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FUZZY ADAPTIVE FUNCTION PROJECTIVE COMBINATION SYNCHRONIZATION OF A CLASS OF FRACTIONAL-ORDER CHAOTIC AND HYPERCHAOTIC SYSTEMS

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Abstract. *In this paper we study the fuzzy adaptive function projective combination synchronization of a class of fractional-order chaotic and hyperchaotic systems. Based on T-S fuzzy model, we reconstruct the considered three fractioned-order chaotic and hyperchaotic systems into T-S fuzzy systems. Based on the stability theory of fractional-order systems and tracking control, we propose a fuzzy adaptive function projective combination controller for the synchronization of two fractional-order drive and one fractional-order response systems. Finally, we obtain stability criteria for the error dynamic systems. We present numerical simulation results to illustrate the effectiveness of the developed synchronization control scheme for the three fractional-order chaotic and hyperchaotic systems in the presence of parameter uncertainty and external disturbances.*
Keywords: Function projective combination synchronization, Adaptive control, Fractional-order chaotic systems, Hyperchaotic systems, T-S fuzzy model

1. **Introduction.** Since Pecora and Carroll [1] introduced a method to synchronize two identical chaotic systems with different initial conditions, chaos synchronization has received considerable attention among scientists due to its importance in many applications such as secure communication, chemical systems, biological systems, and human heartbeat regulation. Since then, a variety of chaotic synchronization methods have been developed, which include adaptive control [2], nonlinear control [3], finite-time synchronization [4], sliding mode control [5], neural network-based synchronization [6], and recurrent hierarchical type-2 fuzzy neural networks-based synchronization [7]. For integer-order chaotic systems, projective synchronization was first reported by Mainieri and Rehacek [8] and then developed by many authors [9,10]. Later, a new synchronization method called modified projective synchronization was proposed [11], and the concept of function projective synchronization was introduced [12,13]. Du et al. [14] discussed a new type of synchronization, called modified function projective synchronization. Combining the adaptive theory, an adaptive function projective synchronization for unified chaotic systems is investigated in [15].

However, in the above and many other references, results are obtained based on traditional integer-order chaotic systems. Since fractional-order calculus provides more accurate models of systems than integer-order calculus does, recently, synchronization control schemes for fractional-order chaotic systems have been proposed, see active control [16], active sliding mode control [17], adaptive-impulsive control [18], fuzzy adaptive control [19], generalized projective synchronization [20] and the references therein. Furthermore, a projective synchronization based on sliding mode control was discussed in [21] for fractional order chaotic systems, and later, a modified projective synchronization for fractional order hyperchaoric systems was proposed in [22]. Function projective synchronization scheme was investigated in [23], and a modified function projective synchronization for a class of partial linear fractional-order chaotic system was studied in [24]. In recent years in control theory, adaptive control has become an area of several investigations. During the past two decades, significant progress on adaptive control for various system has been witnessed. Especially, for fractional-order systems, several results have been obtained [25,26] and an adaptive function projective synchronization for different fractional-order chaotic systems was reported in [27,28].

Actually, Takagi-Sugeno (T-S) fuzzy systems which can provide a method of using local linear systems combined with fuzzy IF-THEN rules to achieve nonlinearities has been widely studied by many authors in the past decades, in fact a great number of results on the stability analysis and controller design problem for integer-order T-S fuzzy systems have been reported and various approaches have been proposed in [29-31]. While for fractional-order T-S fuzzy systems, only a few papers studied the stability analysis and control of it. In [32,33], the stability and stabilization problems were addressed for fractional-order systems in terms of a T-S fuzzy model, sufficient conditions of asymptotic stability for the fractional-order uncertain T-S fuzzy model are given and a state feedback controller is designed to asymptotically stabilize the model. Based on the theory of fractional-order interval system, fractional-order T-S fuzzy model is applied to a wide class of fractional-order chaotic systems with uncertain parameters, and the state feedback controller is developed for fractional-order chaotic systems based on T-S fuzzy systems in [34]. However, the control and/or synchronization problem for fractional-order chaotic systems based on T-S fuzzy model still has room for the deeper research investigation. To the best of our knowledge, there are few results on the adaptive function projective combination synchronization for fractional-order chaotic systems based on T-S fuzzy model.

Inspired by the above discussions, based on the T-S fuzzy modelling theory, we reconstruct the chaotic system to T-S fuzzy model, and then develop a new fuzzy adaptive function projective combination synchronization scheme for three fractional-order chaotic systems and three fractional-order hyperchaotic systems with uncertain parameters and external disturbances. The new scheme has the advantages of function projection synchronization and combination synchronization. To illustrate the effectiveness of the given synchronization control method, modified fractional-order financial chaotic systems and modified fractional-order hyperchaotic systems are analyzed by using the proposed synchronization control scheme.

The organization of this paper is as follows. In Section 2, some fundamentals of fractional calculus are briefly reviewed. Section 3 presents the fuzzy adaptive function projective combination synchronization scheme for fractional-order chaotic systems, and a stability criterion for fractional-order chaotic systems is introduced. Then the proposed synchronization scheme is applied to synchronizing three fractional-order hyperchaotic systems in Section 4. Numerical simulations are carried out in Section 5, and the conclusions are finally drawn in Section 6.

2. **Preliminaties.**

Fractional-order integrations and differentiations are generalizations of integrations and differentiations of integer-order. There are three commonly used definitions of fractional-order derivatives, namely Grunwald-Letnikov (GL), Riemann-Liouville, and Caputo. Among these, the GL derivative provides the most direct numerical methods to solve fractional differential equations [35]. For this reason, we define our problem below in terms of GL fractional derivative. This derivative of a function $f(t)$ of order q, $q \geq 0$ is given by

$$ {}_{t_0}D_t^q f(t) = h^{-q} \sum_{j=0}^{k} \omega_j^{(q)} f(t - jh) \tag{1} $$

where t_0 and t are the initial and current times, h is the sampling time, $k = \frac{t-t_0}{h}$ is the integer part of $\frac{t-t_0}{h}$, and

$$ \begin{aligned} \omega_j^{(q)} &= (-1)^j \begin{pmatrix} q \\ j \end{pmatrix}, \quad j = 0, 1, 2, \dots \\ \begin{pmatrix} q \\ j \end{pmatrix} &= \frac{q(q-1)\cdots(q-j+1)}{j!} \end{aligned} \tag{2} $$

Based on the above GL definition for fractional-order derivative, we give the following synchronization scheme for fractional-order chaotic and hyperchaotic systems.

3. Fuzzy Adaptive Function Projective Combination Synchronization of Three Fractional-Order Chaotic Systems.

To present the synchronization scheme in detail, first we give the description of the systems and the definition of function projective combination synchronization.

3.1. Problem formulation.

Adaptive control has the characteristics of automatic adaptation to the uncertainty of the system. Thus, adaptive stability theory can ensure the stability of the system in the case of unknown system parameters. For the integer-order system, this theory has been continuously improved. However, its application in fractional-order nonlinear differential systems is not well established, and the subject is still being investigated. In this section, the scheme of adaptive function projective combination synchronization between two drive systems and one response system will be considered. These systems are described as follows.

Drive systems:

$$ \begin{aligned} D^\alpha x_1 &= x_3 + (x_2 - a)x_1 \\ D^\alpha x_2 &= 1 - bx_2 - x_1^2 \\ D^\alpha x_3 &= -x_1 - cx_3 \end{aligned} \tag{3} $$

and

$$ \begin{aligned} D^\alpha y_1 &= y_3 + (y_2 - a)y_1 \\ D^\alpha y_2 &= 1 - by_2 - y_1^2 \\ D^\alpha y_3 &= -y_1 - cy_3 \end{aligned} \tag{4} $$

Response system:

$$ \begin{aligned} D^\alpha z_1 &= z_3 + (z_2 - a)z_1 \\ D^\alpha z_2 &= 1 - bz_2 - z_1^2 \\ D^\alpha z_3 &= -z_1 - cz_3 \end{aligned} \tag{5} $$

where $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$ are the state variables of the drive systems (3) and (4), $z = (z_1, z_2, z_3)^T$ is the state variables of the response system (5), and a, b, c are constants. Systems (3), (4) and (5) exhibit chaotic behavior when we choose $a = 1$, $b = 0.1$, $c = 1$ and $\alpha = 0.95$, and for other parameters if chosen appropriately, systems (3), (4) and (5) will also exhibit chaotic behavior.

Now, we give a definition of function projective combination synchronization.

Definition 3.1. *For the three systems, if there exist three matrices $\Upsilon(t), \Psi(t), C(t) \in R^{3\times 3}$, $C(t) \neq 0$, the following inequality holds*

$$\lim_{t\to\infty} \|C(t)z(t) - \Upsilon(t)x(t) - \Psi(t)y(t)\| = 0 \tag{6}$$

where $\|\cdot\|$ represents the matrix norm, and then the two drive systems are said to achieve function projective combination synchronization with the response system.

Define the synchronization errors as

$$e(t) = C(t)z(t) - \Upsilon(t)x(t) - \Psi(t)y(t) \tag{7}$$

For the convenience of the ensuing discussions, we assume that $C(t) = I$, where $I \in R^{3\times 3}$ is an identify matrix. Hence, we obtain

$$e(t) = z(t) - \Upsilon(t)x(t) - \Psi(t)y(t) \tag{8}$$

where matrices $\Upsilon(t), \Psi(t), C(t) \in R^{3\times 3}$ are called the scaling function matrices.

Remark 3.1. *If $\Upsilon(t), \Psi(t), C(t) \in R^{3\times 3}$ are constants, then the function projective combination synchronization reduces to the combination synchronization.*

Remark 3.2. *If $\Upsilon(t) = 0$ and $\Psi(t)$ is constant or $\Psi(t) = 0$ and $\Upsilon(t)$ is constant, then the function projective combination synchronization problem reduces to the modified projective synchronization problem.*

Remark 3.3. *If $\Upsilon(t) = \Psi(t) = 0$, the function projective combination synchronization problem reduces to a chaos control problem.*

Next, to design the fuzzy adaptive function projective combination synchronization scheme, we reconstruct systems (3)-(5) into T-S fuzzy systems, which is suitable for the synchronization controller design.

3.2. **T-S fuzzy systems.** Consider the following fractional order chaotic system

$$D^\alpha x(t) = f(x) \tag{9}$$

We reconstruct it into a T-S fuzzy model based on the T-S fuzzy modelling theory as follows:
Rule i:
IF $s_j(t)$ is M_{ij} $(j = 1, 2, \ldots, p)$, then

$$D^\alpha x(t) = A_i x(t)$$

where $i = 1, 2, \ldots, n$, n is the number of IF-THEN rules, $s_1(t), \ldots, s_p(t)$ are the antecedent variables, each M_{ij} is a fuzzy set, and $A_i \in R^{3\times 3}$ are constant matrices, $x(t) = [x_1(t)\ x_2(t)\ x_3(t)]^T$.

Then, the overall fuzzy system is represented as

$$D^\alpha x(t) = \left\{ \sum_{i=1}^{n} h_i(t) A_i \right\} x(t) \tag{10}$$

where $h_i(t) = \frac{\omega_i(t)}{\sum_{i=1}^n \omega_i(t)}$, $\omega_i(t) = \Pi_{j=1}^p M_{ij}(s_j(t))$ and $M_{ij}(s_j(t))$ is the grade of membership of $s_j(t)$ in M_{ij}. Of course, $h_i(t) \geq 0$ and $\sum_{i=1}^n h_i(t) = 1$.

Note that system (10) is equivalent to system (9). Therefore, suppose $x_1 \in [M_1, M_2]$, for drive system (3), we can set rules as follows:
IF $x_1(t)$ is M_1, Then $D^\alpha x(t) = A_1 x(t) + K$
IF $x_1(t)$ is M_2, Then $D^\alpha x(t) = A_2 x(t) + K$

where

$$A_1 = \begin{bmatrix} -a & -M_1 & 1 \\ -M_1 & -b & 0 \\ -1 & 0 & -c \end{bmatrix},\ A_2 = \begin{bmatrix} -a & -M_2 & 1 \\ -M_2 & -b & 0 \\ -1 & 0 & -c \end{bmatrix},\ K = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

After reconstructing system (3) based on T-S fuzzy model, we obtain the overall fuzzy system as

$$D^{\alpha}x(t) = \sum_{i=1}^{2} h_i(t)\left[A_i x(t) + K\right] \tag{11}$$

where

$$h_1(t) = \frac{x_2 - M_1}{M_2 - M_1},\quad h_2(t) = \frac{M_2 - x_2}{M_2 - M_1}$$

For drive system (4) and response system (5), using the same process, we obtain their fuzzy systems as:

$$D^{\alpha}y(t) = \sum_{i=1}^{2} h_i(t)\left[A_i y(t) + K\right] \tag{12}$$

$$D^{\alpha}z(t) = \sum_{i=1}^{2} h_i(t)\left[A_i z(t) + K\right] + U = h(z(t)) + U \tag{13}$$

where $U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ are control inputs.

Remark 3.4. *We assume that the controller $U(x,y,z)$ consists of two parts: $U(x,y,z) = \tilde{u}(x,y,z) + \hat{u}(x,y,z)$, where $\tilde{u}(x,y,z)$ is a fuzzy controller and $\hat{u}(x,y,z)$ is an adaptive function projective combination synchronization controller.*

For response system (5), we define a compensation function $G(x,y)$ as:

$$G(x,y) = D^{\alpha}(Ax + By) - h(Ax + By) \tag{14}$$

based on Equation (14), we design

$$\tilde{u}(x,y,z) = G(x,y) + \theta(x,y,z) \tag{15}$$

where $\theta(x,y,z)$ is a vector-value function which will be designed later.

Using Equations (13) and (14), we obtain

$$\begin{aligned} D^{\alpha}e(t) &= \sum_{i=1}^{2} h_i(t)[A_i z + K] - \sum_{i=1}^{2} h_i(t)[A_i(Ax + By) + K] \\ &\quad + \theta(x,y,z) + \hat{u}(x,y,z) \\ &= \sum_{i=1}^{2} h_i(t)[A_i e(t)] + \theta(x,y,z) + \hat{u}(x,y,z) \\ &= \Omega(e(t)) + \hat{u}(x,y,z) \end{aligned} \tag{16}$$

where $\Omega(e(t)) = \sum_{i=1}^{2} h_i(t)[A_i e(t)] + \theta(x,y,z)$.

Based on Equation (16), we transform the problem of the synchronization between two drive systems with one response system into the following problem: find an adaptive control law $\hat{u}(x,y,z)$ and a vector-value function $\theta(x,y,z)$, so that system (16) asymptotically converges to zero.

3.3. **Synchronization controller design.** The adaptive function projective combination synchronization controller is designed as follows:

$$\hat{u}(x, y, z) = -ke(t) \tag{17}$$

where $k = diag(k_1, k_2, k_3)$ and the adaptive update law is given as

$$D^{\alpha} k_i(t) = \lambda \left| e_i(t) \right|, \ \lambda > 0, \ i = 1, 2, 3 \tag{18}$$

Lemma 3.1. [28]. *For fractional-order system (16), assume $\Omega(e(t))$ is the nonlinear vector-valued function satisfied local Lipschitz condition, then the following inequality holds*

$$\left\| \Omega(e(t)) \right\| = \left\| \Omega(e(t)) - \Omega(0) \right\| \le l \left\| e(t) \right\|, \ l > 0 \tag{19}$$

where $e(t) = 0$ is the equilibrium points of the system, that is, $\Omega(0) = 0$.

Theorem 3.1. *For fractional-order error dynamic system (16), if we adopt the adaptive controller (17), and the parametric update law (18), the following equation holds*

$$\theta(x, y, z) = \Phi(x, y, z) e(t) \tag{20}$$

where $\Phi(x, y, z)$ is a vector-valued function to be designed. Then, the synchronization error dynamic system (16) is asymptotically stable.

Proof: Since $\theta(x, y, z) = \Phi(x, y, z)e(t)$, system (16) can be rewritten as

$$\begin{aligned} D^{\alpha} e(t) &= \sum_{i=1}^{2} h_i(t)[A_i e(t)] + \Phi(x, y, z) e(t) + \hat{u}(x, y, z) \\ &= \Omega(e(t)) + \hat{u}(x, y, z) \end{aligned} \tag{21}$$

According to the update law (18), we get $k_i > 0$ $(i = 1, 2, 3)$, then the following inequality can be established

$$\sum_{i=1}^{3} D^{-\alpha} \left(k_i \left| e_i(t) \right| \right) \ge 0 \tag{22}$$

We construct a Lyapunov function as

$$V = \sum_{i=1}^{3} \left| e_i(t) \right| + \sum_{i=1}^{3} D^{-\alpha} (k_i \left| e_i(t) \right|) + \sum_{i=1}^{3} \left| (k_i - k^*) k^* \right| \tag{23}$$

where $k^* > \max\left(k_i, \frac{l}{\lambda}\right)$. Using the definition of fractional derivative, we get

$$\begin{aligned} D^{\alpha} V &= D^{\alpha} \sum_{i=1}^{3} \left| e_i(t) \right| + D^{\alpha} \sum_{i=1}^{3} D^{-\alpha} (k_i \left| e_i(t) \right|) + D^{\alpha} \sum_{i=1}^{3} \left| (k_i - k^*) k^* \right| \\ &= \operatorname{sgn}^T(e(t)) D^{\alpha} e(t) + \sum_{i=1}^{3} k_i \left| e_i(t) \right| + \sum_{i=1}^{3} \operatorname{sgn}[(k_i - k^*) k^*] k^* D^{\alpha} k_i \\ &= \operatorname{sgn}^T(e(t)) (\Omega(e(t)) - ke(t)) + \sum_{i=1}^{3} k_i \left| e_i(t) \right| - k^* \sum_{i=1}^{3} \lambda \left| e_i(t) \right| \end{aligned}$$

$$\begin{aligned}
&= \operatorname{sgn}^T(e(t))(\Omega(e(t))) - \sum_{i=1}^{3} k_i\,|e_i(t)| + \sum_{i=1}^{3} k_i\,|e_i(t)| - k^* \sum_{i=1}^{3} \lambda\,|e_i(t)| \\
&= \operatorname{sgn}^T(e(t))(\Omega(e(t))) - k^* \sum_{i=1}^{3} \lambda\,|e_i(t)| \\
&\le l\,\|e(t)\| - k^* \sum_{i=1}^{3} \lambda\,|e_i(t)| \\
&\le l \sum_{i=1}^{3} |e_i(t)| - k^* \sum_{i=1}^{3} \lambda\,|e_i(t)| \\
&= (l - k^*\lambda) \sum_{i=1}^{3} |e_i(t)|
\end{aligned} \tag{24}$$

Since $k^* > \max\left(k_i, \frac{l}{\lambda}\right)$, we obtain

$$D^\alpha V \le (l - k^*\lambda) \sum_{i=1}^{3} |e_i(t)| < 0 \tag{25}$$

Thus, the synchronization error dynamic system (16) is asymptotically stable. This completes the proof.

Next, we consider the extension of the proposed method to accomplish the fuzzy adaptive function projective combination synchronization for three fractional-order hyperchaotic systems.

4. **Fuzzy Adaptive Function Projective Combination Synchronization of Three Fractional-Order Hyperchaotic Systems.** The mathematical model of a three fractional-order hyperchaotic systems is described as follows.
Drive system:

$$\begin{aligned}
D^\alpha x_1 &= x_3 + (x_2 - a)x_1 + x_4 \\
D^\alpha x_2 &= 1 - bx_2 - x_1^2 \\
D^\alpha x_3 &= -x_1 - cx_3 \\
D^\alpha x_4 &= -0.05x_1x_3 - nx_4
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
D^\alpha y_1 &= y_3 + (y_2 - a)y_1 + y_4 \\
D^\alpha y_2 &= 1 - by_2 - y_1^2 \\
D^\alpha y_3 &= -y_1 - cy_3 \\
D^\alpha y_4 &= -0.05y_1y_3 - ry_4
\end{aligned} \tag{27}$$

Response system:

$$\begin{aligned}
D^\alpha z_1 &= z_3 + (z_2 - a)z_1 + z_4 \\
D^\alpha z_2 &= 1 - bz_2 - z_1^2 \\
D^\alpha z_3 &= -z_1 - cz_3 \\
D^\alpha z_4 &= -0.05z_1z_3 - rz_4
\end{aligned} \tag{28}$$

where $x = (x_1, x_2, x_3, x_4)^T$ and $y = (y_1, y_2, y_3, y_4)^T$ are the state variables of drive systems (26) and (27), $z = (z_1, z_2, z_3, z_4)^T$ is the state variables of response system (28), and a, b, c, r are constants. Systems (26)-(28) exhibit chaotic behaviors when $a = 1$, $b = 0.1$, $c = 1$, $r = -0.6$.

To design the fuzzy adaptive function projective combination synchronization scheme, we reconstruct systems (26)-(28) into T-S fuzzy systems. We obtain overall fuzzy system using methods similar to those in Section 3 based on T-S fuzzy model. The resulting equations are

$$D^\alpha x(t) = \sum_{i=1}^{2} h_i(t)\left[A_i x(t) + K\right] \tag{29}$$

$$D^\alpha y(t) = \sum_{i=1}^{2} h_i(t)\left[A_i y(t) + K\right] \tag{30}$$

$$D^\alpha z(t) = \sum_{i=1}^{2} h_i(t)\left[A_i z(t) + K\right] + U \tag{31}$$

where $U = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T$ are control input, $K = [\ 0 \quad 1 \quad 0 \quad 0\ ]^T$, and

$$A_1 = \begin{bmatrix} -a & -M_1 & 1 & 1 \\ -M_1 & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ 0 & 0 & -0.05 * M_1 & r \end{bmatrix}, \quad A_2 = \begin{bmatrix} -a & -M_1 & 1 & 1 \\ -M_1 & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ 0 & 0 & -0.05 * M_2 & r \end{bmatrix}$$

Based on systems (26)-(28), we get synchronization error dynamic system as follows:

$$\begin{aligned} D^\alpha e(t) &= \sum_{i=1}^{2} h_i(t)[A_i z + K] - \sum_{i=1}^{2} h_i(t)[A_i(Ax + By) + K] \\ &\quad + \theta(x, y, z) + \hat{u}(x, y, z) \\ &= \sum_{i=1}^{2} h_i(t)[A_i e(t)] + \theta(x, y, z) + \hat{u}(x, y, z) \\ &= \Omega(e(t)) + \hat{u}(x, y, z) \end{aligned} \tag{32}$$

where $\Omega(e(t)) = \sum_{i=1}^{2} h_i(t)[A_i e(t)] + \theta(x, y, z)$.

Using the same process as that in Section 3, the adaptive function projective combination synchronization controller is designed as follows:

$$\hat{u}(x, y, z) = -ke(t) \tag{33}$$

where $k = diag(k_1, k_2, k_3, k_4)$ and adaptive update law is given as

$$D^\alpha k_i(t) = \lambda \left|e_i(t)\right|, \ \lambda > 0, \ i = 1, 2, 3, 4 \tag{34}$$

Then for error dynamic system (32), we have the following theorem.

Theorem 4.1. *For fractional-order error dynamic system (32), if we use the adaptive controller (33), and the parametric update law (34), the following equation holds:*

$$\theta(x, y, z) = \Phi(x, y, z)e(t) \tag{35}$$

then, the synchronization error dynamic system (32) is asymptotically stable.

Proof: Using the same proof method in Theorem 3.1, we can derive Theorem 4.1 directly and easily. This completes the proof.

5. Simulation Results.

Example 5.1. *Fuzzy adaptive function projective combination synchronization of three fractional-order chaotic system.*

To verify the effectiveness of the proposed methods, we choose $\Upsilon(t)$, $\Psi(t)$ as follows:

$$\Upsilon(t) = diag\{\gamma_1(t),\ \gamma_2(t),\ \gamma_3(t)\},\ \Psi(t) = diag\{\eta_1(t),\ \eta_2(t),\ \eta_3(t)\}.$$

Case 1: *When $\Upsilon(t)$ and $\Psi(t)$ are constants, we select parameters as follows:*

$$\gamma_1(t) = \gamma_2(t) = \gamma_3(t) = 0.5,\ \eta_1(t) = \eta_2(t) = \eta_3(t) = 0.2$$

In our synchronization scheme, based on Equation (16), we choose $\theta(x, y, z)$ as

$$\theta(x, y, z) = \Phi(x, y, z)e(t) = \sum_{i=1}^{2} h_i B_i e(t)$$

where

$$B_1 = \begin{bmatrix} -a & M_1 & -1 \\ M_1 & -b & 0 \\ 1 & 0 & -c \end{bmatrix}, \quad B_2 = \begin{bmatrix} -a & M_2 & -1 \\ M_2 & -b & 0 \\ 1 & 0 & -c \end{bmatrix}.$$

Then we obtain $\|\Omega(e(t))\| \le l \cdot \|e\| = \max\{|-2a|, |-2b|, |-2c|\}\|e\|$.

For simulation purpose, we take initial value as follows:

$$\begin{array}{l} x_1(0) = -0.2,\ \ x_2(0) = 0.2,\ x_3(0) = -0.5, \\ y_1(0) = 0.5,\ \ y_2(0) = 0.5,\ y_3(0) = -0.5, \\ z_1(0) = 10,\ \ z_2(0) = 5,\ z_3(0) = -10,\ M_1 = -20,\ M_2 = 20 \end{array} \tag{36}$$

In practical applications, the system uncertainty and external disturbance are unavoidable. Therefore, we add uncertainty and external disturbance into drive system (11), (12) and response system (13). New drive systems and response system based on fuzzy model can be described as follows:

$$D^{\alpha_i}x(t) = \sum_{i=1}^{2} h_i(t)\left[A_i x(t) + K\right] + \Delta f(x) + d_x(t) \tag{37}$$

$$D^{\alpha_i}y(t) = \sum_{i=1}^{2} h_i(t)\left[A_i y(t) + K\right] + \Delta f(y) + d_y(t) \tag{38}$$

$$D^{\alpha_i}z(t) = \sum_{i=1}^{2} h_i(t)\left[A_i z(t) + K\right] + \Delta f(z) + d_z(t) + U \tag{39}$$

where we assume

$$\begin{array}{l} \Delta f(x) + d_x(t) = \left\{ \begin{array}{l} 0.1\sin(0.2t) + 1.5\sin(0.5x_2) \\ 0.1\sin(0.2t) + 0.5\sin(0.5x_3) \\ 0.1\cos(0.2t) + 0.1\sin(0.5x_1) \end{array} \right. \\ \Delta f(y) + d_y(t) = \left\{ \begin{array}{l} 0.3\sin(2t) + 0.1\sin(0.5y_2^2) \\ 0.5\sin(t) + 0.1\cos(2y_3) \\ 0.15\cos(0.2t) + 0.1\sin(y_1 y_2) \end{array} \right. \\ \Delta f(z) + d_z(t) = \left\{ \begin{array}{l} 0.2\sin(0.3t) + 0.2\sin(z_1 z_3) \\ 0.1\cos(0.2t) + 0.1\sin(0.5z_2) \\ 0.2\sin(0.5t) + 0.2\sin(z_3) \end{array} \right. \end{array} \tag{40}$$

Now, for the error system in (16), considering the above proposed uncertainty, disturbance and the initial values in (36), we obtain simulation results shown in Figures 1-3. Figure 1 shows the phase trajectories of Equation (16) without control input. Figure 2 displays the synchronization error $e(t) \to 0$ with $t \to \infty$. This implies that the synchronization error dynamic system between two drive systems and one response system is asymptotically stable. The control inputs $u_i(t)$, $i = 1, 2, 3$, are shown in Figure 3, which shows that the proposed method is successful in synchronizing the three systems in the presence of uncertainties and disturbances.

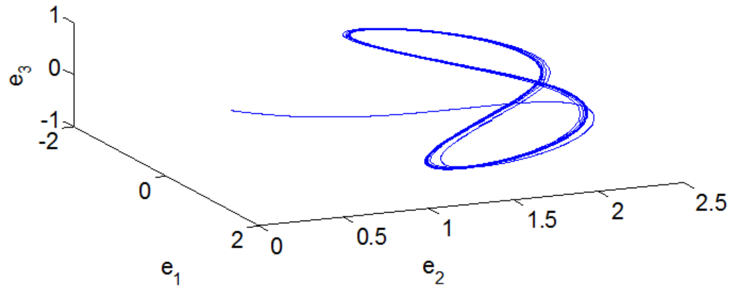

FIGURE 1. Phase trajectories of Equation (16)

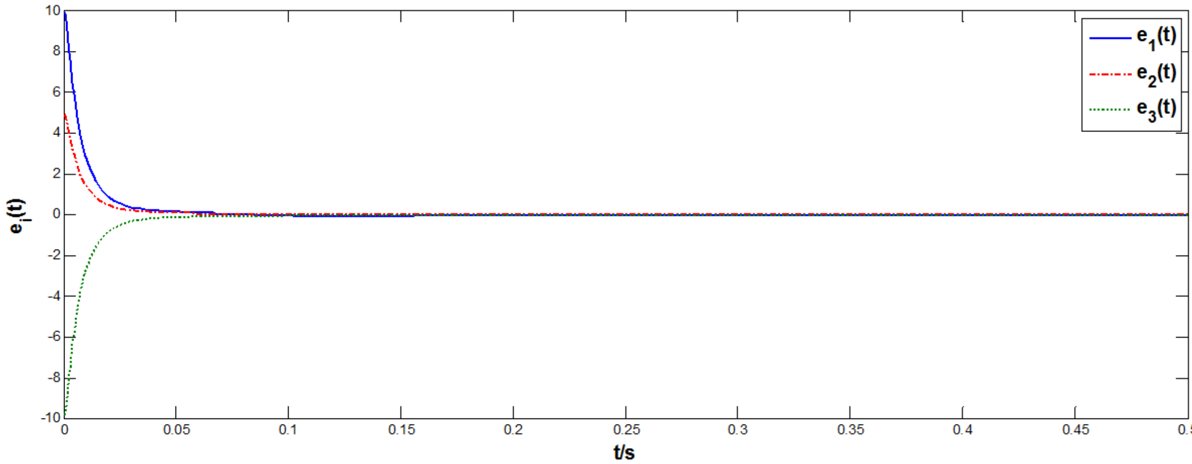

FIGURE 2. Synchronization errors $e_i(t)$

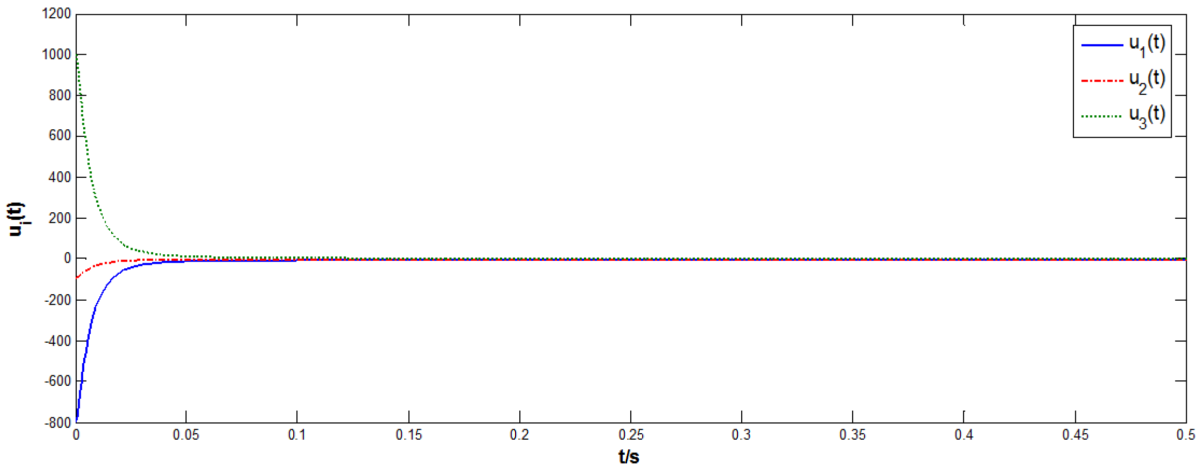

FIGURE 3. Control inputs $u_i(t)$

Case 2: *When $\Upsilon(t)$ and $\Psi(t)$ are time-varying, we select the parameters as:*

$$\begin{cases} \gamma_1(t) = 0.5 + 0.1\sin(0.3t) \\ \gamma_2(t) = 0.5 + 0.2\cos(0.5t) \\ \gamma_3(t) = 0.3 + 0.3\sin(t) \end{cases} \quad \begin{cases} \eta_1(t) = 0.3 + 0.2\sin(0.1t) \\ \eta_2(t) = 0.3 + 0.3\cos(0.5t) \\ \eta_3(t) = 0.3 + 0.3\sin(t) \end{cases}$$

Considering the proposed system uncertainty and disturbance in (40), and the initial conditions as in (36) for system (16), we get simulation results shown in Figure 4, which *show that the proposed method is valid in synchronizing the three systems in presence of the uncertainties and disturbances when the scaling function matrices are time-varying.*

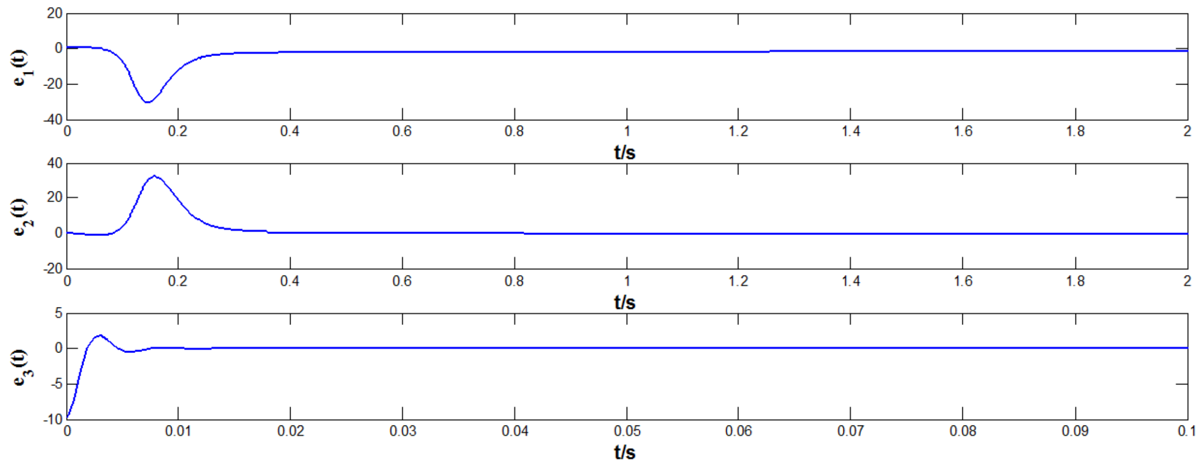

FIGURE 4. Synchronization errors $e_i(t)$

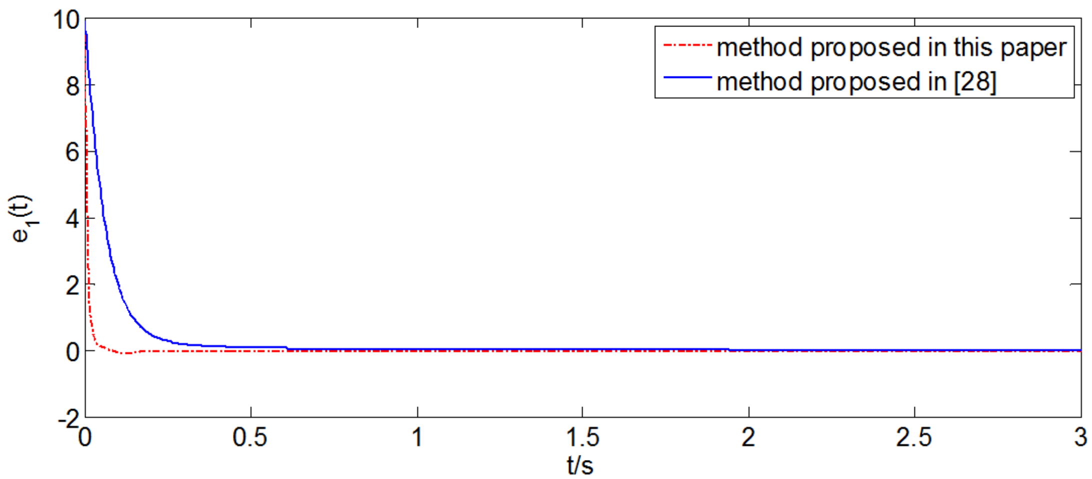

FIGURE 5. Synchronization error $e_1(t)$

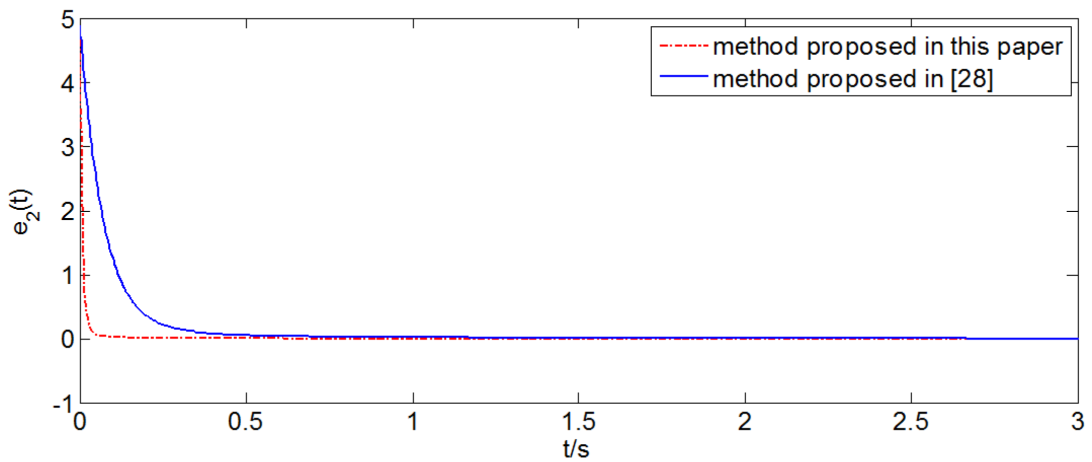

FIGURE 6. Synchronization error $e_2(t)$

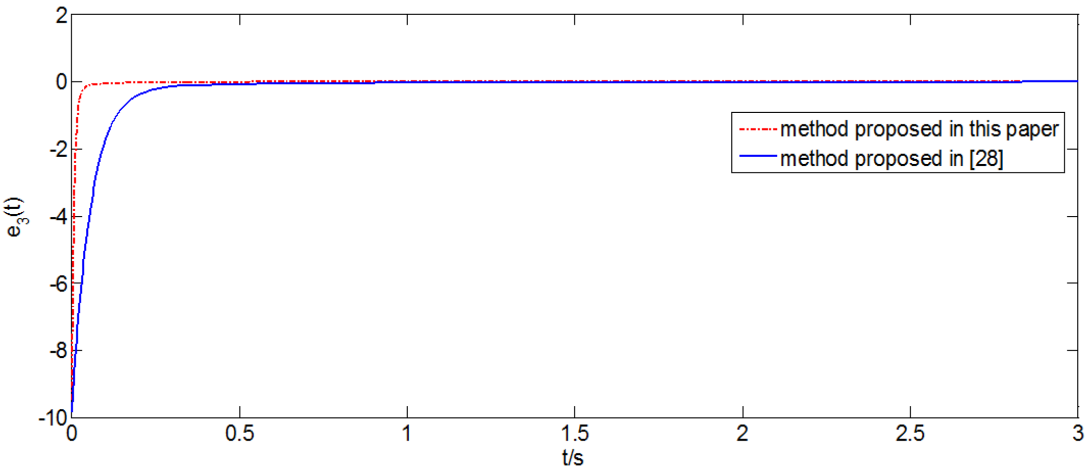

FIGURE 7. Synchronization error $e_3(t)$

To show the advantages of our proposed method, for the above example of Case 1 with the proposed uncertainty and disturbance in (40), we use the method proposed in [28] to design the synchronization controller, and the synchronization errors are shown in Figures 5-7. From the simulation results, we obtain that in comparison with the method proposed in [28], our proposed method has good control effect, and the errors which are under the control of our proposed method have better performance.

Example 5.2. *Fuzzy adaptive function projective combination synchronization of three fractional-order hyperchaotic systems.*

To verify the effectiveness of the proposed method for hyperchaotic systems, we choose $\Upsilon(t)$, $\Psi(t)$ as follows

$$\Upsilon(t) = diag\{\gamma_1(t),\ \gamma_2(t),\ \gamma_3(t),\ \gamma_4(t)\},$$
$$\Psi(t) = diag\{\eta_1(t),\ \eta_2(t),\ \eta_3(t),\ \eta_4(t)\}.$$

Case 1: *When $\Upsilon(t)$ and $\Psi(t)$ are constants, we select parameters as follows:*

$$\gamma_1(t) = \gamma_2(t) = \gamma_3(t) = \gamma_4(t) = 0.1$$
$$\eta_1(t) = \eta_2(t) = \eta_3(t) = \eta_4(t) = 0.2$$

Adopting the same synchronization scheme, based on Equation (32), we choose $\theta(x, y, z)$ as follows

$$\theta(x, y, z) = \Phi(x, y, z)e(t) = \sum_{i=1}^{2} h_i B_i e(t)$$

where

$$B_1 = \begin{bmatrix} -a & M_1 & 1 & 1 \\ -M_1 & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ 0 & 0 & 0.05 * M_1 & r \end{bmatrix},\ B_2 = \begin{bmatrix} -a & M_2 & 1 & 1 \\ -M_2 & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ 0 & 0 & 0.05 * M_2 & r \end{bmatrix}$$

Then we obtain $\|\Omega(e(t))\| \le l \cdot \|e\| = \max\{|-2a|, |-2b|, |-2c|, 0\} \|e\|$. Besides, considering the uncertainty and disturbance are unavoidable in practical application, following

Example 5.1, we assume

$$\Delta f(x)+d^x(t)=\begin{cases}0.1\sin(0.2t)+1.5\sin(0.2x_2)\\0.1\sin(0.2t)+0.5\sin(0.5x_3)\\0.1\cos(0.2t)+0.1\sin(0.2x_1)\\0.1\cos(0.2t)+0.1\sin(0.5x_4)\end{cases}$$
$$\Delta f(y)+d^y(t)=\begin{cases}0.3\sin(2t)+0.1\sin(0.5y_2^2)\\0.5\sin(t)+0.1\cos(2y_3)\\0.15\cos(0.2t)+0.1\sin(y_1y_2)\\0.2\cos(0.5t)+0.2\cos(y_1y_4)\end{cases} \tag{41}$$
$$\Delta f(z)+d^z(t)=\begin{cases}0.2\sin(0.3t)+0.2\sin(z_1z_3)\\0.1\cos(0.2t)+0.1\sin(0.5z_2)\\0.2\sin(0.5t)+0.2\sin(z_3)\\0.1\cos(t)+0.5\sin(z_2z_4)\end{cases}$$

and set the initial values as

$$\begin{array}{l}a=1,\ b=0.3,\ c=1,\ r=-0.6,\ \alpha_1=\alpha_2=\alpha_3=\alpha_4=0.95\\x_1(0)=0.2,\ x_2(0)=0.2,\ x_3(0)=0.2,\ x_4(0)=-0.2;\\y_1(0)=0.5,\ y_2(0)=0.5,\ y_3(0)=0.5,\ y_4(0)=-0.5;\\z_1(0)=10,\ z_2(0)=5,\ z_3(0)=5,\ z_4(0)=-10,\ M_1=-30,\ M_2=30\end{array} \tag{42}$$

The simulation results for this case are shown in Figure 8, which displays the synchronization error $e(t)\to 0$ with $t\to\infty$. This implies that the synchronization error dynamic system between two fractional-order hyperchaotic drive system and one fractional-order hyperchaotic response system is asymptotically stable. The control inputs $u_i(t)$, $i=1,2,3,4$, are shown in Figure 9.

Case 2: *When $\Upsilon(t)$ and $\Psi(t)$ are time-varying, we select parameters as*

$$\begin{cases}\gamma_1(t)=0.2+\sin(0.2t)\\\gamma_2(t)=0.2+\sin(0.5t)\\\gamma_3(t)=0.1+0.3\cos(0.2t)\\\gamma_4(t)=0.1+0.3\cos(0.5t)\end{cases}\quad\begin{cases}\eta_1(t)=0.3+0.2\sin(0.1t)\\\eta_2(t)=0.3+0.5\cos(0.5t)\\\eta_3(t)=0.3+0.1\sin(0.2t)\\\eta_3(t)=0.3+0.3\cos(0.5t)\end{cases}$$

Considering the uncertainty and disturbance in (41), the initial values in (42), we can get the simulation results shown in Figure 10, which demonstrates that the proposed method succeeds in achieving synchronization of the three fractional-order hperchaotic systems.

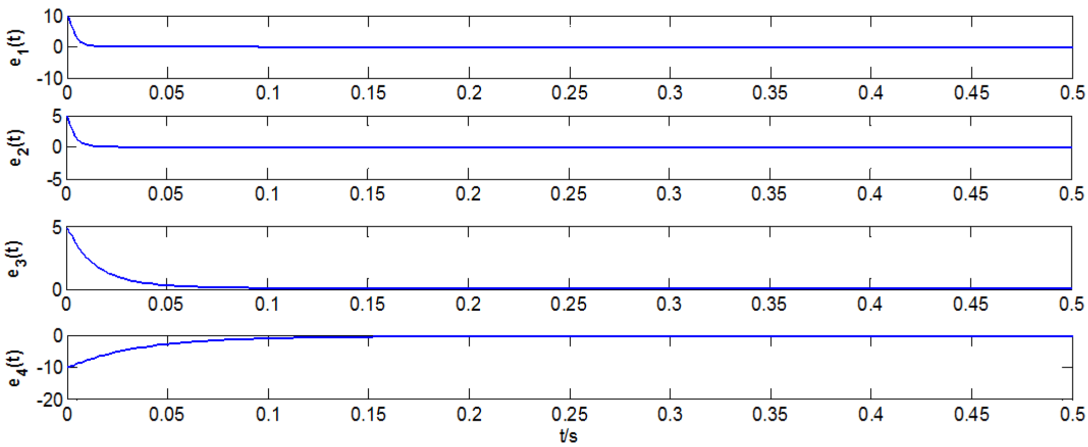

FIGURE 8. Synchronization errors $e_i(t)$

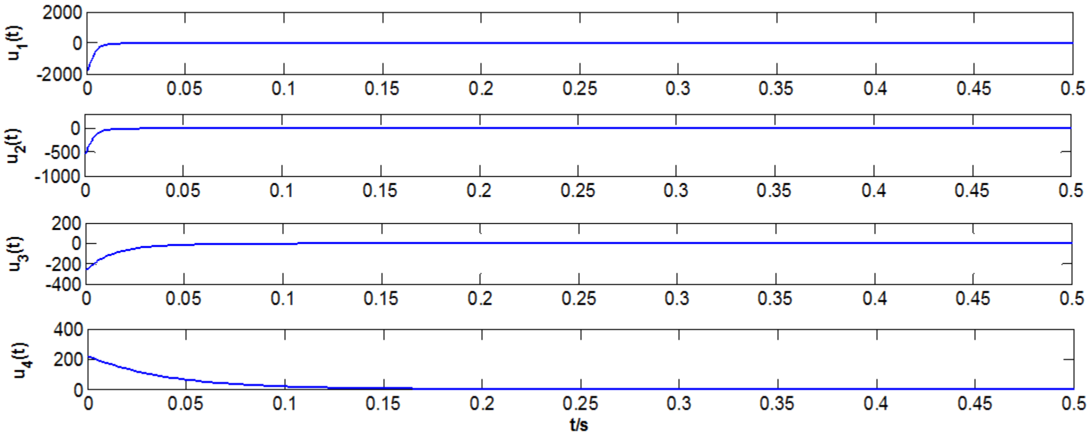

FIGURE 9. Control inputs $u_i(t)$

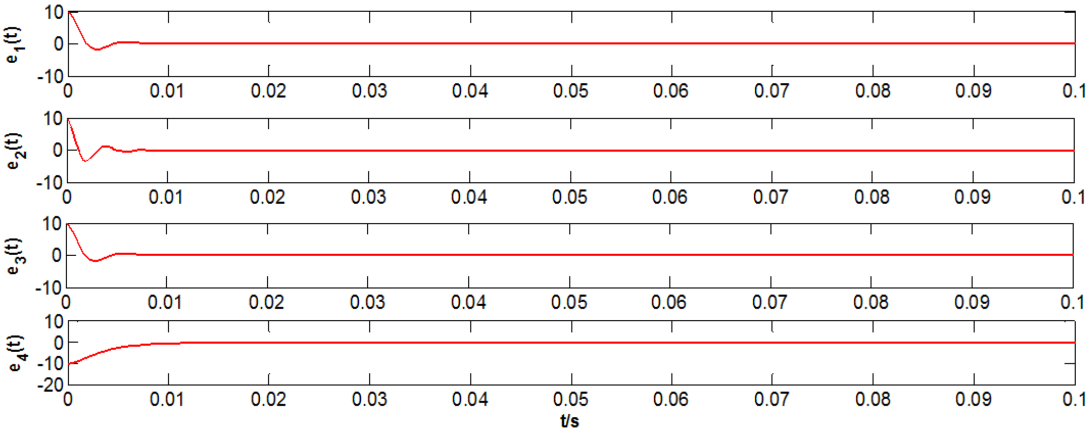

FIGURE 10. Synchronization errors $e_i(t)$

6. **Conclusions.** In this paper, we studied a fuzzy adaptive function projective combination synchronization control scheme for three fractional-order chaotic systems and three fractional-order hyperchaotic systems in the presence of parameter uncertainties and external disturbances. Based on T-S fuzzy model, the considered fractional-order systems are reconstructed into T-S fuzzy systems, and then the stability criteria for fuzzy adaptive function projective combination synchronization of fractional-order chaotic and hyperchaotic systems are proposed. Finally, two examples are given to show the effectiveness of the proposed synchronization control scheme.

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