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APPLICATION OF AN AUTONOMOUS DISTRIBUTED SYSTEM TO THE PRODUCTION PROCESS

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Abstract. *In this study, we use the phase oscillator model proposed by Kuramoto and apply it to the production process for the first time. In our previous study, the production process was itself described by the diffusion equation. In this study, a joint propagation model, which represents the connections between different processes, is described by the diffusion equation. We applied the joint model of an autonomous distributed system to the production process under the diffusion equation for the propagation of production elements and the same physical quantity of propagation. From the obtained data of the production flow process, we propose the phase oscillator model for the analysis of production process. We verify that the offset value identifies asynchronous or synchronous production processes.*

Keywords: Offset value, Autonomous distributed system, Throughput, Graph theory, Diffusion process

1. Introduction.

Several studies have addressed the problem of increasing the productivity of production processes used in the production industry [1, 2]. Moreover, in the field of production, various theories have been applied to improve and reform production processes and increase productivity.

In a previous study [3], we addressed the problem of reducing construction work and inventory in the steel industry. Specifically, we investigated the relationship between variations in the rate of construction and delivery rate. In this study, we perform analysis using the queuing model and apply log-normal distribution to modeling the system in the steel industry [3].

Moreover, several studies have reported approaches that lead to shorter lead times [4, 5]. From order products, lead time occurs on the work required preparation of the members for production.

Many aspects can potentially affect lead time. For example, from order products, the lead time from the start of development to the completion of a product is called the time-to-finish time, such as the work required preparation of the members for production equipments.

Moreover, several studies have focused on reducing customer lead times. In [6], the author addresses the problem of reducing the production lead time.

In [7], the authors propose a method that increases both production efficiency and production of a greater diversity of products for customer use. Their proposed approach results in shortened lead times and reduces the uncertainty in demand. Their method captures the stochastic demand of customers and produces solutions by solving a nonlinear stochastic programming problem.

In summary, several studies have considered uncertainty and proposed practical approaches to shorten the lead time. The demand is treated as a stochastic variable and applied mathematical programming. To our knowledge, previous studies have not treated lead time as a stochastic variable.

Because fluctuations in the supply chain and market demand and the changes in the production volume of suppliers are propagated to other suppliers, their effects are amplified. Therefore, because the amounts of stock are large, an increase or decrease of the suppliers' stock is modeled using differential equation. This differential equation is said as Billwhip model, which represents a stock congestion [8, 9].

The theory of constraints (TOC) describes the importance of avoiding bottlenecks in production processes [10]. When using production equipment, delays in one production step are propagated to the next. Hence, the use of production equipment may lead to delays. In this study, we apply a physical approach and regard each step as a continuous step. By applying this approach, we can mathematically analyze the delay of each step and obtain methods to address it. To the best of our knowledge, previous studies have not applied physical approaches to analyze delays.

In a previous study [11], we constructed a state in which the production density of each process corresponds to the physical propagation of heat [26]. Using this approach, we showed that a diffusion equation dominates the production process.

In other words, when minimizing the potential of the production field (stochastic field), the equation, which is defined by the production density function $S_i(x,t)$ and the boundary conditions, is described using the diffusion equation with advection to move in transportation speed ρ. The boundary conditions means a closed system in the production field. The adiabatic state in thermodynamics represents the same state [11].

To achieve the goal of a production system, we propose using a mathematical model that focuses on the selection process and adaptation mechanism of the production lead time. We model the throughput time of the production demand/production system in the production stage using a stochastic differential equation of log-normal type, which is derived from its dynamic behavior. Using this model and the risk-neutral integral, we define and compute the evaluation equation for the compatibility condition of the production lead time. Furthermore, we apply the synchronization process and show that the throughput of the production process is reduced [12, 13].

In a company, it is important to determine a rational throughput rate for continuation of production under an incomplete information state. In the previous research, aiming at rationally performing start date management in the manufacturing industry, a mathematical model of throughput is formulated based on data, and a mathematical structure of start date management is made clear to some extent [15].

According to this result, it is shown that Kalman filter theory having been used in a state estimation problem in the control theory conventionally can be applied under an incomplete information state. In addition, by applying a theory of ongoing assessment in Real Option, a determination condition of a throughput rate is made clear and is confirmed by numerical value calculation [15].

In this study, we apply the phase oscillator model proposed by Kuramoto [17]. To the best of our knowledge, the application of the phase oscillator model to the production process has not been previously proposed.

Kuramoto's model has been widely used for analyses of systems involving elements with autonomous dynamics such as electrical circuits, chemical reactions, heart muscle cells, and nerve cells [17, 18]. We can introduce the phase response function devised by

Winfree [18]. The phase is treated mathematically as the mapping of $\boldsymbol{\theta}(\mathbf{X})$, which is derived from the state vector of the oscillator $\mathbf{X} \in \mathbf{R}$ to the real number $\boldsymbol{\theta} \in [0, 2\pi]$. The phase $\boldsymbol{\theta}(t) = \boldsymbol{\theta}(\mathbf{X}(\mathbf{t}))$ increases from zero to 2π.

We propose an idea that is composed of the elements of product equipment, i.e., the elements are production units such as parts that form the whole product, which is ordered by the customer in the manufacturing industry. These production units are involved at each stage of the production process and correspond to the propagation of a state, such as heat in physics.

In this study, the joint propagation model, which represents the connections between different processes (i.e., interprocess connections), referred to as "joints", is described by the diffusion equation. To apply the joint model of an autonomous distributed system, it is necessary to satisfy the following two conditions:

1. Propagation between processes is represented by the diffusion equation (diffusion interaction) [11].
2. The elements propagated are described by the same physical quantity (in this case, represented by production units).

Here to analyze the production process, we introduce a potential function. The potential function is described by an arithmetic sum of potentials, which include both the current and joint stages of the production process. Therefore, we can define the gradient of the potential function, and using this gradient, we can expand the theoretical analysis to include revenues after the shipment of the final product equipment.

2. **Distribution System and Diffusion Equation of the Production Process.**

From Figure 1, we refer to the network capacity (i.e., a statically acceptable amount of production) in an interprocess network (a production field) as R. An interprocess network indicates a sequential flow from one process to the other after the completion of the current process. Here assuming that the production density function for the i-th equipment is $S_i(x,t)$, $S_i(x,t)$ is expressed by

$$[J(x,t)dt - J(x+dx,t)dt]R = [S_i(x,t+dt) - S_i(x,t)]Rdx \tag{2.1}$$

where J is the production flow [11].

Next, we define the production flow as the displacement of a production density function in the unit production direction. In other words, the production density function is proportional to the cost necessary for production, and thus, it can be considered as the production cost per unit production. Furthermore, because production leads to a return, the production density function can be considered as a return density function

$$\frac{\partial S_i(x,t)}{\partial t} = D\frac{\partial^2 S_i(x,t)}{\partial x^2} \tag{2.2}$$

where D is the diffusion coefficient, t is the time variable, and x is the spatial variable.

This equation is equivalent to the diffusion equation derived from the minimization condition of free energy in a production field, indicating that the connections between processes can be treated as a diffusive propagation of products (refer to Figure 1) [11].

A model of the production process, which is connected in one dimension, is described as follows. The process of production is indicated by the movement of production units from one process (node) to another. This production flow is equivalent to transmission rate, which is defined as the rate of data flow between connected nodes in communication engineering. Accordingly, we formulate the production model in a manner similar to heat propagation in physics. Thus, the production process is modeled mathematically using a continuous diffusion type of partial differential equation consisting of time and spatial variables [11].

Setting the network capacity (the available static production volume) to R in an inter-process network (production field, equivalent to a stochastic field), we obtain the following:

$$[J(x)dt - J(X + dx)dt]R = [S(t + dt) - S(t)]Rdx \tag{2.3}$$

where J is the production flow and S is the production density.

In the present model, the production flow indicates the displacement of production processes in the direction related to the production density. In other words, the production cost per production is as follows:

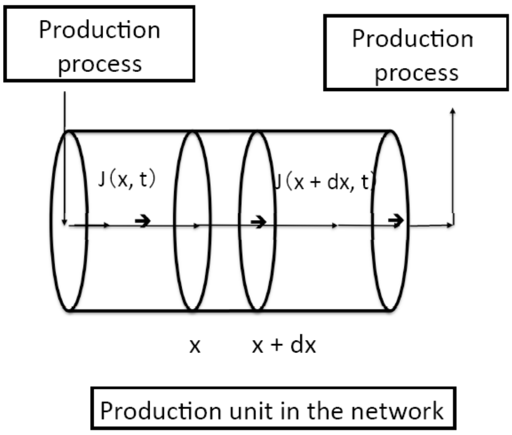

Figure 1. Network inter-process division of worker

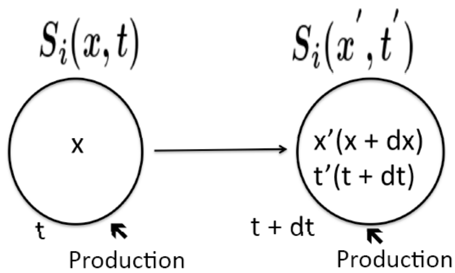

Figure 2. Unit of production by changing the excitation force

Definition 2.1. *Production cost per unit production*

$$J = -D\frac{\partial S}{\partial x} \tag{2.4}$$

where D is a diffusion coefficient.

From Equation (2.3), we obtain

$$-\frac{\partial J}{\partial x} = \frac{\partial S}{\partial t} \tag{2.5}$$

From Equations (2.4) and (2.5), we obtain

$$\frac{\partial S}{\partial t} = D\frac{\partial^2 S}{\partial x^2} \tag{2.6}$$

where $t \in [0, T]$, $x \in [0, L] \equiv \Omega$, $S(0, x) = S_0(x)$, $B_x S(t, x)|_{x=\partial\Omega}$.

This equation is equivalent to the diffusion equation derived from the minimization condition of free energy in a production field [11]. The connections between processes can be treated as a diffusive propagation of products (refer to Figure 1).

As shown in Figure 2, X represents the production elements that constitute a unit production and varies $X \to X^{'}$ at $[t + dt]$. In other words, the unit production varies by exciting the external force and is the basis for revenue generation (an increase of potential energy). Therefore, in the transition $S_i(t, x) \to S_i(t, x^{'})$, the production cost, which is the cumulated external force, increases. The connections between production processes are referred to as "joints".

In the general idea of production flow, we define the joint propagation model at multiple stages in the production process and the potential energy in the production field.

Thereafter, we can construct a control system, which increases the process throughput, by calculating the gradient function in the autonomous distributed system. The gradient function is described in the next section.

3. **Propagation Model of "Joints" and the Potential Function.** To analyze the unit production process, we mathematically model both intraprocess and interprocess networks. We apply the diffusion equation using the graph theory. We can design the control system for production because each modeled production unit corresponds to the diffusion reaction system in physics. The unit production process indicates one of the production processes for the production equipment.

In Figure 3, the set V of the vertex (node) on the finite graph $G = (V, E)$ is divided into the set $\tilde{V}$ of internal points and set ∂V of boundary points. A set of edges (Link) is divided into the sets of internal edges $\tilde{E}$ and boundary edges ∂E. The function $\{f(e) : V \to \mathscr{R}\}$ assigns a process amount u_1, u_2. $C(V) := \{f | f : V \to \mathscr{R}\}$ is the real function on X, where $f(e)$ is the rate of return and $C(V)$ is the production cost function.

Figure 3 shows the unit production model of joints between the processes, and each process f(u_1) and f(u_2) is coupled spatially, as represented in the figure. In other words, according to the process network that divides several processes, each process is coupled by a diffusion interaction, i.e., in this case, using the same physical quantity (for instance, production unit flow), the production process propagates as follows:

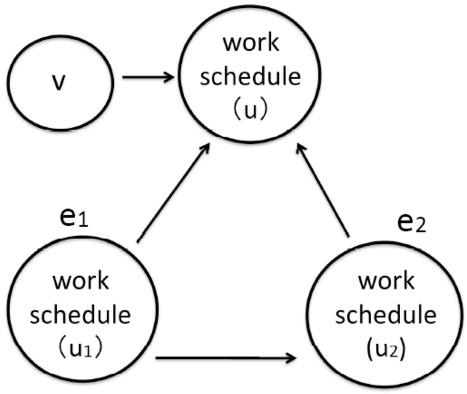

FIGURE 3. Single-stage production of multi-stage model

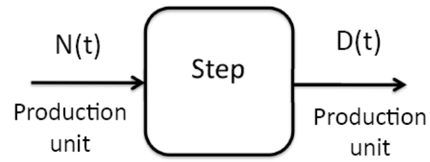

FIGURE 4. Single-stage model

Definition 3.1. *Production function in Figure 3.*

$$S \in C(X), \ \ ^{\forall}S \in L^2 \tag{3.1}$$

The production function is a functional of cumulated production cost function.

The joints between processes are a finite graph coupled by the diffusion propagation of physically equivalent quantities.

V is the set of the process of vertex r and represents the set of each process. $C(V)$ is the real function on V, and $C(E)$ is the space of the real function on edge E, i.e., the edge E is the set of edges (set of diffusion interactions).

Definition 3.2. *The gradient df of f defines $dS \in C(E)$ from $S \in C(V)$.*

$$df(e) \equiv f(t(e)) - f(o(e)) \tag{3.2}$$

where $o(e)$ (the originating node) is the vertex that is the starting point of link e, and $t(e)$ (the terminating node) is the vertex that is the destination point of link e. $d : C(V) \leftarrow C(E)$ is the exterior differential form, and df is the gradient of f.

We represent the total process related to the unit production of the equipment in Figure 6, i.e., the process of each vertex can monitor only the state of neighboring subprocesses.

Here $C \in C(V)$, $C(E)$ are defined in a Hilbert space, and they satisfy the following condition. The inner product $\{S_1, S_2, \cdots, S_n\} \in C(V)$ is as follows:

$$(S_{n-1}, S_n) = \sum_{k=1}^{n-1} S_k S_{k+1}$$

The norm is as follows:

$$||S_{n-1}|| = \sqrt{(S_{n-1} S_n)}$$

At this time, S_n satisfies the following condition:

$$L^2(V) = \{||S_n|| < \infty, n = 1, 2, \cdots, n\}$$

where we assume that $S \in C(V)$ is the time evaluation value currently.

The processes are connected by cascade coupling from process (1) through (n) and each has independent throughput in Figure 7. Here the dynamic model of throughput for any process is used by the oscillation model to work autonomously [21].

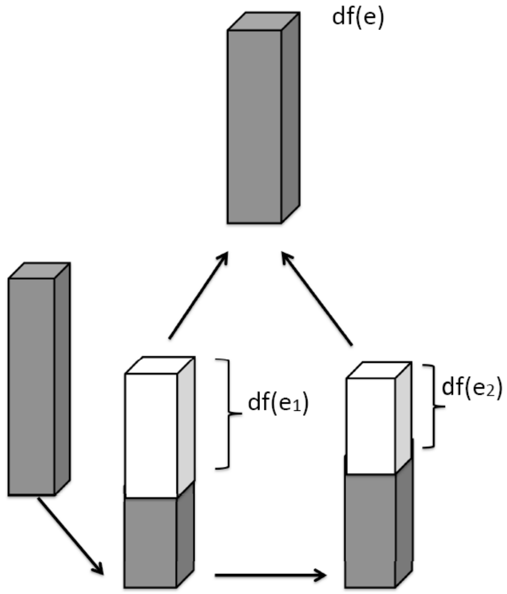

Figure 5. Joint of a finite graph between processes

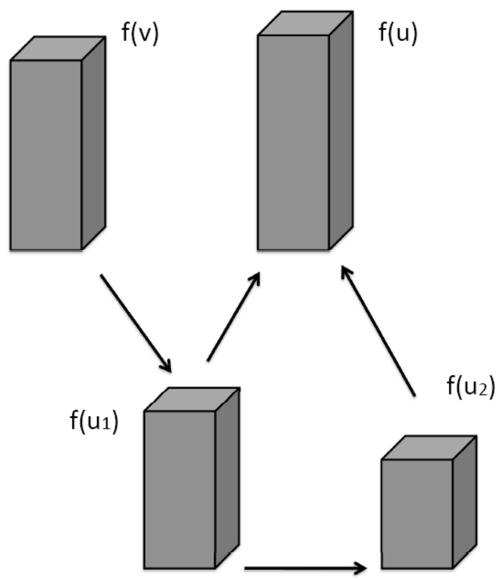

Figure 6. Interaction between process models

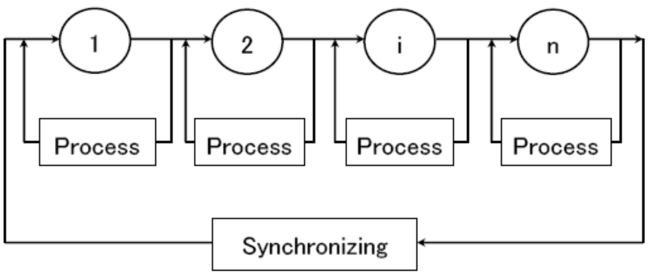

Figure 7. Connection of production system cascaded by N processes

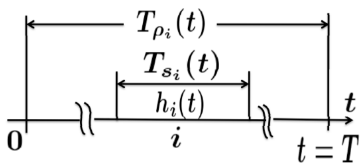

Figure 8. Model of the process cycle period and duration

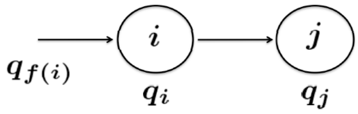

FIGURE 9. Transition of production stage

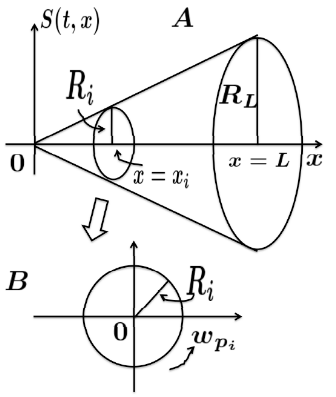

FIGURE 10. Production density function and progress of production process

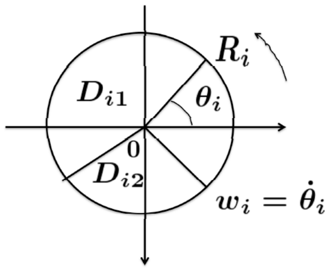

FIGURE 11. Relationship between angular frequency and angular

Definition 3.3. *The progress of time per unit time*

$$T_i(t) = \frac{1}{T_{\rho_i(t)}} \tag{3.3}$$

where $T_{\rho_i(t)}$ is the total read time (process cycle number).

Figure 10(A) indicates the conceptual diagram related to the dynamic production function. R_i $(i = 0, 1, \cdots, L)$ defines the size of the production factor on x_i $(i = 0, 1, \cdots, L)$.

In Figure 10(B), ω_i $(i = 0, 1, \cdots, L)$ is the progress rate of process according to each production element and is indicative of the throughput. Here to define the throughput according to the process size in each process, we use the unit R_i, which corresponds to the progress rate of the process according to the definition of angular velocity in electrical engineering. When the process makes the progress R_i, the process is considered as having made one unit of progress, and the process is referred as a "unit process".

We introduce the phase concept proposed by Winfree [18]. When progressing by one unit, we define the throughput $\omega_i(t)$ after passing time $T_i(t)$ as follows:

Definition 3.4. *Angular frequency* ω_i

$$\omega_i = \frac{1}{T_i(t)} \times (2\pi) = (2\pi) \times f_{\rho_i}(t) \tag{3.4}$$

Moreover, the cycle length and angular frequency are as follows:

$$T_R = \frac{2\pi}{\omega_i}, \quad \theta_i \in [0, 2\pi] \tag{3.5}$$

In Figure 11, D_{i1} and D_{i2}, which are the angles between each region, are derived as follows:

$$|D_{i1}| : |D_{i2}| = \sigma_i : (1 - \sigma_i) \tag{3.6}$$

where σ_i is the free time between each process and corresponds to the split time to be used in a traffic signal control.

4. **Applying Offset in a Traffic Control System.** We represent the offset time between processes i and j as well as the traffic signal control as follows. From Figure 10,

$$\frac{d\theta_i}{dt} = \omega_i \tag{4.1}$$

From Equation (4.1),

$$\frac{d}{dt}(\theta_i - \theta_j) = \omega_i - \omega_j \tag{4.2}$$

If $\omega_i = \omega_j$, $d/dt(\theta_i - \theta_j) = 0$. Here the relationship between the offset T_{ij} and phase difference $(\theta_i - \theta_j)$ is

$$T_{ij} = \frac{\theta_i - \theta_j}{\omega_i} \tag{4.3}$$

Definition 4.1. *Offset* T_{ij}

$$T_{ij} = \frac{\phi(i,j)}{\omega_d} \tag{4.4}$$

where $\phi(i,j)$ is the phase difference and ω_d is the target throughput (read time).

For these considerations, Yuasa and Ito have proposed the structural theory. We have attempted to apply our proposed method to a production system in line with this theory [21, 22]. We briefly summarize the structural theory of an autonomous distributed system proposed by Yuasa and Ito.

The system model for the process i is

$$\frac{d\theta_i}{dt} = f_i(\theta_{i1}, \theta_{i2}, \theta_{i3}, \cdots, \theta_{im}) \tag{4.5}$$

where $\theta_{i1}, \theta_{i2}, \cdots, \theta_{im}$ is the phase of process coupling with θ_i.

The matrix $\mathbf{A}$ between the state of the phase difference $\varphi \equiv (\varphi_1, \varphi_2, \cdots, \varphi_N)$ and the state $\theta \equiv \{\theta_1, \theta_2, \cdots, \theta_n\}$ is assumed as follows:

Assumption 1.

$$\varphi = \mathbf{A}^{\mathbf{t}}\theta \tag{4.6}$$

where $^{\forall}\mathbf{t}$ indicates the matrix transposed, and $\mathbf{A}$ is referred to as the "incident matrix" in graph theory.

From Equations (4.5) and (4.6),

$$\frac{d\varphi}{dt} = \mathbf{A}^{\mathbf{t}}\mathbf{f} \tag{4.7}$$

We obtain the dynamic model of the state difference φ.

For this discussion, φ is autonomous, i.e., the necessary and sufficient condition under which $\mathbf{A}^{\mathbf{t}}\mathbf{f}$ satisfies only the function of φ are as follows [20, 21, 22].

Lemma 4.1. *Any* i, j

$$\sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial \theta_k} = \sum_{k=1}^{n} \frac{\partial \varphi_j}{\partial \theta_k} \tag{4.8}$$

To prove this lemma, θ is defined in n-dimensional space. However, φ in $(n-1)$-dimensional space is defined as follows:

$$\phi = \sum_{i=1}^{n} \theta_i \tag{4.9}$$

ϕ, which is orthogonal to space φ, does not become an element of $\mathbf{A^t f}$, i.e., $\mathbf{A^t f}$ does not become a function of ϕ.

Definition 4.2. *Variable of state difference between* i_k

$$\varphi_{i_k} = \theta_{ik} - \theta_i \tag{4.10}$$

At this time, the necessary and sufficient condition which φ satisfies the gradient system, are as follows:

$$\frac{d\varphi_i}{dt} = f_i(\varphi_i)$$
$$\varphi_i = \sum_{k=1}^{m_i} (\theta_{ik} - \theta_i) \tag{4.11}$$

The potential function to be formed in space φ is as follows:

$$V(\varphi) = \sum_{i=1}^{n} \int f_i(\varphi_i) d\varphi_i \tag{4.12}$$

The potential structure is derived from the sum of the local potential. Therefore, in the production of equipment, such as in a flow production system, each process (subsystem) has a unique potential structure and each process model becomes a nonlinear structure through interaction between processes. The total potential is the sum of the local potentials of each process, and the structure of the autonomous distributed system can only be unique when the total potential is the sum of the local potentials [21, 22].

We now describe the basic production model such as the production flow process shown in Figure 12. If this phase difference is measurable in Figure 11, the offset between processes can be defined as in Equation (4.4).

On the basis of the above gradient theory, the production process is constructed as follows.

Assumption 2.

$$\frac{d\theta_i}{dt} = -\frac{\delta W_i^p(\theta_i)}{\delta\theta_i} + \omega_d \tag{4.13}$$

where W_i^p is as follows:

$$W_i^p(\theta_i) = -\left[\beta\left(\frac{q_{f(i)} + q_j}{q_i}\right)\right] \cos\{\phi(i,j) - D(i,j)\} \tag{4.14}$$

where $q_{f(i)}$ indicates the production volume of the forward process and q_j indicates the production volume of next process after process i.

$$\phi(i,j) = \theta_i - \theta_j \tag{4.15}$$

Therefore, we can construct the gradient system as follows:

$$\frac{d\theta_i}{dt} = \omega_d - 2\left[\beta\left(\frac{q_{f(i)} + q_j}{q_i}\right)\right] \sin\{\phi(i,j) - D(i,j)\} \tag{4.16}$$

where, whenever $\phi(i,j) = D(i,j)$, W_i^p obtains its minimum value, and we obtain

$$\frac{d\theta_i}{dt} = \omega_d \tag{4.17}$$

Definition 4.3. $D(i,j)$

$$D(i,j) = \frac{q_j}{q_{f(i)} + q_i} \cdot \omega_d \cdot \frac{|L(i,j)|}{\rho} \tag{4.18}$$

where $q_{f(i)}$ represents the throughput per unit process of process $f(i)$, q_i represents the throughput per unit process of process i, q_j represents the throughput per unit process, and ρ represents the standard working time of each process.

Definition 4.4. *Average offset time value*

$$\text{Average } D(i,j) = \left\{ \frac{\max(L(i,j)) + \min(L(i,j))}{2} \right\} \tag{4.19}$$

$$D(i,j) = \begin{cases} 0 & (q_i = q_j = 0) \\ \frac{q_j}{q_{f(i)}+q_i} D(i,j) & (\text{Other}) \end{cases} \tag{4.20}$$

We consider the one-way propagation.

$$D(i,j) = D(j,i) \tag{4.21}$$

At this time,

$$\sigma_i = \frac{|D_1|}{|D_1| + |D_2|} \tag{4.22}$$

and

$$\phi(i,j) = \left[\left\{ \theta_i - \xi(i,(i,j)) \right\} - \left\{ \theta_j - \xi(i,(j,i)) \right\} \right] \tag{4.23}$$

where if we assume that the production flow and production volume have no sudden fluctuations, the throughput converges to the local minimum point of the potential.

In other words, the phase difference $\phi(i,j)$ converges to $D(i,j)$. Therefore, if we set the value of $D(i,j)$, we can estimate the degree of process synchronization.

5. **Production Flow Process.** Figure 12 depicts a manufacturing process that is termed as a production flow process. This manufacturing process is employed in the production of control equipment. In this example, the production flow process consists of six stages. In each step S1−S6 of the manufacturing process, materials are being produced.

The direction of the arrows represents the direction of the production flow. In this process, production materials are supplied through the inlet and the end-product is shipped from the outlet.

5.1. **Synchronous model.**

Definition 5.1. *The role of the synchronization model is to reduce the process throughput, i.e.,*

$$dS(t,x) = rS(t,x)dt + \sigma S(t,x)dW(t) \tag{5.1}$$

where $S(t,x)$ represents the production density function as a function of the synchronous status.

Synchronization minimizes the risk in the production process. To realize synchronization, we set the throughput of each stage to the same value. Because we set the working time for the workers in each work stage, there is no volatility in the working time between processes.

Here, $S(t,x)$ represents the production density function as a function of the synchronous status when the equipment is manufactured. t represents the manufacturing time. x

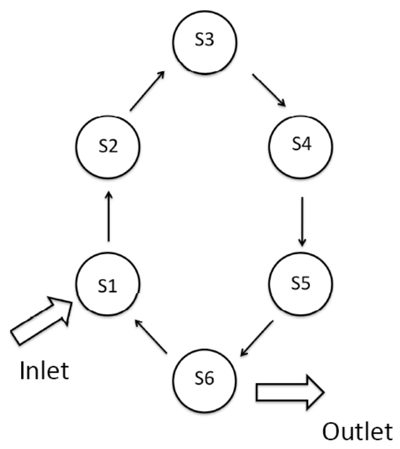

FIGURE 12. Production flow process

represents the production process term when products are manufactured continuously. σ represents the volatility at each stage, and $W(t)$ represents the Wiener process.

5.2. **Asynchronous model.**

Definition 5.2. *When we use the asynchronous model to represent a dynamical system, the throughput is not reduced.*

$$d\tilde{S}(t,x) = \bar{C}(t,x)\tilde{S}(t,x)dt + \tilde{\sigma}\tilde{S}(t,x)dW(t) \tag{5.2}$$

where $\bar{C}(t,x)$ represents the average working time of the total processes when the equipment is manufactured using an asynchronous process.

$$\bar{C}(t,x) = E[C(t,x)] = E\left[\sup_{t\in[0,T]} ||C(t,x)||^p\right] < \infty, \quad p > 2 \tag{5.3}$$

where $C(t,x)$ exists uniquely. Therefore, it is clear that Equation (5.3) is established. $C(t,x)$ is the arbitrage-free term under the equivalent martingale measure. Therefore, each stage of the production flow process can be represented by the Wiener process. Because, the working time in each stage fluctuates stochastically. Then, the relative production density $\tilde{S}(t,x)$ is expressed as follows [27]:

$$\tilde{S}(t,x) = \tilde{S}(0,x) - \int_0^t \tilde{S}(u,x)\sigma_u^* \mathbf{1}_{T-u} d\hat{W}(t) \tag{5.4}$$

That is, the volatility σ_u^* exists. Then $\tilde{S}(t,x)$ is

$$\tilde{S}(t,x) = \frac{S(t,x)}{S(t,0)} \exp\left\{\int_0^t r_u du - \int_0^t C(t,u)du\right\} \tag{5.5}$$

Definition 5.3. *According to the asynchronous model, the average time of working is as follows:*

$$r_u^c \equiv r_u - E[C(t,x)] \tag{5.6}$$

From Equation (5.6), we obtain

$$\tilde{S}(t,x) = \frac{S(t,x)}{S(t,0)} \exp\left\{\int_0^t r_u^c du\right\} \tag{5.7}$$

where $\tilde{S}(t,x)$ represents the production density of the asynchronous model. In the asynchronous model, the production workers at each stage do not complete the assigned work within the allocated time period. Therefore, at each stage, there is a volatility in the working time.

The solution of Equation (5.1) is

$$S(t,x) = S(0,x)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\} \tag{5.8}$$

where r indicates the total average working time when manufacturing using a synchronous process.

According to Equation (5.8), the production density $\tilde{S}(t,x)$ of the asynchronous model is as follows:

$$\tilde{S}(t,x) = \tilde{S}(0,x)\exp\left\{\left(r_u^c - \frac{1}{2}\sigma_c^2\right)t + \sigma_c \hat{W}(t)\right\} \tag{5.9}$$

where from Girsanov theorem, $\hat{W}(t)$ is

$$\hat{W}(t) = W(t) + \int_0^t \lambda(u)du \tag{5.10}$$

Therefore, according to Equations (5.3) and (5.4), the solution of $\tilde{S}(t,x)$ is as follows (Asynchronous model):

$$\tilde{S}(t,x) = \tilde{S}(t,0)\exp\left\{\left(r_c - \frac{1}{2}\sigma_c^2\right)t + \sigma_c W(t)\right\} \tag{5.11}$$

$$d\tilde{S}(t,x) = r_c\tilde{S}(t,x)dt + \sigma_c\tilde{S}(t,x)d\hat{W}(t) \tag{5.12}$$

$\tilde{S}(t,x)$ is a martingale with respect to F_t [27].

Therefore, $\tilde{S}(t,x)$ satisfies Equation (5.2) (Asynchronous model).

6. Results of Test−Run.

6.1. Result of Test−run1.

Test−run1 is asynchronous process. Therefore, the throughput at each step of Test−run1 is different, and the throughput of the entire stage becomes stochastic. Moreover, the stochastic throughput, which is a function of the current time and time remaining until the end of the stage, affects the performance of the entire system. In Tables 2 and 3, we present data that validates our findings presented above.

Therefore, the ratio of the measured throughput to the target throughput is considered as the drift term r_u^c in Equation (5.12). The fluidity of the system is affected by the throughput at each stage. In other words, because the manufacturing progress is affected by bottlenecks, the drift term r_u^c can be defined using the stochastic throughput (Equations (5.7)-(5.9)).

Here the drift term r_u^c is

$$r_u^c = \frac{4.4}{6}(0.73) \tag{6.1}$$

$$r_u^c = \frac{5.5}{6}(0.92) \tag{6.2}$$

The required theoretical throughput for six pieces of equipment/day is computed in Equation (6.1). However, the actual throughput corresponds to 4.4 pieces of equipment/day.

Furthermore, we can use the same approach to compute the volatility of the throughput at each stage. This average of volatility is given as follows:

$$\sigma_s \approx 0.29 \left(= \frac{1}{N} \sum_{i=1}^{N} \sigma^i(x) \right) \tag{6.3}$$

Therefore,

Definition 6.1. *The system throughput in this model (Production evaluation model)*

$$d\tilde{S}(t,x) = 0.73\tilde{S}(t,x)dt + 0.29\tilde{S}(t,x)d\hat{W}(t) \tag{6.4}$$

6.2. **Result of Test−run2.** Next, we consider the case of Test−run2.

Test−run2 is synchronous process. In this case, the process is set in such a way that each stage has the same throughput. Therefore, no risks are introduced as the process progresses. Hence, in principle, the throughput at each stage satisfies the condition. Moreover, because the manufacturing processes require synchronization, we can easily define the "synchronization throughput".

This system has essentially no risk. However, in Tables 4 and 5 we do not observe any values of volatility equal to zero. Therefore, in Equation (5.1), the term σ is equal to the average volatility.

Here, r, σ in Equation (5.1) are

$$r^1 = \frac{5.5}{6} = 0.92$$
$$r^2 = 1 - 0.06 = 0.94$$

r^1 and r^2 are not much different. The volatility is

$$\sigma = 0.06$$

Therefore, the throughput model of this system is defined as follows.

Definition 6.2.

$$dS(t,x) = 0.92S(t,x)dt + 0.06S(t,x)dW(t) \tag{6.5}$$

If the system approaches the synchronization, $\sigma \to 1$. If $\sigma \to$ small data ($\sigma = 0.01$), this system becomes stationary.

For the case of a fully synchronized system, see Figure 13. In Figure 14, the integrated

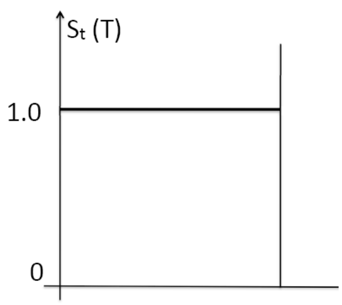

FIGURE 13. Perfect synchronization system

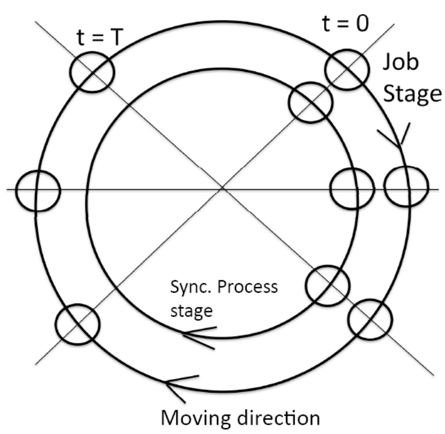

FIGURE 14. Perfect synchronization system

finite number of processing stages progress depending on the synchronization throughput of each stage (stationary system).

Specifically, the synchronous production system is the principle, and the processing stages progress in a cycle, i.e., we set the throughput at T_1, T_2, and T_3 in Figure 15, and synchronize the stages in a cycle.

If Equation (6.6) is satisfied,

$$\frac{1}{N}\sum_{i=1}^{N} r_i^c \leq \sup\ r_i^c:\ (i = 1, 2, \cdots, N) \tag{6.6}$$

A risk reduction system was constructed, where $N = kM$ $(k = 1, 2, \cdots, N)$ (k is a positive integer). Because we set the working time for the workers in each work stage, there is no volatility in the working time between processes. Next, we applied the throughput model and used the results of the test runs to perform numerical calculations. Our model shows that the throughput for each process at each stage is satisfied. If Equation (6.7) is satisfied, r_i^c $(i = 1, 2, \cdots, N)$ is a real number. This process is a type of bottleneck synchronization. The bottleneck synchronization means a recommendation from the famous "The theory of constraints (TOC)" [10].

1. If $r_i^c \neq r_j^c$, $i \neq j$, synchronization of every stage.
2.
$$\frac{1}{N}\sum_{i=1}^{c} r_i^c \leq \sup\ r_i^c,\ i = 1, 2, \cdots, N \tag{6.7}$$
if Equation (6.7) is satisfied, the process is a type of bottleneck synchronization.
3. $r_i^c = r_j^c$, $i = j < N$, the synchronization of some stages.

Here Figure 15 can be considered for item 3.

Definition 6.3. *Evaluation of the relative production density function $\tilde{S}_T(x)$ at $t = T$.*

$$d\tilde{S}(T, x) = r_t^c \tilde{S}(T, x)dt + \sigma_s^* \tilde{S}(T, x)d\hat{W}_t \tag{6.8}$$

In this case, the reduction of σ_s^* is a key point of building the system. Therefore, we named to "Synchronization with preprocess" method as to reduce this σ_s^*.

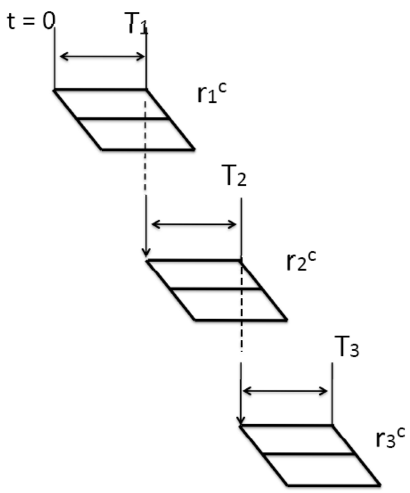

Figure 15. Cyclic synchronization

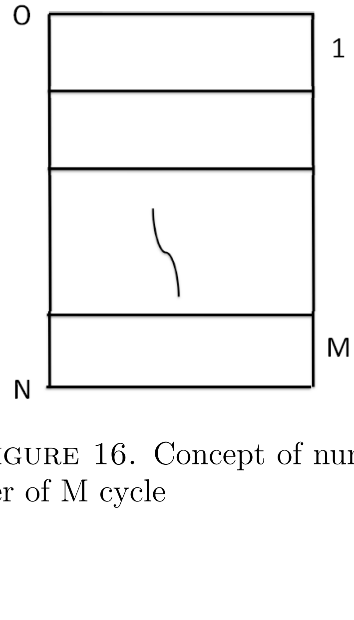

Figure 16. Concept of number of M cycle

7. **Analysis of the Test−Run Results.**

- (Test−run1): Because the throughput of each process (S1−S6) is asynchronous, the overall process throughput is asynchronous. In Table 2, we list the manufacturing time (min) of each process. In Table 3, we list the volatility in each process performed by the workers. Finally, Table 2 lists the target times. The theoretical throughput is obtained as $3 \times 199 + 2 \times 15 = 627$ (min). In addition, the total working time in stage S3 is 199 (min), which causes a bottleneck. In Figure 17, we plot the measurement data listed in Table 2, which represents the total working time of each worker (K1−K9). In Figure 18, we plot the data contained in Table 2, which represents the volatility of the working times.
- (Test−run2): Set to synchronously process the throughput. The target time listed in Table 4 is 500 (min), and the theoretical throughput (not including the synchronization idle time) is 400 (min). Table 5 presents the volatility of each working process (S1−S6) for each worker (K1−K9).

TABLE 1. Correspondence between the table labels and the Test−run number

	Table Number	Production process	Working time	volatility
Test−run1	Table 2	Asynchronous process	627 (min)	0.29
Test−run2	Table 4	Synchronous process	500 (min)	0.06

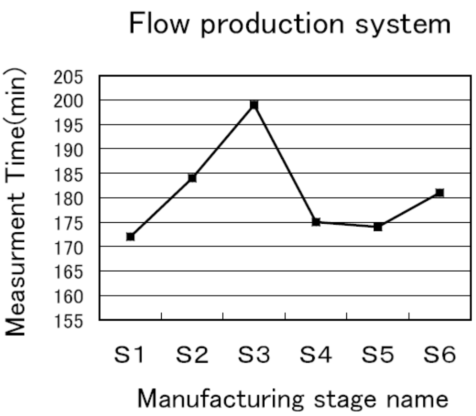

FIGURE 17. Total production time at each stage for each worker

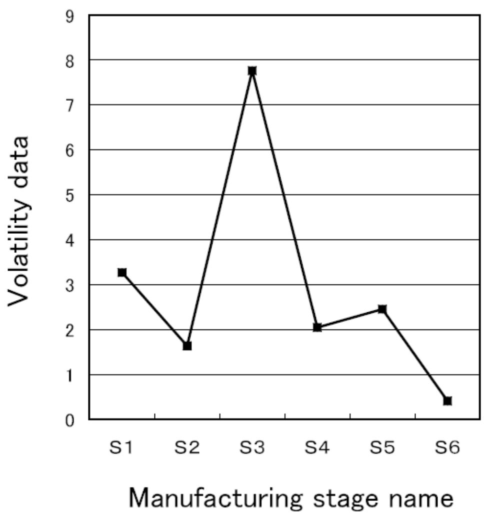

FIGURE 18. Volatility data at each stage for each worker

In Table 1, Test−run2 is a good method in throughput clearly than Test−run1, and also that the volatility in the work (Test−run2) is less than the volatility (Test−run1).

8. **Numerical Results of the Production Flow System.** In this section, we examine the volatility of throughput in asynchronous and synchronous processes using the offset value and average offset value based on Equations (4.18) and (4.19) [13].

8.1. **Offset value of Test−run1.** Table 6 indicates the subtotal working time for working processes S1−S2−S3 in Table 2. The calculation based on this data is as follows:

$$\omega_d = \frac{x_0}{20 \times 9} = \frac{145}{180} \simeq 0.8$$

where "20" is the maximum read-time, and "9" is the number of workers.

$$D(i,j) = \frac{0.73}{0.84+0.79} \times 0.8 \times \frac{199-184}{20} \simeq 0.27$$

where $\rho = 20$ and $AVE\ \Delta S = 199 - 184$.

Table 7 indicates the subtotal working time for working processes S3−S4−S5 in Table 2. The calculation based on this data is as follows:

$$D(i,j) = \frac{0.83}{\frac{0.84+0.79}{1.57}} \times 0.8 \times \frac{199-175}{20} \simeq 0.51$$

where $\rho = 20$ and $AVE\ \Delta S = 199 - 175$.

$$D(i,j) = \frac{0.83}{\frac{0.84+0.79}{1.57}} \times 0.8 \times \frac{175-174}{20} \simeq 0.02$$

The average offset value is as follows:

$$Average\ \ D(i,j) = (0.51+0.02)/2 \simeq \mathbf{0.27} \tag{8.1}$$

Table 2. Total production time at each stage for each worker (Asynchronous)

	WS	S1	S2	S3	S4	S5	S6
K1	15	20	20	25	20	20	20
K2	20	22	21	22	21	19	20
K3	10	20	26	25	22	22	26
K4	20	17	15	19	18	16	18
K5	15	15	20	18	16	15	15
K6	15	15	15	15	15	15	15
K7	15	20	20	30	20	21	20
K8	20	29	33	30	29	32	33
K9	15	14	14	15	14	14	14
Total	145	172	184	199	175	174	181

Table 3. Volatility of Table 2

K1	1.67	1.67	3.33	1.67	1.67	1.67
K2	2.33	2	2.33	2	1.33	1.67
K3	1.67	3.67	3.33	2.33	2.33	3.67
K4	0.67	0	1.33	1	0.33	1
K5	0	1.67	1	0.33	0	0
K6	0	0	0	0	0	0
K7	1.67	1.67	5	1.67	2	1.67
K8	4.67	6	5	4.67	5.67	6
K9	0.33	0.33	0	0.33	0.33	0.33

Table 4. Total production time at each stage for each worker (Synchronous)

	WS	S1	S2	S3	S4	S5	S6
K1	20	20	24	20	20	20	20
K2	20	20	20	20	20	22	20
K3	20	20	20	20	20	20	20
K4	20	25	25	20	20	20	20
K5	20	20	20	20	20	20	20
K6	20	20	20	20	20	20	20
K7	20	20	20	20	20	20	20
K8	20	27	27	22	23	20	20
K9	20	20	20	20	20	20	20
Total	180	192	196	182	183	182	180

Table 5. Volatility of Table 4

K1	0	1.33	0	0	0	0
K2	0	0	0	0	0.67	0
K3	0	0	0	0	0	0
K4	1.67	1.67	0	0	0	0
K5	0	0	0	0	0	0
K6	0	0	0	0	0	0
K7	0	0	0	0	0	0
K8	2.33	2.33	0.67	1	0	0
K9	0	0	0	0	0	0

TABLE 6. Example of Test−run1 (S1−S2−S3)-Asynchronous-

x_0	x_1(S1)	x_2(S2)	x_3(S3)
145	172	184	199
x_0/x_i	0.84	0.79	0.73

TABLE 7. Example of Test−run1 (S3−S4−S5)-Asynchronous-

x_0	x_1(S1)	x_2(S2)	x_3(S3)
145	199	175	174
x_0/x_i	0.73	0.84	0.83

TABLE 8. Example of Test−run2 (S1−S2−S3)-Synchronous-

x_0	x_1(S1)	x_2(S2)	x_3(S3)
180	192	196	182
x_0/x_i	0.94	0.92	0.99

TABLE 9. Example of Test−run2 (S3−S4−S5)-Synchronous-

x_0	x_3(S3)	x_4(S4)	x_5(S5)
180	182	183	182
x_0/x_i	0.99	0.99	0.99

8.2. **Offset value of Test−run2.** Table 8 indicates the subtotal working time for working processes S1−S2−S3 in Table 4. The calculation based on this data is as follows. We calculate the maximum and minimum values of the offset value between S1−S2.

The working time difference is

$$|L(i,j)| = |\text{S2} - \text{S1}| = |196 - 192| = 4$$

$$\omega_d = \frac{x_0}{20 \times 9} = \frac{180}{180} = 1.0$$

where "20" is the standard number of working hours and "9" is the number of workers.

Offset value is

$$Minimum\ \ D(i,j) = \frac{0.99}{0.92 + 0.94} \times 1.0 \times \frac{4}{20} \simeq 0.11$$

From the standard number of working hours (time), the working time differences at working stage "2" and "3" are as follows:

$$|L(i,j)| = |\text{S3} - \text{S2}| = |182 - 196| = 14$$

$$\omega_d = \frac{x_0}{20 \times 9} = \frac{180}{180} = 1.0$$

where "20" is the standard working times, and "9" is the number of workers.

The offset value is

$$D(i,j) = \frac{0.99}{0.92 + 0.94} \times 1.0 \times \frac{14}{20} \simeq 0.11$$

Table 9 indicates the subtotal working time for working processes S3−S4−S5 in Table 4. The calculation based on this data is as follows:

$$D(i,j) = \frac{0.99}{0.99 + 0.99} \times 0.99 \times \frac{|182 - 183|}{20} \simeq 0.03$$

where $\rho = 20$ and $AVE\Delta S = |182 - 183|$. Therefore, the average offset value is

$$Average\ \ D(i,j) = (0.11 + 0.03)/2 = \mathbf{0.07} \quad (8.2)$$

According to comparison between the Test−run1, 2 and the Offset Value, the Test−run1 corresponds to the Offset Value in Equation (8.2), and Test−run2 corresponds to the

Offset Value in Equation (8.1). These calculation results are consistent with the volatility in Table 1 [13].

Here the determination of ω_d is as follows:

$$\omega_d = \frac{x_0}{180}$$

where x_0 sets the target total read time. The important thing is a method of determining ω_d.

9. **Conclusions.** We applied a structure theory of autonomous distributed system. We introduced the offset of the traffic signals control for a phase difference between processes. We made clear that the offset value corresponds to the results of Test−run. Using a phase difference in the unit elements of the process, we utilized the phase difference for process throughput analysis.

A joint model of an autonomous distributed system was applied to a production process. As a prerequisite, each process was described by the diffusion equation and the propagation of production elements was described by the same physical quantity (production unit). We verified the joint model on the basis of the results of Test−run in the production flow process.

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