FURTHER RESULTS ON GLOBAL STATE FEEDBACK STABILIZATION OF NONLINEAR SYSTEMS WITH LOW-ORDER AND HIGH-ORDER NONLINEARITIES

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Abstract. This paper discusses the state feedback control problem for a class of nonlinear systems with low-order and high-order nonlinearities, and multiple time-varying delays. The introduction of sign function together with the method of adding a power integrator and Lyapunov-Krasovskii theorem makes the closed-loop system globally asymptotically stable.

Keywords: Nonlinear systems, Low-order and high-order nonlinearities, Multiple time-varying delays, State feedback, Sign function

1. Introduction. Consider a class of nonlinear systems:

\[ \begin{align*}
\dot{x}_i(t) &= x_{i+1}^{p_i}(t) + f_i(t, x(t), x_1(t - \tau_1(t)), \ldots, x_n(t - \tau_n(t))), \ i = 1, \ldots, n - 1, \\
\dot{x}_n(t) &= u^{p_n}(t) + f_n(t, x(t), x_1(t - \tau_1(t)), \ldots, x_n(t - \tau_n(t))), 
\end{align*} \]

where \( x(t) = [x_1(t), \ldots, x_n(t)]^\top \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R} \) are the system state and control input, respectively. For \( i = 1, \ldots, n, \tau_i(t) : R^+ \to R^+ \) is time-varying delay with \( 0 \leq \tau_i(t) \leq \varepsilon_i \), where \( \varepsilon_i \) is a positive constant, \( p_i \in R_{odd}^{\geq 1} \triangleq \{ \frac{p}{q} \in R^+: p \text{ and } q \text{ are odd integers, } p \geq q \} \), \( f_i : R^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is an unknown continuous function. The initial value \( x_0(\theta) \) is a continuous function on \( \theta \in [-\tau, 0] \) with \( \tau = \max\{\varepsilon_1, \ldots, \varepsilon_n\} \).

In recent years, the global stabilization for system (1) with both lower-order and higher-order nonlinearities has become a hot topic being studied. In the latest papers [1, 2], the authors discussed that these two papers encompass and substantially generalize the existing results in [3-9].

In [1, 2], the following condition on the uncertain term \( f_i \) is assumed:

\[ |f_i(\cdot)| \leq c \sum_{j=1}^{i} \left( |x_j|^{\frac{1}{r_j}} r_j^{-\frac{1}{r_j}} + |x_j|^{\frac{r_j+\omega}{r_j}} \right), \]  

where \( r_1 = 1, r_i p_i - 1 = r_{i-1} + \omega, i = 1, \ldots, n \). However, (2) needs the condition of \( \omega = \frac{s}{o} \) with \( s \) being an even integer and \( o \) being an odd integer, which results in \( \frac{r_i+\omega}{r_i} \) in (2) being always a ratio of odd integers. Naturally, an interesting problem may be proposed:

Is it possible to relax the assumption on \( \omega \) in (2)? Under the weaker assumption, can one design a stabilizing controller?
In this paper, by introducing the sign function approach, and overcoming several troublesome obstacles in the design and analysis procedure, we focus on solving the above problem under the assumption of the restriction on $\omega$ being relaxed to any real number.

2. Mathematical Preliminaries. The following notations and lemmas are to be used throughout the paper.

Notations: $R^+$ stands for the set of all the nonnegative real numbers. For any vector $x = [x_1, \cdots, x_n]^T \in R^n$, denote $\bar{x}_i = [x_1, \cdots, x_i]^T \in R^i$, $i = 1, \cdots, n-1$, $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$, $x_i = x(t + \theta)$, $\theta \in [-\tau, 0]$, and $\|x_i\| = \sup_{-\tau \leq \theta \leq 0} \|x(t + \theta)\|$, $\forall t \geq 0$. Use $c$ or $c(h_i)$ to represent any positive constant or constant dependent on $[h_1, \cdots, h_i]^T$, which may be implicitly changed from place to place. A sign function $sgn(x)$ is defined as: $sgn(x) = 1$ if $x > 0$, $sgn(x) = 0$ if $x = 0$, and $sgn(x) = -1$ if $x < 0$. The arguments of functions (or functionals) are sometimes omitted or simplified; for instance, we sometimes denote a function $f(x(t))$ by $f(x)$, $f(r)$, or $f$.

Lemma 2.1. [10] For $x, y \in R$, $p \geq 1$ is a constant, then $|x + y|^p \leq 2^{p-1}|x|^p + |y|^p$, $(|x| + |y|)\frac{1}{p} \leq \frac{1}{p}|x|^p + \frac{1}{p}|y|^p$. If $p \in R_{odd}^+$, then $|x - y|^p \leq 2^{p-1}|x|^p - |y|^p$, $|x|^\frac{1}{p} - |y|^\frac{1}{p} \leq 2^{1 - \frac{1}{p}}|x - y|^\frac{1}{p}$.

Lemma 2.2. [10] Let $m, n$ be positive constants. Given any positive number $\gamma > 0$, then $|x|^m |y|^n \leq \frac{m}{m + n} \gamma |x|^{m+n} + \frac{n}{m + n} \gamma^{-\frac{1}{n}} |y|^{m+n}$.

Lemma 2.3. [11] For the continuous function $f : [a, b] \rightarrow R$ $(a \leq b)$, if it is monotonically increasing and satisfies $f(a) = 0$, then $|\int_{a}^{b} f(x) dx| \leq |f(b)||b - a|$.

Lemma 2.4. $f(x) = sgn(x)|x|^a$ is continuously differentiable and satisfies $\dot{f}(x) = a|x|^{a-1}$, where $a \geq 1$, $x \in R$.

Proof: See the Appendix. □

Lemma 2.5. [2] Let $0 \leq \mu_1 \leq \cdots \leq \mu_n$ be real numbers and $c_j > 0$, $j = 1, \cdots, n$. Then for any $x \in R$, one has $c_1|x|^\mu_1 + c_n|x|^\mu_n \leq \sum_{j=1}^n c_j |x|^{\mu_j} \leq (\sum_{j=1}^n c_j)(|x|^{\mu_1} + |x|^{\mu_n})$.

3. Design of State Feedback Controller.

3.1. Problem formulation. In this paper, we need the following assumptions:

Assumption 2.1. For each $i = 1, \cdots, n$ and any $\omega > 0$, there exists a known constant $M$ such that

$$
|f_i(t, x(t), x_1(t - \tau_1(t)), \cdots, x_n(t - \tau_n(t)))|
\leq M \sum_{j=1}^i \left( |x_j(t)|^{\frac{1}{\rho_j-\rho_{i-1}}} + |x_j(t - \tau_j(t))|^{\frac{1}{\rho_j-\rho_{i-1}}} + |x_j(t)|^{\frac{\rho_{i+\omega}}{\rho_j}} + |x_j(t - \tau_j(t))|^{\frac{\rho_{i+\omega}}{\rho_j}} \right),
$$

where

$$
\rho_1 = 1, \quad \rho_{i+1} = \frac{\rho_i + \omega}{\rho_i}, \quad i = 1, \cdots, n.
$$

(3)

Assumption 2.2. For $\tau_i(t)$, $i = 1, \cdots, n$, there is a constant $\delta_i$ such that $\tau_i(t) \leq \delta_i < 1$.

Remark 3.1. Compared with Assumption 1 in [2], two ingredients make the assumption in this paper much weaker. One is that the restriction on $\omega$ is removed. In (2), $\omega = \frac{r}{o}$ with $s$ being an even integer and $o$ being an odd integer results in $\frac{r+\omega}{r}$ always being a ratio of odd integers. The other is the appearance of multiple time-varying delays in state variables. □
3.2. State feedback controller design. We now design a state feedback controller to stabilize system (1) under Assumptions 3.1 and 3.2.

Step 1: Define $z_1(t) = x_1(t)$ and $\sigma = r_{n+1}p_1 \cdots p_n$, and choose

$$V_1(t, z_{1t}) = \frac{z_1^2}{2} + \frac{r_1}{2\sigma - \omega} |z_1|^2 + \frac{b_1}{1 - \delta_1} \int_{t}^{t_{-\tau_1}} \left(z_1^2(s) + \frac{z_1^2}{r_1}(s)\right) ds + \frac{c_1}{1 - \delta_2} \int_{t_{-\tau_2}}^{t} \left(z_1^2(s) + \frac{z_1^2}{r_1}(s)\right) ds,$$

(4)

where $b_1, c_1$ are positive constants. (4) and $0 \leq \tau_i(t) \leq \varepsilon_i$ imply that

$$V_1(t, z_{1t}) \geq \frac{z_1^2}{2} + \frac{r_1}{2\sigma - \omega} |z_1|^2 + \frac{b_1 \varepsilon_1}{1 - \delta_1} + \frac{c_1 \varepsilon_2}{1 - \delta_2},$$

$$\triangleq \pi_{12}(\|z_{1t}\|),$$

(5)

where $\pi_{11}(\cdot), \pi_{12}(\cdot)$ are class $\mathcal{K}_\infty$ functions.

From Lemma 2.2 and Assumption 3.1, it follows that

$$\dot{V}_1 \leq -b_1 \left(\frac{z_1^2}{r_1}(t - \tau_1(t)) + z_1^2(t - \tau_2(t))\right) - c_1 \left(\frac{z_1^2}{r_1}(t - \tau_2(t)) + z_1^2(t - \tau_2(t))\right)$$

$$+ \left(\frac{b_1}{1 - \delta_1} + \frac{c_1}{1 - \delta_2}\right) \left(z_1^2 + \frac{z_1^2}{r_1}\right) + \left(z_1 + \text{sgn}(z_1)|z_1|\right) \frac{2\alpha_{p_1}}{r_1} \alpha_1^{p_1}$$

$$+ \left(z_1 + \text{sgn}(z_1)|z_1|\right) \left(x_2^{p_1} - \alpha_1^{p_1}\right) + \left(z_1 + \text{sgn}(z_1)|z_1|\right) \frac{2\alpha_{r_1}}{r_1} f_1,$$

(6)

where $0 < b_{11} < b_1, \lambda_1(b_{11}) > 0$. By (1), (4), (6) and Assumption 3.2, we have

$$\dot{V}_1 \leq -b_1 \left(\frac{z_1^2}{r_1}(t - \tau_1(t)) + z_1^2(t - \tau_1(t))\right) - c_1 \left(\frac{z_1^2}{r_1}(t - \tau_2(t)) + z_1^2(t - \tau_2(t))\right)$$

$$+ \left(\frac{b_1}{1 - \delta_1} + \frac{c_1}{1 - \delta_2}\right) \left(z_1^2 + \frac{z_1^2}{r_1}\right) + \left(z_1 + \text{sgn}(z_1)|z_1|\right) \frac{2\alpha_{p_1}}{r_1} \alpha_1^{p_1}$$

$$+ \left(z_1 + \text{sgn}(z_1)|z_1|\right) \left(x_2^{p_1} - \alpha_1^{p_1}\right) + \left(z_1 + \text{sgn}(z_1)|z_1|\right) \frac{2\alpha_{r_1}}{r_1} f_1,$$

(7)

where $b_{11} = b_1 - \tilde{b}_{11}, c_{11} = c_1$. Then the first virtual controller $\alpha_1$ defined by

$$\alpha_1(z_1) = -h_1^{p_1} \left(z_1 + \text{sgn}(z_1)|z_1|\right) \frac{2\alpha_{r_1}}{r_1},$$

(8)

results in

$$\dot{V}_1 \leq -a_{11} \left(z_1^2 + \frac{z_1^2}{r_1}\right) - b_{11} \left(\frac{z_1^2}{r_1}(t - \tau_1(t)) + \frac{z_1^2}{r_1}(t - \tau_1(t))\right)$$

$$- c_{11} \left(\frac{z_1^2}{r_1}(t - \tau_2(t)) + \frac{z_1^2}{r_1}(t - \tau_2(t))\right) + \left(z_1 + \text{sgn}(z_1)|z_1|\right) \frac{2\alpha_{r_1}}{r_1} \left(x_2^{p_1} - \alpha_1^{p_1}\right),$$

(9)

where $h_1 = a_{11} + \lambda_1(b_{11}) + \frac{b_{11}}{1 - \delta_1} + \frac{c_{11}}{1 - \delta_2}, a_{11} > 0$. 

Step $k (k = 2, \cdots, n)$: We start with the following proposition:
Proposition 3.1. Suppose that there is a continuously differentiable Lyapunov-Krasovskii functional $V_{k-1}(t, \zeta_{k-1,t})$ satisfying

$$\pi_{k-1,1} (||\zeta_{k-1,t}||) \leq V_{k-1}(t, \zeta_{k-1,t}) \leq \pi_{k-1,2} (||\zeta_{k-1,t}||c),$$

and a series of continuous virtual controllers $\alpha_2, \ldots, \alpha_{k-1}$ defined by

$$z_i(t) = x_i^{p_1 \cdots p_{k-1}} - \alpha_{i-1}^{p_1 \cdots p_{k-1}} (z_{i-1}(t)), \quad \alpha_i(z_i(t)) = -h_i^{1/p_i} \left( z_i(t) + \text{sgn}(z_i(t)) |z_i(t)|^{\frac{\gamma_{k+1}p_k}{p_k}} \right)^{1 \over p_i}, \quad i = 2, \ldots, k-1,$$

such that

$$\dot{V}_{k-1}(t, \zeta_{k-1,t}) \leq -\sum_{j=1}^{k-1} a_{k-1,j} \left( z_j^2 + z_j^{2\gamma_j p_1 \cdots p_j} \right) - \sum_{j=1}^{k-1} b_{k-1,j} \left( z_j^{2\gamma_j p_1 \cdots p_j} (t - \tau_j(t)) + z_j^{2\gamma_j p_1 \cdots p_j} (t - \tau_j(t)) \right)$$

$$- \sum_{j=1}^{k-1} c_{k-1,j} \left( z_j^2 (t - \tau_{j-1}(t)) + z_j^{2\gamma_j p_1 \cdots p_{j-1}} (t - \tau_{j-1}(t)) \right) + \left( \text{sgn}(z_{k-1}) |z_{k-1}|^{2\gamma_k p_1 \cdots p_{k-2}} \right),$$

where $\pi_{k-1,1} \cdot \pi_{k-1,2} \cdot c$ are class $K_\infty$ functions, $h_i$, $i = 2, \ldots, k-1$, $a_{k-1,j}$, $b_{k-1,j}$, $c_{k-1,j}$, $j = 1, \ldots, k-1$, are positive constants. Then, by defining $z_k = x_k^{p_1 \cdots p_k} - \alpha_k^{p_1 \cdots p_k}$, the $k$th functional

$$V_k(t, \zeta_{kt}) = V_{k-1}(t, \zeta_{k-1,t}) + W_{Lk}(\zeta_k) + W_{Hk}(\zeta_k) + W_{Dk}(t, \zeta_{kt})$$

is continuously differentiable and satisfies

$$\pi_{k1} (||\zeta_k||) \leq V_k(t, \zeta_{kt}) \leq \pi_{k2} (||\zeta_{kt}||c),$$

and one can design a controller $\alpha_k(z_k) = -h_k^{1/p_k} \left( z_k + \text{sgn}(z_k)|z_k|^{\gamma_k p_k p_k} \right)^{1 \over p_k}$ such that

$$\dot{V}_k(t, \zeta_{kt}) \leq -\sum_{j=1}^{k} a_{kj} \left( z_j^2 + z_j^{2\gamma_j p_1 \cdots p_j} \right) - \sum_{j=1}^{k} b_{kj} \left( z_j^{2\gamma_j p_1 \cdots p_j} (t - \tau_j(t)) + z_j^{2\gamma_j p_1 \cdots p_j} (t - \tau_j(t)) \right)$$

$$- \sum_{j=1}^{k} c_{kj} \left( z_j^2 (t - \tau_{j-1}(t)) + z_j^{2\gamma_j p_1 \cdots p_{j-1}} (t - \tau_{j-1}(t)) \right) + \left( \text{sgn}(z_k) |z_k|^{2\gamma_k p_1 \cdots p_k} \right),$$

$\pi_{k1} \cdot \pi_{k2} \cdot c$ are class $K_\infty$ functions, $a_{kj}, b_{kj}, c_{kj}, j = 1, \ldots, k$, are positive constants,

$$W_{Lk} = \int_{\alpha_{k-1}}^{\gamma_k} \left( s^{p_1 \cdots p_k} - \alpha_k^{p_1 \cdots p_k} \right)^{2 \gamma_{k+1} p_k}{p_k} ds,$$

$$W_{Hk} = \int_{\alpha_{k-1}}^{\gamma_k} \text{sgn} \left( s^{p_1 \cdots p_k} - \alpha_k^{p_1 \cdots p_k} \right) \left| s^{p_1 \cdots p_k} - \alpha_k^{p_1 \cdots p_k} \right|^{2 \gamma_{k+1} p_k}{p_k} ds,$$

$$W_{Dk} = \left( b_k \int_{t - \tau_k(t)}^{t} \left( z_k^2 + z_k^{2\gamma_k p_1 \cdots p_k} \right) ds + c_k \int_{t - \tau_{k+1}(t)}^{t} \left( z_k^2 + z_k^{2\gamma_k p_1 \cdots p_k} \right) ds \right), \quad b_k, c_k > 0.$$
Hence, at step \( n \), by choosing \( V_n(\cdot) = V_{n-1}(\cdot) + W_{Ln}(\cdot) + W_{Hn}(\cdot) + W_{Dk}(\cdot) \), designing appropriate constants such that \( a_{nj} > 0, b_{nj} = c_{nj} = 0, j = 1, \cdots, n \), and constructing the controller as

\[
\dot{z}_n(t) = -h_n \sum_{j=1}^{n} a_{nj} \left( z_j^2 + z_j \frac{2^a}{(n+1)^{1-p}} \right),
\]

it is easy to get

\[
\dot{V}_n \leq -\sum_{j=1}^{n} a_{nj} \left( z_j^2 + z_j \frac{2^a}{(n+1)^{1-p}} \right),
\]

4. Stability Analysis. We state the main result in this paper.

**Theorem 4.1.** If Assumptions 3.1 and 3.2 hold for system (1), under the continuous state feedback controller (17), then the equilibrium at the origin of the closed-loop system is globally asymptotically stable.

**Proof:** Firstly, by the existence and continuation of the solutions, the states \( x(t) \) and \( z(t) \) are defined on \([-\tau, t_M]\), where the number \( t_M \) may be infinite or not. The following analysis focuses on \([-\tau, t_M]\). Secondly, from Proposition 3.1, we know that

\[
\pi_{n1}(\|z(t)\|) \leq V_n(t, z_i) \leq \pi_{n2}(\|z_i\|),
\]

where \( \pi_{n1}(\cdot) \) and \( \pi_{n2}(\cdot) \) are class \( K_\infty \) functions. (18) and Lemma 4.3 in [12] imply that there exists a class \( K_\infty \) function \( \pi_{n3}(\cdot) \) such that

\[
\dot{V}(t, z_i) \leq -\pi_{n3}(\|z(t)\|).
\]

Thirdly, for any \( \varepsilon > 0 \), since \( \pi_{n1}(\cdot) \) is a class \( K_\infty \) function, one can always find a \( \beta = \beta(\varepsilon) \) satisfying \( \beta > \varepsilon > 0 \) such that \( \pi_{n2}(\varepsilon) \leq \pi_{n1}(\beta) \). If \( \|z_0(\theta)\| < \varepsilon \), (19) and (20) yield

\[
\pi_{n1}(\|z(t)\|) \leq V_n(t, z_i(\theta)) \leq V_n(0, z_0(\theta)) \leq \pi_{n2}(\|z_0(\theta)\|) \leq \pi_{n2}(\varepsilon) \leq \pi_{n1}(\beta), \quad \forall t \in [0, t_M],
\]

which means that \( \|z(t)\| \leq \beta, \forall t \in [-\tau, t_M] \). Suppose that \( t_M \) is finite, then \( \lim_{t \to t_M} \|z(t)\| = +\infty \), which contradicts \( \|z(t)\| \leq \beta, \forall t \in [-\tau, t_M] \). Hence, the state \( z(t) \) is well defined on \([-\tau, +\infty) \), so is \( x(t) \).

(19), (20) and Lyapunov-Krasovskii theorem in [13] result in the transformed closed-loop \( z \)-system being globally asymptotically stable at the equilibrium \( z = 0 \), which, together with (11), directly leads to the globally asymptotic stability of the closed-loop system at the origin \( x = 0 \).

5. Conclusions. By combining the method of adding a power integrator together and the sign function design approach, this paper further discusses the global stabilization for a class of high-order nonlinear systems with multiple time-varying delays.

Recently, [14-41] discuss different control problems for stochastic nonlinear systems. However, all these references only consider the systems with linear or higher-order growth condition. An important problem is how to give the design and analysis of controller for stochastic nonlinear systems with both lower-order and higher-order nonlinearities by adopting this method in this paper.

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REFERENCES

Lemmas 2.1, 2.3 and Assumption 3.2 implies that \( W \) and Assumption 3.2 implies that \( W \) is clear that \( W \) holds. Similarly, it can be shown that (21) still holds when \( x_k \leq \alpha_{k-1} \). (16) together with (11), Lemmas 2.1, 2.3 and Assumption 3.2 implies that

\[ W_{lk} \geq c \int_{\alpha_{k-1}}^{x_k} (s - \alpha_{k-1})^2 p_{l+k} \, ds \geq c (x_k - \alpha_{k-1})^2 p_{l+k} \]

Similarly, it can be shown that (21) still holds when \( x_k \leq \alpha_{k-1} \). (16) together with (11), Lemmas 2.1, 2.3 and Assumption 3.2 implies that
\[
W_{tk}(\cdot) \leq |z_k|^{2 - \frac{1}{2} \gamma_{tk1} - 1} x_k - \alpha_k - 1 | \leq c z_k^2 \leq c \|z_{kt}\|_C^2,
\]
\[
W_{tk}(\cdot) \leq |z_k|^{2 - \frac{1}{2} \gamma_{tk1} - 1} x_k - \alpha_k - 1 | \leq c z_k^{2 - \frac{1}{2} \gamma_{tk1} - 1} \leq c \|z_{kt}\|_C^{2 - \frac{1}{2} \gamma_{tk1} - 1},
\]
\[
W_{dk}(\cdot) \leq \left( \frac{b_k}{1 - \delta_k} + \int_{t-\varepsilon_k}^t + \frac{c_k}{1 - \delta_k} \right) \left( z_k^2 + \int_{t-\varepsilon_k}^t \right) ds
\]
\[
\leq \left( \frac{b_k \varepsilon_k}{1 - \delta_k} + \frac{c_k \varepsilon_k}{1 - \delta_k} \right) \left( \|z_{tk}\|_C^2 + \|z_{kt}\|_C^{2 - \frac{1}{2} \gamma_{tk1} - 1} \right)
\]
\[
\triangleq \pi_{k2}(\|z_{tk}\|_C),
\]
where \(\pi_{k2}(\cdot)\) is a class \(\mathcal{K}_\infty\) function obviously. It follows from (10), (13), (21)-(23) and \(W_{dk}(\cdot) \geq 0\) that
\[
V_k(\cdot) \leq \pi_{k-1,2}(\|z_{tk-1}\|_C + c \|z_{kt}\|_C^2 + c \|z_{kt}\|_C^{2 - \frac{1}{2} \gamma_{tk1} - 1} + \pi_{k2}(\|z_{tk}\|_C))
\]
\[
\leq \pi_{k-1,2}(\|z_{tk}\|_C + c \|z_{kt}\|_C^2 + c \|z_{kt}\|_C^{2 - \frac{1}{2} \gamma_{tk1} - 1} + \pi_{k2}(\|z_{tk}\|_C))
\]
\[
\triangleq \pi_{k2}(\|z_{tk}\|_C),
\]
\[
V_k(\cdot) \geq \pi_{k-1,1}(\|z_{tk-1}\|) + c(x_k - \alpha_k - 1)^2 \gamma_{tk1} - 1
\]
\[
= \pi_{k-1,1}(\|z_{tk-1}\|) + c \left( z_k + \alpha_k \gamma_{tk1} - 1(z_k) \right)^{\gamma_{tk1} - 1} - \alpha_k(z_k)^{\gamma_{tk1} - 1}
\]
\[
\triangleq \pi_{k2}(\|z_{tk}\|),
\]
Obviously, \(\pi_{k2}(\cdot)\) is a class \(\mathcal{K}_\infty\) function. Next we show that \(W_k(\z_k)\) is positive definite and radially unbounded. On one hand, it is easy to know that \(W_k(\z_k) \geq 0\) and \(W_k(\z_k) = 0\) if and only if \(\z_k = 0\. On the other hand, \(\|z_k\| \to +\infty\) means \(\|z_{tk-1}\| \to +\infty\) or \(|z_k| \to +\infty\). When \(\|\z_k\| \to +\infty\), we obtain from (25) that \(W_k(\z_k) \geq \pi_{k-1,1}(\|z_{tk-1}\|) \to +\infty\), \(\|z_k\| \to +\infty\). As for the case of \(|z_k| \to +\infty\) and \(\|z_{tk-1}\| \leq M\) for a finite positive constant \(M\), since \(\alpha_k(z_k-1)\) is continuous on \(z_k-1\) and \(\|z_{tk-1}\| \leq \|z_k\| \leq M\), we have
\[
W_k(\z_k) \geq c \left( (z_k + \alpha_k \gamma_{tk1} - 1(z_k-1)) \right)^{\gamma_{tk1} - 1} - \alpha_k(z_k-1) \right)^{\gamma_{tk1} - 1} \to +\infty, \|z_k\| \to +\infty
\]
Above analysis implies that \(W_k(\z_k)\) is radially unbounded. Hence, there is a class \(\mathcal{K}_\infty\) function \(\pi_{k1}(\cdot)\) such that \(\pi_{k1}(\|z_{tk}(\cdot)\|) \leq W_k(\z_k) \leq V_k(\cdot)\), which and (24) arrive at (14).

(ii) Using (1), (13), (14), (16) and Assumption 3.2, we have
\[ f_k + \sum_{j=1}^{k-1} \left( \frac{\partial W_{Lk}}{\partial x_j} + \frac{\partial W_{Hk}}{\partial x_j} \right) \cdot (x_{j+1}^{p_j} + f_j). \] (27)

Next, we estimate the last three terms on the right-hand side of (27).

First of all, it follows from (11), Lemmas 2.1, 2.2, 2.5 that

\[ \left( \frac{2-\frac{1}{p_{k-1}-p_k}}{z_{k-1}^2} + \text{sgn}(z_{k-1})\frac{2^{\sigma-p_k}p_{k-1}}{z_{k-1}^{p_{k-1}-p_k}} \right) (a_k^{p_{k-1}} - a_k^{p_k}) \]

\[ \leq \bar{a}_{k,k-1,1} \left( \frac{z_{k-1}^2}{z_k^2} + \frac{z_{k-1}^{p_{k-1}-p_k}}{z_k^{2\sigma}} \right) + \lambda_{k1}(\bar{a}_{k,k-1,1}) \left( \frac{z_k^2 + z_k^{2\sigma}}{z_{k-1}^{p_{k-1}-p_k}} \right), \] (28)

where \( \bar{a}_{k,k-1,1} > 0 \) and \( \lambda_{k1}(\bar{a}_{k,k-1,1}) > 0 \) are constants.

In view of (3), one can deduce that

\[ \frac{1}{p_1 \cdots p_k} \leq \frac{r_{j+1}p_j}{r_{j}p_1 \cdots p_k}, \quad \frac{r_{k+1}p_k}{r_{j}p_1 \cdots p_j} \leq \frac{r_{k+1}p_k}{r_{j}p_1 \cdots p_j}, \quad 1 \leq j \leq k-1 \] (29)

which and (11), (29), Lemmas 2.1, 2.2, 2.5 yield

\[ f_k(\cdot) \leq c(\bar{h}_{k-1}) \left( \sum_{j=1}^{k-1} \left( |z_j(t - \tau_{j+1}(t))| \frac{1}{p_{j+1}-p_j} + |z_j(t - \tau_{j+1}(t))| \frac{1}{r_{j}p_1 \cdots p_{j-1}} \right) \right) \]

\[ + \sum_{j=1}^{k} \left( |z_j(t)| \frac{1}{p_{j+1}-p_j} + |z_j(t)| \frac{r_{k+1}p_k}{r_{j}p_1 \cdots p_{j-1}} + |z_j(t - \tau_j(t))| \frac{1}{p_{j+1}-p_j} \right) \]

\[ + |z_j(t - \tau_j(t))| \frac{r_{k+1}p_k}{r_{j}p_1 \cdots p_{j-1}} \right) \] (30)

With the help of (30), Lemmas 2.2, 2.5, and \( r_{k+1}p_1 \cdots p_k \leq \sigma, k = 1, \ldots, n, \) we get

\[ \left( \frac{2-\frac{1}{p_{k+1}-p_k}}{z_{k}^2} + \text{sgn}(z_{k})\frac{2^{\sigma-p_k}p_{k-1}}{z_{k}^{p_{k-1}-p_{k}}} \right) f_k \]

\[ \leq c(\bar{h}_{k-1}) \left( |z_{k}^2| \frac{2\sigma-p_k}{p_{k+1}-p_k} + |z_{k}| \frac{2^{\sigma-p_k}p_{k-1}}{r_{k+1}p_k} \right) \left( \sum_{j=1}^{k} \left( |z_j(t)| \frac{1}{p_{j+1}-p_j} + |z_j(t)| \frac{r_{k+1}p_k}{r_{j}p_1 \cdots p_{j-1}} \right) \right) \]

\[ + |z_j(t - \tau_j(t))| \frac{1}{p_{j+1}-p_j} + |z_j(t - \tau_j(t))| \frac{r_{k+1}p_k}{r_{j}p_1 \cdots p_{j-1}} \right) + \sum_{j=1}^{k-1} \left( |z_j(t - \tau_{j+1}(t))| \frac{1}{p_{j+1}-p_j} \right) \]

\[ + |z_j(t - \tau_{j+1}(t))| \frac{r_{k+1}p_k}{r_{j}p_1 \cdots p_{j-1}} \right) \] (31)

where \( \bar{a}_{k11, \ldots, \bar{a}_{k,k-1,2}, \bar{b}_{k11, \ldots, \bar{b}_{kk1, \ldots, c_{k,k-1,1,1}, \bar{h}_{k-1}} \left( z_k^2 + z_k^{2\sigma} \right) \frac{2^{\sigma-p_k}p_{k-1}}{r_{k+1}p_k} \right) \) are some positive constants to be designed.
By (11), (16) and Lemmas 2.1, 2.3, we have
\[
\sum_{j=1}^{k-1} \left( \frac{\partial W_{Lk}}{\partial x_j} + \frac{\partial W_{Hk}}{\partial x_j} \right) (x_{j+1}^p + f_j) \\
= - \int_{\alpha_{k-1}}^{\alpha_k} \left( 2p_1 \cdots p_{k-1} - 1 \cdot \left( s^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{k-1}} \right) 1 - \frac{1}{p_{k-1}} \right) + \frac{2\sigma - r_{k+1}}{r_{k} p_1 \cdots p_{k-1}} \\
\cdot \left( s^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{k-1}} \right) \frac{1}{r_{k} p_1 \cdots p_{k-1}} \right) ds \cdot \sum_{j=1}^{k-1} \frac{\partial \alpha_{k-1}^{p_1 \cdots p_{k-1}}}{\partial x_j} (x_{j+1}^p + f_j) \\
\leq c \left( |z_k| + |z_k| \frac{2\sigma - \omega}{r_k p_1 \cdots p_{k-1}} \right) \cdot \sum_{j=1}^{k-1} |x_j| \left( |x_{j+1}^p| + |f_j| \right) \prod_{l=j}^{k-1} \left( 1 + |z_l|^{\frac{\omega}{r_l}} \right). \quad (32)
\]

Next, we will prove that for \( j = 1, \cdots, k - 1, k = 2, \cdots, n \) or \( j = 1, \cdots, n - 1, k = j + 1, \cdots, n \),
\[
|x_j|^{p_1 \cdots p_{j-1} - 1} (|x_{j+1}^p| + |f_j|) \prod_{l=j}^{k-1} \left( 1 + |z_l|^{\frac{\omega}{r_l}} \right)
\leq c(\tilde{h}_{j-1}) \left( \sum_{i=1}^{k} \left( |z_i| + |z_i| \frac{r_{k+1} \cdots p_{k-1} + \omega}{r_{k+1} \cdots p_{k-1}} \right) + \sum_{i=1}^{j} \left( |z_i(t - \tau_i(t))| + |z_i(t - \tau_i(t))| \frac{r_{k+1} \cdots p_{k-1} + \omega}{r_{k+1} \cdots p_{k-1}} \right)
\right)
\]
\[
\cdot \sum_{i=1}^{j} \left( |z_i(t - \tau_i(t))| + |z_i(t - \tau_i(t))| \frac{r_{k+1} \cdots p_{k-1} + \omega}{r_{k+1} \cdots p_{k-1}} \right)
\]
\[
\cdot \prod_{l=j}^{k-1} \left( 1 + |z_l|^{\frac{\omega}{r_l}} \right). \quad (33)
\]

The conclusion can be proved by an inductive argument on \( k \).

First of all, by (11), (30) and Lemmas 2.2, 2.5, then for \( j = 1, \cdots, n - 1, k = j + 1, \cdots, n \),
\[
|x_j|^{p_1 \cdots p_{j-1} - 1} (|x_{j+1}^p| + |f_j|) \prod_{l=j}^{k-1} \left( 1 + |z_l|^{\frac{\omega}{r_l}} \right)
\leq c(\tilde{h}_{j-1}) \left( |z_j|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} + |z_j - 1|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} + |z_j - 1|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} \right)
\]
\[
\cdot \left( \sum_{i=1}^{j} \left( |z_i|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} + |z_i|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} + |z_i(t - \tau_i(t))|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} + |z_i(t - \tau_i(t))|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} \right)
\right)
\]
\[
\cdot \prod_{l=j}^{k-1} \left( 1 + |z_l|^{\frac{\omega}{r_l}} \right). \quad (34)
\]

For any \( j \in \{1, 2, \cdots, n - 1\} \), when \( k = j + 1 \), (34) and Lemmas 2.2, 2.5 imply that
\[
\left( 1 + |z_j|^{\frac{\omega}{r_j}} \right) |x_j|^{p_1 \cdots p_{j-1} - 1} (|x_{j+1}^p| + |f_j|)
\leq c(\tilde{h}_{j-1}) \left( 1 + |z_j|^{\frac{\omega}{r_j}} \right) \left( |z_j|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} + |z_j - 1|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} + |z_j - 1|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} \right)
\]
\[
\cdot \left( \sum_{i=1}^{j} \left( |z_i|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} + |z_i|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} + |z_i(t - \tau_i(t))|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} + |z_i(t - \tau_i(t))|^{1 - \frac{1}{p_1 \cdots p_{j-1}}} \right)
\right)
\]
\[
\cdot \prod_{l=j}^{k-1} \left( 1 + |z_l|^{\frac{\omega}{r_l}} \right). \quad (35)
\]
which means that (33) holds for \( k = j + 1 \).

Suppose that (33) holds for \( k = m, j + 1 \leq m \leq n \). Then when \( k = m + 1 \), it can be shown from (34) and Lemmas 2.2, 2.5 that

\[
|x_j|^{p_1 \cdots p_{j-1}\cdot (1 + |x_j|^{\frac{1}{r-1}})} \prod_{l=j}^m \left( 1 + |z_l|^{\frac{1}{r}} \right)
\leq c(\bar{h}_{j-1}) (1 + |z_m|^{\frac{1}{r_m}}) \left( \sum_{i=1}^{j-1} \left( |z_i(t - \tau_i(t))| + |z_i(t - \tau_{i+1}(t))| \right)^{\frac{r_m p_1 \cdots p_{m-1} + \omega}{r_1 p_1 \cdots p_{j-1}}} \right)
\]

\[
+ \sum_{i=1}^{j} \left( |z_i(t - \tau(t))| + |z_i(t - \tau(t))| \right)^{\frac{r_m p_1 \cdots p_{m} + \omega}{r_1 p_1 \cdots p_{j}}} \right)
\]

\[
+ \sum_{i=1}^{j} \left( |z_i(t - \tau_{i+1}(t))| + |z_i(t - \tau_{i+1}(t))| \right)^{\frac{r_m p_1 \cdots p_{m-1} + \omega}{r_1 p_1 \cdots p_{j-1}}} \right),
\]

which implies that (33) holds for \( k = m + 1 \).

(32) and (33) together with Lemmas 2.2, 2.5 arrive at

\[
\frac{\partial W_{Lk}}{\partial x_j} + \frac{\partial W_{Hk}}{\partial x_j} (x_{j+1}^p + f_j)
\]

\[
\leq c(\bar{h}_{k-2}) \left( \sum_{j=1}^{k-1} \left( |z_k| + |z_k|^{\frac{2^{a_0}}{r_m p_1 \cdots p_{k-1}} - 1} \right) \left( \sum_{i=1}^{j} \left( |z_i(t - \tau_i(t))| + |z_i(t - \tau_i(t))| \right)^{\frac{r_k p_1 \cdots p_{k-1} + \omega}{r_1 p_1 \cdots p_{j}}} \right)
\]

\[
+ \sum_{i=1}^{j} \left( |z_i(t - \tau(t))| + |z_i(t - \tau(t))| \right)^{\frac{r_k p_1 \cdots p_{k} + \omega}{r_1 p_1 \cdots p_{j}}} \right)
\]

\[
+ \sum_{j=1}^{k-2} \bar{a}_{kj} \left( z_j^2 + z_j^{\frac{2^{a_0}}{r_1 p_1 \cdots p_{j-1} - 1}} \right) + \sum_{j=1}^{k} \bar{b}_{kj} \left( z_j^2 (t - \tau_j(t)) + z_j^{\frac{2^{a_0}}{r_1 p_1 \cdots p_{j-1} - 1}} (t - \tau_j(t)) \right)
\]

\[
+ \tilde{a}_{k,k-1,3} \left( z_{k-1}^2 + z_{k-1}^{\frac{2^{a_0}}{r_1 p_1 \cdots p_{k-2} - 2}} \right) + \sum_{j=1}^{k-1} \tilde{c}_{kj} 2 \left( z_j^2 (t - \tau_{j+1}(t)) + z_j^{\frac{2^{a_0}}{r_1 p_1 \cdots p_{j-1} - 1}} (t - \tau_{j+1}(t)) \right)
\]

\[
+ \lambda_{k3} (\bar{a}_{k12}, \cdot \cdot \cdot, \bar{a}_{k,k-1,3} + \tilde{b}_{k12}, \cdot \cdot \cdot, \tilde{b}_{kk2}, \bar{c}_{k12}, \cdot \cdot \cdot, \bar{c}_{k,k-1,2}, \bar{h}_{k-2}) \left( z_k^2 + z_k^{\frac{2^{a_0}}{r_1 p_1 \cdots p_{k-1} - 1}} \right)
\]

\[
(37)
\]
where $\bar{a}_{k12}, \cdots, \bar{a}_{k,k-2,2}, \bar{a}_{k,k-1,3}, b_{k12}, \cdots, b_{kk2}, \bar{c}_{k12}, \cdots, \bar{c}_{k,k-1,2} > 0$ and $\lambda_{k3}(\bar{a}_{k12}, \cdots, \bar{a}_{k,k-2,2}, \bar{a}_{k,k-1,3}, b_{k12}, \cdots, b_{kk2}, \bar{c}_{k12}, \cdots, \bar{c}_{k,k-1,2}, \bar{h}_{k-2}) > 0$ are some constants to be designed. Substituting (28)-(31) and (37) into (27), we get

$$\dot{V}_k \leq - \sum_{j=1}^{k-1} a_{kj} \left( \frac{1}{2} \left( z_j^2 + z_{j+1} \right) + \left( \frac{2}{z_k^{1-p_{k-1}}} + |z_k|^{2-p_{k-1}} \right) \alpha_k^{p_k} \right)$$

\[ + \left( \frac{1}{1-\delta_k} + \frac{c_k}{1-\delta_{k+1}} \right) \left( \frac{2}{z_k^{1-p_{k-1}}} + |z_k|^{2-p_{k-1}} \right) ^2 \right) \alpha_k^{p_k}, \] (38)

and constants satisfy

$$a_{kj} = \begin{cases} a_{k-1,j} - a_{kj1} - a_{kj2} > 0, & j = 1, \cdots, k-2, \\ a_{k-1,j} - a_{k,k-1,1} - a_{k,k-1,2} - a_{k,k-1,3} > 0, & j = k-1, \\ a_{k-1,j} - a_{k,k-1,1} - a_{k,k-1,2} - a_{k,k-1,3} > 0, & j = k, \\ \end{cases}$$

$$b_{kj} = \begin{cases} b_{k-1,j} - b_{kj1} - b_{kj2} > 0, & j = 1, \cdots, k-1, \\ b_{k-1,j} - b_{k,k-1} - b_{kk2} > 0, & j = k, \\ \end{cases}$$

\[ c_{kj} = \begin{cases} c_{k-1,j} - c_{kj1} - c_{kj2} > 0, & j = 1, \cdots, k-1, \\ c_{k} > 0, & j = k. \\ \end{cases} \]

Choosing the virtual controller

$$\alpha_k(z_k) = -h_k^{\frac{1}{p_{k-1}}} \left( z_k + \text{sgn}(z_k)|z_k|^{\frac{r_{k+1}p_k}{p_{k-1}}} \right)^{\frac{1}{p_{k-1}}},$$ (39)

where $h_k = 2^{p_{k-1}-1} \left( a_{kk} + \lambda_k(\bar{a}_{k11}, \cdots, \bar{c}_{k,k-1,2}, \bar{h}_{k-1}) + \frac{b_k}{1-\delta_k} + \frac{c_k}{1-\delta_{k+1}} \right)^{p_{k-1}-1}, a_{kk} > 0,$

and noticing from Lemmas 2.1, 2.2, 2.5 that

$$-h_k^{\frac{1}{p_{k-1}}} \left( z_k + \text{sgn}(z_k)|z_k|^{\frac{r_{k+1}p_k}{p_{k-1}}} \right)^{\frac{1}{p_{k-1}}} \left( z_k^{2-p_{k-1}} + |z_k|^{2-p_{k-1}} \right) \alpha_k^{p_k} \] \leq - \left( a_{kk} + \lambda_k(\bar{a}_{k11}, \cdots, \bar{c}_{k,k-1,2}, \bar{h}_{k-1}) + \frac{b_k}{1-\delta_k} + \frac{c_k}{1-\delta_{k+1}} \right) \left( z_k^{2} + z_k^{2-p_{k-1}} \right) \alpha_k^{p_k},$$ (40)

we obtain (15).