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STABILITY ANALYSIS FOR SAMPLED-DATA SYSTEMS BASED ON MULTIPLE LYAPUNOV FUNCTIONAL METHOD

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Abstract. *This paper develops a novel method to deal with the stability problem of sampled-data systems. The multiple Lyapunov function method used in switched systems is extended to sampled-data systems and a multiple Lyapunov functional method is developed for nonlinear sampled-data systems. An application is presented where two different Lyapunov functionals are constructed in order to determine the stability of a linear sampled-data system for variable sampling frequency. The method in this paper needs neither the restriction of connecting adjacent Lyapunov functions at switching point nor the condition pertaining to dwell time, and thus the results provide an effective example of an application of multiple Lyapunov function technique. If more Lypunov functionals are employed, much less conservative results may be obtained.*
Keywords: Multiple Lyapunov functions, Switched systems, Sampled-data systems, Stability analysis, Maximum allowable transfer interval (MATI)

1. **Introduction.** Sampled-data systems have been extensively studied during the past decades, where for problems of stability analysis and controller synthesis, there are mainly fourth approaches developed. The first one is based on lifting technique [1, 2], in which the problem is transformed to an equivalent finite-dimensional discrete problem. This approach may not well work in the case of uncertain sampling times or uncertain system matrices. The second method is based on modeling the sampled-data system as a continuous system with delayed control input, where the derivative of the delay is equal to one [3-5]. Then, methods used to study normal delay systems can be applied to study sampled-data systems [6]. However, for this method, as Mirkin pointed out in [7], the sawtooth structure of sampled-data system and all the available information about the actual sampling pattern are neglected. The third method is based on the impulsive modeling of sampled-data in which a time-varying periodic Lyapunov function is used [8]. In [9, 10], based on the third method, a Lyapunov function with discontinuity is introduced and a less conservative result is obtained. The fourth method is emulation method [11], in which one first designs a continuous-time controller based on a continuous-time plant model. At this step the sampling is completely ignored. Then, the obtained continuous-time controller is discretized and implemented using a sampler and hold device.

Notice that sampled-data systems can be seen as a switched (or hybrid) system. For switched systems, the single Lyapunov function and multiple Lyapunov function method are two basic tools for stability analysis [12-15]. The multiple Lypunov function method often leads to a less conservative result than the single Lyapunov function method. However, for those three main methods mentioned above for sampled-data systems, all of them in fact belong to the category of the single Lyapunov function (functional) [16-18]. Therefore, it follows that if the multiple Lyapunov function method is applied to the stability analysis of sampled-data systems, a less conservative result may be obtained. Of course, it may be a hard problem to solve since when using this method, one needs to know the values of suitable Lyapunov functions at switching times, which in general requires knowledge of the state at these times. ([12], p.55). As a result, in order to apply this method, connecting adjacent Lyapunov functions at switching points is a commonly accepted strategy by using "min-switching" strategy of all Lyapunov functions [19]. Although easy to design and realize, such a realization is a special case of normal multiple Lyapunov function methods [14]. The average dwell time method [20] can be seen as another application of multiple Lyapunov function and this method is often used for stability analysis of switched systems or switched delay systems [20-25]. Applying this method has the restriction of dwell time of the switched system and also normally the change rate of the activated Lyapunov function values. Therefore, it is a challenge problem to avoid such restrictions when multiple Lyapunov functional method is applied.

In this paper, the multiple Lyapunov functional method is presented to study the stability of sampled-data systems. Neither the connecting adjacent Lyapunov functionals at switching points nor dwell time condition is necessary. In the case of the longer sample interval which is divided into two intervals, the sampled-data system can be regarded as a hybrid system with two subsystems. Two Lyapunov functionals are then used. In keeping with the characteristics of sampled-data systems, the multiple Lyapunov functionals are well constructed such that at each switching point, the values of activated Lyapunov functional decrease without the restriction of dwell time. Based on the method in this paper, the maximum sample interval, named maximum allowable transfer interval (MATI) in the networked control systems, can be easily obtained for linear sampled-data systems based on linear matrix inequalities (LMIs). It is shown that the results in this paper contain the existing one as a special case and a numerical example is also given to show the effectiveness of the proposed method. Results obtained in this paper show the effectiveness of using multiple Lyapunov function methods for switched system.

This paper is organized as follows. In Section 2, two theorems are given such that the considered nonlinear sampled-data systems are globally uniformly asymptotically stable (GUAS). Then, in Section 3, LMI conditions guaranteeing GUAS for linear sampled-data system with variable sampling are proposed. Section 4 gives an example to show the effectiveness of the method. Section 5 concludes the paper.

For consistency, we try to adopt the same notations as in [10]. Let $\mathbf{N} = \{0, 1, 2, \ldots\}$. Given an interval $I \subset R$, $B(I, R^n)$ denotes the space of real functions from I to R^n with norm $||\phi|| := \sup_{t\in I} |\phi(t)|$, for all $\phi \in B(I, R^n)$, where $|\cdot|$ denotes any one of the equivalent norms in R^n. For a given signal $x(\cdot)$, x_t denotes the function $x_t : [-r, 0] \to R^n$ defined by $x_t(\theta) = x(t + \theta)$, $\forall\theta \in [-r, 0]$ for some positive constant r. A function $\alpha \in [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{K}$, and we write $\alpha \in \mathcal{K}$ when α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then we say it is of class $\mathcal{K}_\infty$. $\lambda_{\max}(\cdot)(\lambda_{\min})$ denotes the maximum (minimum) eigenvalue of a symmetric matrix.

2. **Stability of Nonlinear Sampled-data Systems.**

Consider the following sampled-data system,

$$\dot{x} = f(x(t), t) + g_1(x(t), t)g_2(x(s_k), s_k), \quad t \in [s_k, s_{k+1}), \quad k \in \mathbf{N}, \tag{1}$$

where $x(t)$ and $g_2(x(s_k), s_k)$ denote the state and control input, respectively; f, g_1 and g_2 are globally Lipschitz functions from $R^n \times R$ to R^n such that $f(0,t) = 0$, $g_i(0,t) = 0$ $(i = 1, 2)$, $\forall t \in R_{[0,+\infty)}$. The sampled-data sequence $\{s_k\}$ forms a strictly increasing sequence in $[s_0, \infty)$ for some initial time s_0. Suppose that $0 < h_{\min} \le s_{k+1} - s_k \le \rho_{1m}$, $\forall k \in \mathbf{N}$ for constants $h_{\min}$ and ρ_{1m}. ρ_{1m} denotes the maximum sampling interval, and is also called MATI. Let $\rho_{12} = \rho_{1m} - \rho_{2m}$, where $0 \le \rho_{2m} \le \rho_{1m}$. Notice that the sequence $\{s_k\}$ is a variable sampling one, thus for some interval $[s_k, s_{k+1})$ with properties $s_{k+1} - s_k \ge \rho_{12}$, we can introduce another sequence g_k defined as

$$g_k = \begin{cases} s_k + \rho_{12}, & \text{if } s_{k+1} - s_k > \rho_{12}, \\ s_{k+1}, & \text{if } s_{k+1} - s_k \le \rho_{12}. \end{cases}$$

Define $I_1[0,\infty) = \bigcup_{k=0}^{\infty}[s_k, g_k)$ and $I_2[0,\infty) = \bigcup_{k=0}^{\infty}[g_k, s_{k+1})$. We call time points s_k and g_k switching points and introduce the switching signal $\sigma(t)$

$$\sigma(t) = \begin{cases} 1, & t \in I_1[0,\infty), \\ 2, & t \in I_2[0,\infty). \end{cases}$$

Then, sign $V_{\sigma(t)}(x_t, t)$ is defined as

$$V_{\sigma(t)}(x_t, t) = \begin{cases} V_1(x_t, t), & t \in I_1[0,\infty), \\ V_2(x_t, t), & t \in I_2[0,\infty). \end{cases}$$

Theorem 2.1. *Suppose that there exist $\psi_i \in \mathcal{K}_\infty$ $(i = 1, 2, 3, 4)$, $\psi_j \in \mathcal{K}$ $(j = 5, 6)$, and the functionals $V_1(\phi, t) : B([-\rho_{12}, 0], R^n) \times I_1[0, +\infty) \to R_{[0,\infty)}$, and $V_2(\phi, t) : B([-\rho_{1m}, 0], R^n) \times I_2[0, +\infty) \to R_{[0,\infty)}$, which are continuously differentiable in $I_1[0,\infty)$ and $I_2[0,\infty)$ respectively, such that*

$$\begin{aligned} &\psi_1(|\phi(0)|) \le V_1(\phi, t) \le \psi_2(\|\phi\|), \quad \forall \phi \in B([-\rho_{12}, 0], R^n), \quad t \in I_1[0,\infty), \\ &\psi_3(|\phi(0)|) \le V_2(\phi, t) \le \psi_4(\|\phi\|), \quad \forall \phi \in B([-\rho_{1m}, 0], R^n), \quad t \in I_2[0,\infty), \end{aligned} \tag{2}$$

and for every $\{s_k,\ g_k\}$, any solution to system (1) is globally defined and satisfies

$$\begin{aligned} &\frac{dV_1(x_t, t)}{dt} \le -\psi_5(|x(t)|), \quad s_k \le t \le g_k, \quad k \in \mathbf{N}, \\ &\frac{dV_2(x_t, t)}{dt} \le -\psi_6(|x(t)|), \quad g_k \le t \le s_{k+1}, \quad k \in \mathbf{N}, \end{aligned} \tag{3}$$

and for any switching point t_k (including s_k and g_k), it holds that

$$V_{\sigma(t)}(x_{t_k}, t_k) \le \lim_{t \to t_k^-} V_{\sigma(t)}(x_t, t). \tag{4}$$

Then, system (1) is GUAS under the MATI ρ_{1m}.

Proof: When $\rho_{2m} = 0$, that is $\rho_{12} = \rho_{1m}$, this theorem reduces to Theorem 1 in [9].

For $\rho_{2m} \ne 0$, the following gives the stability proof. For every s_k, g_k, $k \in \mathbf{N}$, it yields from (3)

$$\begin{aligned} &V_1(x_t, t) \le V_1(x_{s_k}, s_k), \quad \forall t \in [s_k, g_k), \\ &V_2(x_t, t) \le V_2(x_{g_k}, g_k), \quad \forall t \in [g_k, s_{k+1}). \end{aligned} \tag{5}$$

Let $\bar{\psi}_1(\cdot) = \min\{\psi_1(\cdot), \psi_3(\cdot)\}$ and $\bar{\psi}_2(\cdot) = \max\{\psi_2(\cdot), \psi_4(\cdot)\}$. It is easy to verify that they are still of class $\mathcal{K}_\infty$. Suppose that $t \in [g_l, s_{l+1})$, $l \in \mathbf{N}$ then it follows that from (4) and (5)

$$\begin{aligned} \bar{\psi}_1(|x(t)|) &\le V_2(x_t, t) \le V_2(x_{g_l}, g_l) \le V_1(x_{g_l^-}, g_l^-) \\ &\le V_1(x_{s_l}, s_l) \le \ldots \le V_1(x_{t_0}, t_0) \le \bar{\psi}_2(\|x_{t_0}\|). \end{aligned} \tag{6}$$

Thus, it is obtained that $|x(t)| \leq \bar{\psi}_1^{-1}\bar{\psi}_2(||x_{t_0}||)$. For $\forall t \in [s_l, g_l)$, a similar proof can be obtained. Thus, $|x(t)| \leq \alpha(||x_{t_0}||)$, $\forall t \geq t_0$ for $\alpha(\cdot) = \bar{\psi}_1^{-1}\bar{\psi}_2(\cdot) \in \mathcal{K}$, and the stability proof follows from the definition of globally uniformly stability in [9].

The proof of GUAS, for every ε, let $\delta_1 > 0$, be such that $\bar{\psi}_2(\delta_1) \leq \bar{\psi}_1(\varepsilon)$. Then from (6), $||x_{t_0}|| < \delta_1$ implies that $|x(t)| < \varepsilon$, $t \geq t_0$. For this δ_1 and any $\eta > 0$, we show that there exists a $T = T(\delta_1, \eta)$ such that $|x(t)| < \eta$ for $\forall t \geq t_0 + T$. Given $\delta_2 > 0$ such that $\bar{\psi}_2(\delta_2) \leq \bar{\psi}_1(\eta)$ for $t \geq t_0 + T$. Then, it suffices to show that $||x_{t_0+T}|| < \delta_2$ which implies $|x(t)| < \eta$, $\forall t \geq 0$. By using contradiction method, we assume that such a T does not exist and therefore we can construct a sequence $c_k = s_k + \min\{\frac{\rho_{12}}{2}, \frac{h_{\min}}{2}\}$, $k \in \mathbf{N}$ such that $||x_{c_k}|| \geq \delta_2$. Thus, in $[s_k, c_k] \subset [s_k, g_k)$, we can define intervals $I_k = [c_k - \frac{\delta_2}{2L_1}, c_k]$, where $L_1 > \max\left(L, \frac{\delta_2}{\min\left\{\frac{\rho_{12}}{2}, \frac{h_{\min}}{2}\right\}}\right)$ and $|f(x,t) + g_1(x,t)g_2(x_k)| < L$, $\forall k \in \mathbf{N}$ (since f, g_1, g_2 are Lipshitz, there exists $L > 0$ such that $|f(x,t) + g_1(x,t)g_2(x_k)| < L$). It follows for this that, $x(t)$ is continuous for any $t \in I_k$ and we can use the Mean Value Theorem, so for any $t \in I_k$, there exists a $\theta \in [0,1]$ such that

$$\begin{aligned} |x(t)| &= |x(c_k) + \dot{x}(c_k + \theta(t - c_k))(t - c_k)| \\ &\geq |x(c_k)| - |\dot{x}(c_k + \theta(t - c_k))||t - c_k| \\ &\geq \delta_2 - L\frac{\delta_2}{2L_1} \geq \frac{\delta_2}{2}. \end{aligned} \tag{7}$$

Thus, $\frac{dV_1(x_t,t)}{dt} \leq -\psi_3(\frac{\delta_2}{2})$ for any $t \in I_k$, and otherwise $\frac{dV_1(x_t,t)}{dt} \leq 0$ for $t \in I_1[0,\infty)/I_k$ and $\frac{dV_2(x_t,t)}{dt} \leq 0$ for $t \in I_2[0,\infty)$. Therefore, it follows that

$$\begin{aligned} \int_{t_0}^{c_k} \dot{V}_{\sigma(t)}(x_t,t)dt &= \int_{s_k}^{c_k} \dot{V}_1(x_t,t)dt + \int_{g_{k-1}}^{s_k} \dot{V}_2(x_t,t)dt \\ &\quad + \int_{c_{k-1}}^{g_{k-1}} \dot{V}_1(x_t,t)dt + \int_{s_{k-1}}^{c_{k-1}} \dot{V}_1(x_t,t)dt + \ldots \\ &\leq -\psi_3\left(\frac{\delta_2}{2}\right)k\frac{\delta_2}{2L_1}. \end{aligned} \tag{8}$$

On the other hand, noting from (4) it is obtained that

$$\begin{aligned} &\int_{t_0}^{c_k} \dot{V}_{\sigma(t)}(x_t,t)dt \\ &= V_1(x_{c_k}, c_k) - V_1(x_{s_k}, s_k) + V_2(x_{s_k}, s_k) - V_2(x_{g_{k-1}}, g_{k-1}) \\ &\quad + \ldots + V_1(x_{c_1}, c_1) - V_1(x_{t_0}, t_0) \\ &\geq V_1(x_{c_k}, c_k) - V_1(x_{t_0}, t_0). \end{aligned} \tag{9}$$

Thus, it follows from (8) and (9) that

$$V_1(x_{c_k}, c_k) - V_1(x_{t_0}, t_0) \leq -\psi_3\left(\frac{\delta_2}{2}\right)k\frac{\delta_2}{2L_1}. \tag{10}$$

This would imply $V_1(x_{c_k}, c_k) < 0$ for a sufficiently large k. By using the contradiction method, we conclude that the system is GUAS. The proof is completed.

Remark 2.1. *In Theorem 2.1, at each switching point, the Lyapunov functionals jump decreasingly but need not continuous restriction. Between two consecutive switching points, the corresponding activated Lyapunov functional also needs decrease, but the other Lyapunov functional in the same interval is not required, which may increase or even have a*

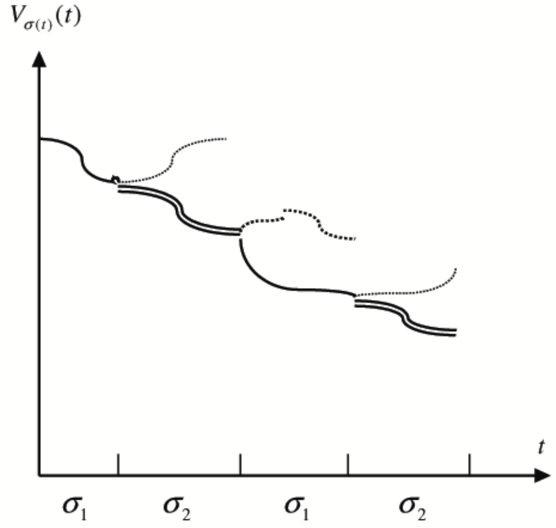

FIGURE 1. Two Lyapunov functionals (solid graphs correspond to $V_1(t)$, dashed graphs correspond to $V_2(t)$), dash-dotted graphs denote other inactivated functionals

point of discontinuity; see Figure 1. Theorem 2.1 can be seen as an extension of multiple Lyapunov functions method used in the switched systems [12, 13].

Remark 2.2. *If $\rho_{2m} = 0$, then $\rho_{12} = \rho_{1m}$ and only switching sequence $\{s_k\}$ and $V_1(x_t, t)$ is maintained. Hence, two Lyapunov functionals reduce to a single Lyapunov functional. For this case, Theorem 2.1 above reduces to Theorem 1 in* [9].

For system (1), replace (2) with the following condition

$$\begin{aligned}
&\psi_1(|\phi(0)|) \leq V_1(\phi, t) \leq \psi_2(||\phi||),\\
&\qquad\qquad \forall\phi \in B([-\rho_{12}, 0], R^n), \quad t \in \{s_k, g_k\}, \quad k \in \mathbf{N},\\
&\psi_3(|\phi(0)|) \leq V_2(\phi, t) \leq \psi_4(||\phi||),\\
&\qquad\qquad \forall\phi \in B([-\rho_{1m}, 0], R^n), \quad t \in \{g_k, s_{k+1}\}, \quad k \in \mathbf{N}
\end{aligned} \tag{11}$$

Then, the following theorem can be obtained.

Theorem 2.2. *For system (1), suppose conditions (3), (4) and (11) hold, then system (1) is GUAS under the MATI ρ_{1m}.*

Proof: First proof stability for $\rho_{12} > 0$. Let $\bar{\psi}_1(\cdot) = \min\{\psi_1(\cdot), \psi_3(\cdot)\}$ and $\bar{\psi}_2(\cdot) = \max\{\psi_2(\cdot), \psi_4(\cdot)\}$. For every s_k, $k \in \mathbf{N}$, similar to the proof of Theorem 2.1, it yields

$$\begin{aligned}
\bar{\psi}_1(|x(s_k)|) &\leq V_1(x_{s_k}, s_k) \leq V_2(x_{s_k^-}, s_k^-)\\
&\leq V_2(x_{g_{k-1}}, g_{k-1}) \leq \dots\\
&\leq V_1(x_{t_0}, t_0) \leq \bar{\psi}_2(||x_{t_0}||).
\end{aligned} \tag{12}$$

Thus, for $\forall k \in \mathbf{N}$, $|x(s_k)| \leq \bar{\psi}_1^{-1}\bar{\psi}_2(||x_{t_0}||)$.

Noting that since f, g_1, g_2 are globally Lipschitz with Lipschitz constant L, it is obtained that

$$|f_1(x(t), t)| \leq L|x(t)|, \tag{13}$$
$$|g_1(x(t), t)| \leq L|x(t)|, \tag{14}$$
$$|g_2(x(s_k), s_k)| \leq L|x(s_k)|, \tag{15}$$

Noting also that $|x(s_k)| \leq \bar{\psi}_1^{-1}\bar{\psi}_2(||x_{t_0}||)$, we obtain that

$$|\dot{x}(t)| \leq |f_1(x(t),t)| + |g_1(x(t),t)||g_2(x(s_k),s_k)| \leq \bar{L}|x(t)|$$

for some positive constant $\bar{L}$. Then, by the equation $x(t) = x(s_k) + \int_{s_k}^{t} \dot{x}(s)ds$, $\forall t \in [s_k, s_{k+1})$, it follows that

$$|x(t)| \leq |x(s_k)| + \bar{L}\int_{s_k}^{t} |x(s)|ds, \quad \forall t \in [s_k, s_{k+1}). \tag{16}$$

From Bellman-Gronwall Lemma ([26], p.101), we have

$$|x(t)| \leq |x(s_k)|e^{\bar{L}(t-s_k)} \leq |x(s_k)|e^{\bar{L}\rho_{1m}}, \quad \forall t \in [s_k, s_{k+1}). \tag{17}$$

Therefore, for $\forall t \in [s_k, s_{k+1})$, $\forall k \in \mathbf{N}$, it follows that

$$|x(t)| \leq e^{\bar{L}\rho_{1m}}\bar{\psi}_1^{-1}\bar{\psi}_2(||x_{t_0}||).$$

The stability proof is completed since $e^{\bar{L}\rho_{1m}}\bar{\psi}_1^{-1}\bar{\psi}_2 \in \mathcal{K}$.

The proof of GUAS is the same as Theorem 2.1 and omitted in this paper.

For $\rho_{2m} = 0$, then only a sequence of $\{s_k\}$ is maintained, which can be seen as a special case of $\rho_{2m} > 0$. The proof is similar to the case of $\rho_{2m} > 0$ and is omitted here. The proof is completed.

Remark 2.3. *Theorem 2.2 only requires that the condition (2) holds at the switching point. The relaxation will be useful for stability analysis of sampled-data systems with constant sampling, as will be seen in the following section.*

3. Stability of Linear Sampled-data Systems with Variable Sampling.

As an application of the proposed multiple Lyapunov functional method, we will give a stability criterion for linear sampled-data system with variable sampling in terms of LMIs in this section. Consider the following linear system with variable sampling

$$\begin{cases} \dot{x} = Ax(t) + B_u u(t), \\ u(t) = Kx(s_k), \quad t \in [s_k, s_{k+1}), \quad k \in \{1, 2, \ldots\}, \end{cases} \tag{18}$$

where A and B_u are constant matrices. Suppose that the controller gain K is a known matrix in this paper and denote $B = B_u * K$.

Introduce the notations $z_1 = z_1(t) = x(s_k)$ and $z_2 = z_2(t) = x(g_k)$, $\forall t \in [g_k, s_{k+1})$, $\forall k \in \mathbf{N}$. During the interval $[s_k, g_k)$ and $[g_k, s_{k+1})$, we introduce the following two different Lyapunov functionals

$$\begin{aligned} V_1(t) = {} & x(t)^T Px(t) + \int_{t-\rho_1}^{t} (\rho_{1m} - t + s)\dot{x}^T(s)R\dot{x}(s)ds \qquad (19) \\ & +(\rho_{1m} - \rho_1)(x(t) - z_1)^T X(x(t) - z_1), \quad \forall t \in [s_k, g_k), \\ V_2(t) = {} & x(t)^T Px(t) + \int_{t-\rho_2}^{t} (\rho_{2m} - t + s)\dot{x}^T(s)\bar{R}_1\dot{x}(s)ds \\ & + \int_{t-\rho_1}^{t-\rho_2} (\rho_{1m} - t + s)\dot{x}^T(s)\bar{R}_2\dot{x}(s)ds \\ & +(\rho_{1m} - \rho_1)(x(t) - z_1)^T \bar{X}_1(x(t) - z_1) \\ & +(\rho_{2m} - \rho_2)(x(t) - z_2)^T \bar{X}_2(x(t) - z_2), \quad \forall t \in [g_k, s_{k+1}), \qquad (20) \end{aligned}$$

where matrices $P > 0$, $R > 0$, $X > 0$, $\bar{R}_1 > 0$, $\bar{R}_2 > 0$, $\bar{X}_1 > 0$, $\bar{X}_2 > 0$ are matrices to be determined.

Construct two Lyapunov functionals

$$V(t) = \begin{cases} V_1(t), & t \in [s_k, g_k), \\ V_2(t), & t \in [g_k, s_{k+1}), \quad \forall k \in \mathbf{N}. \end{cases} \tag{21}$$

Now, we are in the position to give the main result.

Theorem 3.1. *System (18) is GUAS under the MATI of ρ_{1m}, if there exist matrices $P > 0$, $R > 0$, $X > 0$, $\bar{R}_1 > 0$, $\bar{R}_2 > 0$ and $\bar{X}_1 > 0$, $\bar{X}_2 > 0$ and matrices N, L_1, L_2 as well as constants $\rho_{2m} > 0$ such that the following LMIs hold,*

$$M_1 + \rho_{12} M_2 < 0, \tag{22}$$

$$\begin{bmatrix} M_1 & \rho_{12} N \\ * & -\rho_{12} R \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} \Pi_1 + \rho_{2m}\Pi_2 & \rho_{12} L_2 \\ * & -\rho_{12}\bar{R}_2 \end{bmatrix} < 0, \tag{24}$$

$$\begin{bmatrix} \Pi_1 & \rho_2 L_1 & \rho_{12} L_2 \\ * & -\rho_2 \bar{R}_1 & 0 \\ * & * & -\rho_{12}\bar{R}_2 \end{bmatrix} < 0, \tag{25}$$

and

$$X \geq \bar{X}_1, \quad R \geq \bar{R}_2, \tag{26}$$

where $\rho_{12} = \rho_{1m} - \rho_{2m}$,

$$\bar{F} = \begin{bmatrix} A & B \end{bmatrix},$$

$$\begin{aligned} M_1 = {} & \begin{bmatrix} P \\ 0 \end{bmatrix} \bar{F} + \bar{F}^T \begin{bmatrix} P & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} X \begin{bmatrix} I & -I \end{bmatrix} - N \begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N^T \\ & + \rho_{1m} \bar{F}^T R \bar{F} + \rho_{2m} M_2, \\ M_2 = {} & \begin{bmatrix} I \\ -I \end{bmatrix} X \bar{F} + \bar{F}^T X^T \begin{bmatrix} I \\ -I \end{bmatrix}, \\ \Pi_1 = {} & \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} P \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} P \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} - L_1 \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix} \\ & - \begin{bmatrix} I \\ -I \\ 0 \\ 0 \end{bmatrix} L_1^T - L_2 \begin{bmatrix} 0 & I & 0 & -I \end{bmatrix} - \begin{bmatrix} 0 \\ I \\ 0 \\ -I \end{bmatrix} L_2^T + \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \rho_{2m} \bar{R}_1 \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix} \\ & - \begin{bmatrix} I \\ 0 \\ 0 \\ -I \end{bmatrix} \bar{X}_1 \begin{bmatrix} I & 0 & 0 & -I \end{bmatrix} + L_3 \begin{bmatrix} -A & 0 & I & -B \end{bmatrix} + \begin{bmatrix} -A^T \\ 0 \\ I \\ -B^T \end{bmatrix} L_3^T \\ & - \begin{bmatrix} I \\ -I \\ 0 \\ 0 \end{bmatrix} \bar{X}_2 \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\Pi_2 = \begin{bmatrix} I \\ 0 \\ 0 \\ -I \end{bmatrix} \bar{X}_1 \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \bar{X}_1 \begin{bmatrix} I & 0 & 0 & -I \end{bmatrix}$$
$$+ \begin{bmatrix} I \\ -I \\ 0 \\ 0 \end{bmatrix} \bar{X}_2 \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \bar{X}_2 \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix}.$$

Proof: For the constructed Lyapunov functional (21), it is easy to find at each switching point s_k, $\int_{t-\rho_1}^{t}(\rho_{1m} - t + s)\dot{x}^T(s)R\dot{x}(s)ds = 0$ and $x(t) - z_1 = 0$, and thus $V_1(s_k) \leq \lim_{t\to s_k^-} V_2(t)$. On the other hand, at each switching point g_k, in interval $[g_k, s_{k+1})$, $\int_{t-\rho_2}^{t}(\rho_{2m}-t+s)\dot{x}^T(s)\bar{R}_1\dot{x}(s)ds = 0$ and $x(t)-z_2 = 0$, it follows that $V_2(g_k) \leq \lim_{t\to g_k^-} V_1(t)$ under the restriction of (26). Thus, under the condition of (26), it always holds that $V(t_k) \leq \lim_{t\to t_k^-} V(t)$. Hence, condition (4) in Theorem 2.1 is satisfied.

In the time interval $[s_k,\ g_k)$, for Lyapunov functional $V_1(t)$, along the trajectory of system (18), similarly to the proof of Theorem 2 in [10], it can be readily shown that $\dot{V}_1(t) \leq -\psi_5(x(t)) < 0$ holds under the LMI conditions (22) and (23), where $\psi_5(x(t)) = \alpha|x(t)|^2$ for some positive constant α.

In time interval $[g_k,\ s_{k+1})$, for Lyapunov functional $V_2(t)$, along the trajectory of system (18), it follows that

$$\begin{aligned}\dot{V}_2(t) &= 2x^T P\dot{x} - \int_{t-\rho_2}^{t} \dot{x}^T(s)\bar{R}_1\dot{x}(s)ds + \rho_{2m}\dot{x}^T(t)\bar{R}_1\dot{x}(t) - \int_{t-\rho_1}^{t-\rho_2} \dot{x}^T(s)\bar{R}_2\dot{x}(s)ds \\ &+2(\rho_{1m} - \rho_1)(x(t) - z_1))^T\bar{X}_1\dot{x} - (x(t) - z_1)^T\bar{X}_1(x(t) - z_1) \\ &+2(\rho_{2m} - \rho_2)(x(t) - z_2))^T\bar{X}_2\dot{x} - (x(t) - z_2)^T\bar{X}_2(x(t) - z_2).\end{aligned} \tag{27}$$

Noting that in interval $[g_k, s_{k+1})$, it is true that $\rho_{1m} - \rho_1 = \rho_{2m} - \rho_2$.

Let $\zeta = \begin{bmatrix} x(t) & z_2 & \dot{x}(t) & z_1 \end{bmatrix}^T$. It can be shown that

$$\begin{aligned}&2\zeta^T L_1(x - z_2) = 2\zeta^T L_1 \int_{t-\rho_2}^{t} \dot{x}(s)ds \\ &\leq \rho_2\zeta^T L_1\bar{R}_1^{-1}L_1^T\zeta + \int_{t-\rho_2}^{t} \dot{x}^T(s)\bar{R}_1\dot{x}(s)ds, \\ &2\zeta^T L_2(z_2 - z_1) = 2\zeta^T L_2 \int_{t-\rho_1}^{t-\rho_2} \dot{x}(s)ds \\ &\leq \rho_{12}\zeta^T L_2\bar{R}_2^{-1}L_2^T\zeta + \int_{t-\rho_1}^{t-\rho_2} \dot{x}^T(s)\bar{R}_2\dot{x}(s)ds, \\ &2\zeta^T L_3(\dot{x}(t) - Ax(t) - Bz_1) = 0.\end{aligned} \tag{28}$$

Thus, it yields from (27) and (28) that

$$\dot{V}_2(t) \leq \zeta^T(\bar{\Pi}_1 + (\rho_{2m} - \rho_2)\Pi_2 + \rho_2\Pi_3)\zeta,$$

where

$$\begin{aligned}\Pi_3 &= L_1\bar{R}_1^{-1}L_1^T, \\ \bar{\Pi}_1 &= \Pi_1 + \rho_{12}L_2\bar{R}_2^{-1}L_2^T.\end{aligned}$$

Notice that

$$\bar{\Pi}_1 + (\rho_{2m} - \rho_2)\Pi_2 + \rho_2\Pi_3 < 0$$

is equivalent to

$$\bar{\Pi}_1 + \rho_{2m}\Pi_2 < 0,$$
$$\bar{\Pi}_1 + \rho_2\Pi_3 < 0,$$

(The proof can be seen in [10]), which are equivalent to LMIs (24) and (25), respectively. Hence, $\dot{V}_2(t) \leq -\psi_6(x(t))$, where $\psi_6(x(t)) = -\lambda_{\max}(\bar{\Pi}_1 + (\rho_{2m} - \rho_2)\Pi_2 + \rho_2\Pi_3)|x(t)|^2$. Thus, the proof follows from Theorem 2.1.

Remark 3.1. *If $\rho_{2m} = 0$, similar to the analysis of Remark 2.2, the LMI conditions (22) and (23) reduce to the LMI conditions in Theorem 2 in* [10]. *Thus, Theorem 3.1 in this paper contains Theorem 2 in* [10] *as a special case.*

Remark 3.2. *Just as pointed out in* [10], *if system matrices satisfy the polytopic condition, then the stability of the system can be checked for each of the individual vertices by solving the LMIs in Theorem 3.1 with the same matrix variables $P > 0$, $R > 0$, $X > 0$, $\bar{R}_1 > 0$, $\bar{R}_2 > 0$ $\bar{X}_1 > 0$, $\bar{X}_2 > 0$.*

Remark 3.3. *The results in this paper provide a good example for the application of multiple Lyapunov function. In the main theorems of this paper, we are not faced with the restriction of the dwell time method or the restriction of "connecting adjacent Lyapunov functions at switching point"* [12]. *Based on the inherent characteristics of sampled-data systems, we construct multiple Lyapunov functionals such that at each switching point the values of activated Lyapunov functional decrease.*

Remark 3.4. *Noting in this paper, we only divide the sampled-data system into two subsystem and use two Lyapunov functionals. If more subsystems are obtained and more Lyapunov functionals are used, a much less conservative result will be obtained in the cost of calculation.*

4. Example.

Example 4.1. *Consider the process model from* [4] *with*

$$A = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 + g_2 \\ -1 \end{bmatrix},$$

where $|g_1| \leq 0.1$ and $|g_2| \leq 0.3$.

The state-feed back gain is given the same as in [4], $K = -\begin{bmatrix} 2.6884 & 0.6649 \end{bmatrix}$. According to Remark 3.2, for each combination of A_j and B_j, $1 \leq j \leq 2$ defined by

$$A_1 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0.5 \\ 0.1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.7 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.3 \\ -1 \end{bmatrix},$$

solving LMIs defined in Theorem 3.1, we find that MATI for variable sample case is up to 0.5176, when we choose $\rho_{2m} = 0.3676$ and $\rho_{12} = 0.15$. MATI for different methods can be seen in Table 1. However, we must notice that the result may be more conservative than some newest results in the literature, for example, [5, 27]. However, as pointed out in Remark 3.4 above, the less conservative result should be obtained if more Lyapunov functionals are employed in the cost of calculations.

TABLE 1. The $MATI$ for different methods in Example 4.1

	[4]	[10]	theorem of this paper
Variable sampling	0.35	0.4476	0.5176 (Theorem 3.1)

5. **Conclusions.** This paper has dealt with the stability problem of sampled-data systems. For nonlinear sampled-data systems, two stability theorems were firstly developed. And then, based on the inherent characteristics of sampled-data systems, two special Lyapunov functionals are constructed for linear sampled-data systems with variable sampling. This paper provides a good application example of multiple Lyapunov function method used in hybrid system field. It is noted that if more Lyapunov functions are employed, the less conservative results will be obtained in the cost of calculations, which may be a further research topic in the future.

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