the present. Therefore, there is improving room for the issue.

In this paper, we extend the results of [19] to discrete-time systems. Considering some inherent characteristics of discrete-time systems, such as the unavoidable "dimension curse" problem, the recursive characteristic of their solutions and so on, it is certain that the direct extension faces the significant challenge of calculating the performance difference. Thus, from the convenience of visual analysis, we consider the discrete-time linear systems. The main contribution is that, using calculus of variations, we first derive a simple formula for the difference of the cost with respect to the switching time for the one switching time systems. Then, we propose a method to handle systems with multiple switching times. Finally, we verify the feasibility of the proposed method on a numerical example.

The remainder of this paper is organized as follows. Section 2 presents the discrete-time switched linear system model and formulates the optimal switching time control problem. The formulae for the difference of the cost functional and the method of handling the multiple switching times case are given in Section 3, followed by an illustrative example in Section 4. Finally, the conclusions and discussions are stated in Section 5.

2. Problem Formulation.

Consider the discrete-time switched linear autonomous system described by:

$$x(k+1) = A_i x(k),\ k = T_i, \cdots, T_{i+1} - 1 \tag{1}$$

where $x(k) \in R^n$ is the system state, and $i \in \{0, 1, \cdots, M\}$ is the discrete control or switching strategy. $\{A_i\}_{i=0}^{M}$ are a finite sequence of constant matrices of appropriate dimension and initial condition is $x(0) = x_0$. $\{T_i\}_{i=1}^{M}$ is the sequence of switching times, and defining $T_0 = 0$, $T_{M+1} = N$. We assume that there is no internal forced switching, i.e., the system can stay at or switch to any mode at any time instant.

In this paper, the cost function $L(x) : R^n \to R$ is assumed to be taken as the quadratic form:

$$L(x(k)) = \frac{1}{2} x^T(k) Q x(k)$$

where $Q = Q^T > 0$ is the weight for the state. The overall objective functional to be minimized can be defined by:

$$J = \sum_{k=0}^{N-1} L(x(k)) = \frac{1}{2} \sum_{k=0}^{N-1} x^T(k) Q x(k) \tag{2}$$

The goal of this paper is to solve the following discrete-time optimal switching time control problem for the switched linear autonomous system (1).

Problem (Optimal Control Problem)

Consider a discrete-time switched linear autonomous system (1). Assume that a pre-specified sequence of active subsystems $(0, 1, \cdots, M)$ is given. Find optimal switching instants $\{T_i\}_{i=1}^{M}$ such that the corresponding cost (2) is minimized.

3. **Difference of the Cost Function.** We consider the control parameter to consist of the switching times $\{T_i\}_{i=1}^M$ and denote it by the N-dimensional variable $\tau = (T_1, \cdots, T_M)^T$. Note that J is a function of τ via (2). Thus, the above problem is actually a multivariable parameter optimization problem. However, solving it requires the explicit solution of the state equations, which are dependent on the switching times. We therefore solve the problem by classical variational methods.

In this section, for simplicity, we first consider one-dimension switched systems, and then extend the results to the multi-dimension systems. We shall leave the full generality of the problem behind and first consider only systems with one switching time case ($M = 1$). In the following, we derive a formula for the difference ΔJ.

3.1. **One-dimension case.** From (1), one-dimension switched systems are defined by:

$$x(k+1) = a_i x(k), \ k = T_i, \cdots, T_{i+1} - 1 \tag{3}$$

where $x(k) \in R$ is the system state. $\{a_i\}_{i=0}^M$ are a finite sequence of constants and initial condition is $x(0) = x_0 \in R$.

From (2), the corresponding overall cost becomes:

$$J = \sum_{k=0}^{N-1} L(x(k)) = \frac{1}{2} \sum_{k=0}^{N-1} q x^2(k) \tag{4}$$

For one-dimension case, we have the following conclusion.

Theorem 3.1. *Consider the discrete-time switched linear system defined over $\{0, 1, \cdots, N-1\}$, with initial condition $x(0) = x_0 \in R$, and the signal switching time is $\tau \in \{1, \cdots, N-1\}$:*

$$x(k+1) = \begin{cases} a_1 x(k), & k = 1, \cdots, \tau - 1 \\ a_2 x(k), & k = \tau, \cdots, N-1 \end{cases} \tag{5}$$

Given the quadratic cost functional $J = \sum_{k=0}^{N-1} L(x(k)) = \frac{1}{2}\sum_{k=0}^{N-1} qx^2(k)$. We denote the cost with the switching time at $\tau - 1$, τ and $\tau + 1$ by $J_{\tau-1}$, J_τ and $J_{\tau+1}$, respectively. Consider J as a function of the switching time τ, and denote its difference by ΔJ. Define the costate $\bar{\lambda}(k)$, $\lambda(k) \in R$ by the (backwards) difference equation:

$$\begin{aligned} &\lambda(k) = qx(k) + a_2 \lambda(k+1), \quad k = \tau, \cdots, N-2 \\ &\bar{\lambda}(k) = q\bar{x}(k) + a_2 \bar{\lambda}(k+1), \quad k = \tau - 1, \cdots, N-2 \\ &\lambda(N-1) = \bar{\lambda}(N-1) = 0 \end{aligned} \tag{6}$$

where $\bar{x}(k)$ are the solutions of switched systems with switching time $\tau - 1$. Then, ΔJ has the following form

$$\Delta J_{forward} = J_{\tau-1} - J_\tau = \frac{1}{2}\left(a_2^2 - a_1^2\right) x(\tau - 1)\, \bar{\lambda}(\tau - 1) \tag{7}$$

$$\Delta J_{backward} = J_{\tau+1} - J_\tau = \frac{1}{2}\left(a_1^2 - a_2^2\right) x(\tau)\, \lambda(\tau) \tag{8}$$

Proof: Combined with the optimal control theory, it follows immediately from variational principles.

3.2. **Multi-dimension case.** We now turn our attention to the multi-dimension case. Consider the discrete-time switched linear autonomous system (1) and the cost functional (2). Then, we have the following conclusion.

Theorem 3.2. *Consider the discrete-time switched linear system defined over* $\{0, 1, \cdots, N-1\}$*, with initial condition* $x(0) = x_0 \in R$*, and the signal switching time is* $\tau \in \{1, \cdots, N-1\}$*:*

$$x(k+1) = \begin{cases} A_1 x(k), & k = 1, \cdots, \tau - 1 \\ A_2 x(k), & k = \tau, \cdots, N-1 \end{cases} \tag{9}$$

Given a quadratic cost functional $J = \sum_{k=0}^{N-1} L(x(k)) = \frac{1}{2}\sum_{k=0}^{N-1} x^T(k)Qx(k)$*. We denote the cost with the switching time at* $\tau-1$*,* τ *and* $\tau+1$ *by* $J_{\tau-1}$*,* J_τ *and* $J_{\tau+1}$*, respectively. Consider* J *as a function of the switching time* τ*, and denote its difference by* ΔJ*. Define the costate* $\lambda(k) \in R^n$ *by the (backwards) difference equation:*

$$\begin{cases} \lambda(k) = Qy(k) + A_2^T \lambda(k+1), \\ y(k+1) = A_2 y(k), \end{cases} \quad k = \tau, \cdots, N-1 \\ \lambda(T) = 0 \tag{10}$$

where $y(k) \in R^n$ *has the same dimensional vector with* $x(k)$*. Then,* ΔJ *has the following form*

$$\Delta J_{forward} = J_{\tau-1} - J_\tau = \frac{1}{2} x^T(\tau-1)\left(A_2^T \bar{\lambda}(\tau) - A_1^T \lambda(\tau)\right) \tag{11}$$

$$\Delta J_{backward} = J_{\tau+1} - J_\tau = \frac{1}{2} x^T(\tau)\left(A_1^T \tilde{\lambda}_1(\tau+1) - A_2^T \lambda_1(\tau+1)\right) \tag{12}$$

where $\lambda_1(k)$ *satisfies (10) with* $k = \tau+1, \cdots, N-1$*.* $\bar{\lambda}(k)$*,* $\tilde{\lambda}_1(k)$ *are the corresponding solution of (10) with* $\bar{x}(k)$ *and* $\tilde{x}(k)$ *instead of* $y(k)$*, respectively.* $\bar{x}(k)$*,* $\tilde{x}(k)$ *are corresponding solutions of switched systems with switching time* $\tau-1$*,* $\tau+1$*, respectively.*

Proof: By classical variational methods, we first analyze the cost variation between two systems: an unperturbed system and a perturbed system. Specifically, we denote the unperturbed system by x, with the nominal switching time τ. Consider now a perturbed system, for discrete systems, it has two cases, one is with switching time $\tau - 1$, denote system $\bar{x}$, and the other is with switching time $\tau + 1$, denote system $\tilde{x}$.

Now, we analyze the induced variation in the performance index. Note that due to the difference of the switching time, the discrepancy between x and $\bar{x}$ (also between x and $\tilde{x}$) yields to a discrepancy in cost J. For simplicity, we denote the discrepancy in cost as $\Delta J_{forward} = J_{\tau-1} - J_\tau$ and $\Delta J_{backward} = J_{\tau+1} - J_\tau$, respectively (where $J_{\tau-1}$, J_τ and $J_{\tau+1}$ denote the corresponding cost with the switching time at $\tau-1$, τ and $\tau+1$, respectively).

For $\Delta J_{forward}$, by a series of calculations and rearrangements, we have:

$$\begin{aligned} \Delta J_{forward} &= J_{\tau-1} - J_\tau \\ &= \sum_{k=0}^{N-1} L(\bar{x}(k)) - \sum_{k=0}^{N-1} L(x(k)) \\ &= \frac{1}{2}\sum_{k=0}^{N-1}\left[\bar{x}^T(k)Q\bar{x}(k) - x^T(k)Qx(k)\right] \\ &= \frac{1}{2} x^T(\tau-1)\left\{A_2^T\left[\sum_{i=0}^{N-\tau-1}\left(A_2^i\right)^T Q A_2^i\right]\bar{x}(\tau) - A_1^T\left[\sum_{i=0}^{N-\tau-1}\left(A_2^i\right)^T Q A_2^i\right]x(\tau)\right\} \end{aligned} \tag{13}$$

By calculating (10) backwards in time, we can obtain:

$$\lambda(\tau) = \left[\sum_{i=0}^{N-\tau-1} \left(A_2^i\right)^T Q A_2^i\right] y(\tau) \tag{14}$$

Choosing $y(k) = \bar{x}(k)$, we obtain that (11) holds. Similarly, we can prove (12) holds. Thus, Theorem 3.2 is proved.

Remark 3.1. *In the above theorem, the obtained difference formula can be used to find the optimal switching time for discrete-time switched systems; however, as [16-20] consider continuous-time systems, the methods are invalid. Moreover, although [25-30] study the discrete-time switched systems, they are merely concentrated on the case where the switching time is fixed and not the control variable.*

Now, we transfer our attention to multiple switching times case, for the general switched systems with multiple switching times ($N > 1$), defined by:

$$\begin{aligned} &x(k+1) = A_i x(k),\ k = T_i, \cdots, T_{i+1} - 1,\ i \in \{0, 1, \cdots, M\} \\ &x(0) = x_0 \end{aligned} \tag{15}$$

In the research of discrete systems, it cannot avoid the inherent "dimension curse" problem. Meanwhile, for switched systems (15) with M switching points $\{T_i\}_{i=1}^M$, it will have 2^M possibilities disturbance cases according to whether left shift or right shift, so in this optimal control problem, for the given initial series of switching points, we need to repeat comparing at least 2^M performance index to determine next series of switching points until finding the optimal or sub-optimal solution. This is impossible.

Therefore, the above method becomes invalid and we cannot obtain the perfect results. However, we can use the following method. As the finite mode sequence is fixed, we firstly find the optimal switching time $\hat{T}_1$ in $1, \cdots, N-1$ by Theorem 3.2 for mode 1 and 2, and then obtain the optimal switching time $\hat{T}_2$ in $\hat{T}_1, \cdots, N-1$ for mode 2 and 3 by adopting the same method, and so on. The following simulation example verifies the feasibility and effectiveness of the proposed method.

4. Simulation Example.

In order to verify the numerical feasibility of the proposed method, we use the difference formulae in (11) and (12) on a numerical example. This example has been adopted from [19], but here we use its discretization.

Example 4.1. *Consider the discrete-time linear switched system:*

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 0.9 & 0 \\ 0.1 & 1.2 \end{pmatrix} x(k) = A_1 x(k),\ k = 1, \cdots, T_1 \\ x(k+1) &= \begin{pmatrix} 1 & 0.1 \\ 0.1 & 0.8 \end{pmatrix} x(k) = A_2 x(k),\ k = T_1 + 1, \cdots, T_2 \\ x(k+1) &= A_1 x(k),\ k = T_2 + 1, \cdots, N-1 \end{aligned}$$

where $x(k) \in R^2$ is system state. Assume $N = 10$ and the system switches at T_1 from subsystem 1 to 2 and switches at T_2 from subsystem 2 to 3. Find optimal switching times T_1, T_2 such that cost $J = \frac{1}{2}\sum_{k=0}^{N-1} x^T(k)x(k)$ is minimized. Here, the initial condition is chosen as $x(0) = x_0 = [1\ 0]^T$.

Then, based on Theorem 3.2, we can obtain the following results. Firstly, for subsystem 1 and subsystem 2, the first switching time is initialized to $T_{10} = 1$, and the specific data are listed in Table 1. From Table 1, it can be seen that the optimization algorithm

TABLE 1. The switching times $T_1(k)$ and the cost $J_{12}(T_1(k))$

Iteration	0	1	2	3	4
$T_1(k)$	1	2	3	4	5
$J_{12}(T_1(k))$	5.3884	4.6387	4.1633	3.9182	3.8862

TABLE 2. The switching times $T_2(k)$ and the cost $J_{23}(T_2(k))$

Iteration	0	1	2	3	4
$T_2(k)$	5	6	7	8	9
$J_{23}(T_2(k))$	3.7651	2.6165	2.1431	1.9937	1.9781

terminates after 4 iterations. For subsystem 1 and subsystem 2, the first optimal switching time is found to be $T_{1opt} = 5$.

Then, for subsystem 2 and subsystem 3, the second switching time is initialized to $T_{20} = T_{1opt} = 5$, and the specific data are listed in Table 2. From Table 2, it can be seen that the optimization algorithm terminates after 4 iterations. For subsystem 2 and subsystem 3, the second optimal switching time is found to be $T_{2opt} = 9$.

Finally, for this example, we obtain that the local optimal switching time vector is $\bar{\tau} = [5, 9]$ and the corresponding local optimal performance is $J_{opt} = 3.8862$.

In addition, as value of M is small, by comparing all the possible value of the switching instants $\{T_1, T_2\}$ and calculating the corresponding performance cost, we can obtain that the optimal switching times and performance in theory are as follows:

$$\tau_{opt,theory} = [5, 9], \quad J_{opt,theory} = 3.8862$$

From the above data, it can be seen that, for this example, we can achieve the optimal values in theory using the proposed method.

To further verify the results obtained above, we give the performance cost associated with the switching time variables for all their possible values in Figure 1.

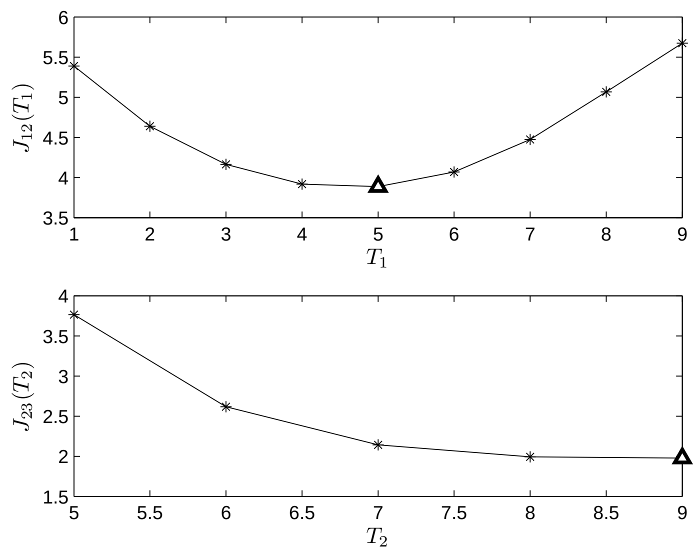

FIGURE 1. The performance cost

In Figure 1, the above curve is displayed as functions of T_1, and it gives the performance cost associated with all possible values of the first switching time of subsystems 1 and subsystems 2. The bottom curve is displayed as functions of T_2, and it gives that of the second switching time of subsystems 2 and subsystems 3. The optimal values are presented by triangular.

From Figure 1, we can see that the optimal values we obtained above are correct.

Remark 4.1. *From the above example, we can see that the proposed method is feasible.*

5. **Conclusions.** In this paper, the optimal switching time control problem for the discrete-time linear switched systems has been studied. Based on the calculus of variations, we have obtained the difference of the cost functional with respect to the switching time for signal switching time case. Then, we have provided a method to deal with the multiple switching time systems. At last, a simulation example is given. Moreover, the number of switching points can be viewed as a part of the control variable instead of being only a given constant, which needs considering. Meanwhile, the optimal timing control of the nonlinear discrete-time switched systems also needs considering further.

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