International Journal of Innovative Computing, Information and Control
Volume **13**, Number **1**, February **2017**
ICIC International ©2017 ISSN 1349-4198
pp. 67–74

A MATHEMATICAL MODEL FOR DIMENSIONING OF SQUARE ISOLATED FOOTINGS USING OPTIMIZATION TECHNIQUES: GENERAL CASE

Sandra López Chavarría, Arnulfo Luévanos Rojas and Manuel Medina Elizondo

Instituto de Investigaciones Multidisciplinaria
Facultad de Contaduría y Administración
Universidad Autónoma de Coahuila
Blvd. Revolución 151 Ote. CP 27000, Torreón, Coahuila, México
arnulfol_2007@hotmail.com

Received June 2016; revised October 2016

Abstract. *This paper presents a mathematical model for dimensioning of square footings using optimization techniques (general case), i.e., the column is localized anywhere of the footing to obtain the most economical contact surface on the soil, when the load that must support said structural member is applied (axial load and moments in two directions). The classical model is developed by test and error, i.e., a dimension is proposed, and the equation of the biaxial bending is used to obtain the pressure acting on the four corners of the square footing, which must meet the conditions as the following: 1) the minimum pressure should be equal or greater than zero, because the soil cannot withstand tensile; 2) The maximum pressure must be equal or less than the allowable capacity that can withstand the soil. Therefore, normal practice to use the classic model will not be a recommended solution. Then, the proposed model is best option, since it is more economic.*
Keywords: Square isolated footings, Most economical contact surface, Allowable capacity of the soil, Biaxial bending

1. **Introduction.** The foundation is the part of the structure which transmits the loads to the underlying soil. The foundations are classified into superficial and deep ones, which have important differences: in terms of geometry, the behavior of the soil, its structural functionality and its constructive systems [1,2].

Superficial foundations may be of various types according to their function: isolated footing, combined footing, strip footing, or mat foundation [1-6].

In the design of superficial foundations, in the specific case of isolated footings, there are three types in terms of the application of loads: 1) footings subjected to concentric axial load, 2) footings subjected to axial load and moment in one direction (uniaxial bending), 3) footings subjected to axial load and moments in two directions (biaxial bending) [1-4]. The hypothesis used in the classical model is developed by trial and error, i.e., a dimension is proposed, and after the equation of the biaxial bending is used to obtain the pressure acting on the four corners of the square footing, which must meet the following conditions: 1) the minimum pressure should be equal to or greater than zero, because the soil is not capable of withstanding tensile; 2) the maximum pressure must be equal to or less than the allowable capacity that can withstand the soil [1-6].

Mathematical models have been presented to obtain the dimensions of rectangular, square and circular isolated footings subjected to axial load and moments in two directions (biaxial bending), but the column is located in the center of the footing [7-9]. Also a comparison between the rectangular, square and circular footings in terms of the contact area with soil has been proposed; this paper considers the column situated in the center of the footing [10]. A new approach for dimensioning of rectangular footings using optimization techniques is presented, and also the column is situated in the center of the footing [11].

Also models have been presented to find the dimensions of rectangular and trapezoidal combined footings subjected to axial load and moments in two directions (biaxial bending) in each column [12,13].

This paper presents a mathematical model for dimensioning of square footings using optimization techniques (general case), i.e., the column is localized anywhere of the footing to obtain the most economical contact surface on the soil, when the load that must support said structural member is applied (axial load and moments in two directions). Also, numerical examples are presented to validate the mathematical model proposed in this paper to obtain the most economical area of square footings under an axial load and moments in two directions.

The paper is organized as follows. Methodology (Section 2) describes the mathematical model for the dimensioning of square footings, shows the equations for the general case, presents the special cases of footings (concentric footings, edge footings, and corner footings), and also shows the equations to obtain the optimal area for each case. In Section 3, numerical examples are presented to validate the proposed mathematical model using optimization techniques. Results and discussion are presented in Section 4. Conclusion (Section 5) completes the paper.

2. Methodology.

Figure 1 shows a square footing subjected to an axial load and moment in two directions (biaxial bending) and the column is localized anywhere of the footing, where pressure is different in the four corners of the contact surface.

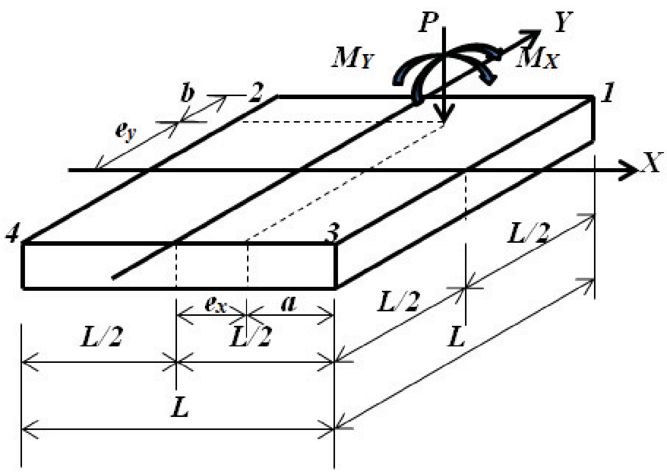

FIGURE 1. Square footing under a column localized anywhere of the footing

The general equation for any type of footings subjected to biaxial bending is [7-14]:

$$\sigma = \frac{P}{A} \pm \frac{M_{xT}y}{I_x} \pm \frac{M_{yT}x}{I_y} \tag{1}$$

where σ is the pressure exerted by the soil on the footing, A is the contact area of the footing, P is the axial load applied at the center of gravity of the footing, M_{xT} is the total moment around the axis "X" ($M_{xT} = M_x + Pe_y$), M_{yT} is the total moment around the axis "Y" ($M_{yT} = M_y + Pe_x$), x is the distance in the direction "X" measured from the axis "Y" up the fiber under study, y is the distance in direction "Y" measured from the axis "X" up the fiber under study, I_y is the moment of inertia around the axis "Y" and I_x is the moment of inertia around the axis "X".

The pressure exerted by the soil anywhere on the footing in function of coordinates (x, y) is obtained [7-14]:

$$\sigma(x, y) = \frac{P}{A} + \frac{M_{xT}y}{I_x} + \frac{M_{yT}x}{I_y} \tag{2}$$

Substituting $M_{xT} = M_x + Pe_y$, $M_{yT} = M_y + Pe_x$, $A = L^2$, $I = \frac{L^4}{12}$, and the corresponding coordinates at each corner into Equation (2), the pressures at each corner of the square footing are presented below.

The pressure in corner 1 with coordinates $(x = L/2,\ y = L/2)$ is:

$$\sigma_1 = \frac{P}{L^2} + \frac{6(M_x + Pe_y)}{L^3} + \frac{6(M_y + Pe_x)}{L^3} \tag{3}$$

The pressure in corner 2 with coordinates $(x = -L/2,\ y = L/2)$ is:

$$\sigma_2 = \frac{P}{L^2} + \frac{6(M_x + Pe_y)}{L^3} - \frac{6(M_y + Pe_x)}{L^3} \tag{4}$$

The pressure in corner 3 with coordinates $(x = L/2,\ y = -L/2)$ is:

$$\sigma_3 = \frac{P}{L^2} - \frac{6(M_x + Pe_y)}{L^3} + \frac{6(M_y + Pe_x)}{L^3} \tag{5}$$

The pressure in corner 4 with coordinates $(x = -L/2,\ y = -L/2)$ is:

$$\sigma_4 = \frac{P}{L^2} - \frac{6(M_x + Pe_y)}{L^3} - \frac{6(M_y + Pe_x)}{L^3} \tag{6}$$

Now, if we consider $e_y = L/2 - b$ and $e_x = L/2 - a$, since in some cases, the column could be located anywhere on the footing (see Figure 1), Equations (3) to (6) are presented as follows:

$$\sigma_1 = \frac{P}{L^2} + \frac{6[M_x + P(L/2 - b)]}{L^3} + \frac{6[M_y + P(L/2 - a)]}{L^3} \tag{7}$$

$$\sigma_2 = \frac{P}{L^2} + \frac{6[M_x + P(L/2 - b)]}{L^3} - \frac{6[M_y + P(L/2 - a)]}{L^3} \tag{8}$$

$$\sigma_3 = \frac{P}{L^2} - \frac{6[M_x + P(L/2 - b)]}{L^3} + \frac{6[M_y + P(L/2 - a)]}{L^3} \tag{9}$$

$$\sigma_4 = \frac{P}{L^2} - \frac{6[M_x + P(L/2 - b)]}{L^3} - \frac{6[M_y + P(L/2 - a)]}{L^3} \tag{10}$$

Figure 2 shows more general cases of square footings according to the location of the column, which are concentric footings, edge footings and corner footings.

The concentric footings: the column is localized in center of the footing (see Figure (2a)). The edge footings: the column is found on one property line of the footing (see Figure 2(b)). The corner footings: the column is located on two property lines attached of the footing (see Figure 2(c)).

2.1. **Objective function to minimize the area.** An area function is defined as the contact surface total area A, which is:

$$A = L^2 \tag{11}$$

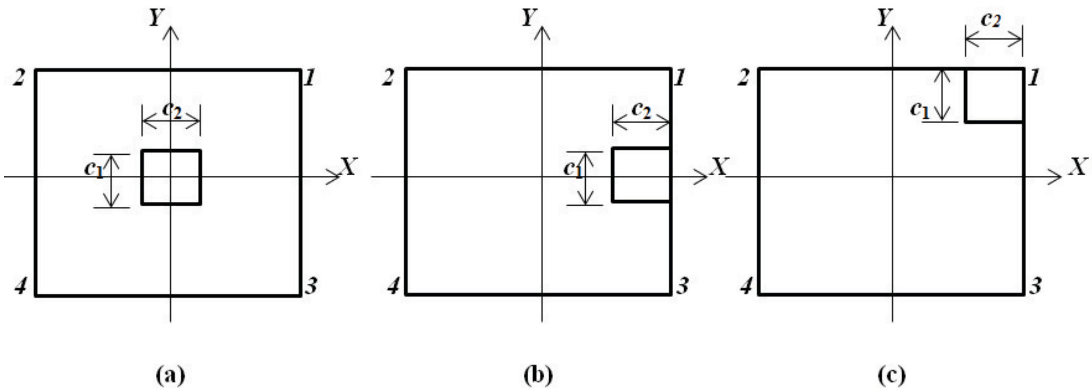

FIGURE 2. General cases of square footings: (a) concentric footings, (b) edge footings, and (c) corner footings

2.2. **Constraint functions (general case).** Equations for dimensioning of square footings are:

$$\sigma_1 = \frac{P}{L^2} + \frac{6[M_x + P(L/2 - b)]}{L^3} + \frac{6[M_y + P(L/2 - a)]}{L^3} \tag{12}$$

$$\sigma_2 = \frac{P}{L^2} + \frac{6[M_x + P(L/2 - b)]}{L^3} - \frac{6[M_y + P(L/2 - a)]}{L^3} \tag{13}$$

$$\sigma_3 = \frac{P}{L^2} - \frac{6[M_x + P(L/2 - b)]}{L^3} + \frac{6[M_y + P(L/2 - a)]}{L^3} \tag{14}$$

$$\sigma_4 = \frac{P}{L^2} - \frac{6[M_x + P(L/2 - b)]}{L^3} - \frac{6[M_y + P(L/2 - a)]}{L^3} \tag{15}$$

$$0 \leq \left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} \right\} \leq \sigma_{\max} \tag{16}$$

where $\sigma_{\max}$ is allowable load capacity of the soil.

2.3. **Special cases.**

2.3.1. *Constraint functions for the concentric footings.* Substituting the values of $a = L/2$ and $b = L/2$ into Equations (12) and (15), constraint functions are presented:

$$\sigma_1 = \frac{P}{L^2} + \frac{6M_x}{L^3} + \frac{6M_y}{L^3} \tag{17}$$

$$\sigma_2 = \frac{P}{L^2} + \frac{6M_x}{L^3} - \frac{6M_y}{L^3} \tag{18}$$

$$\sigma_3 = \frac{P}{L^2} - \frac{6M_x}{L^3} + \frac{6M_y}{L^3} \tag{19}$$

$$\sigma_4 = \frac{P}{L^2} - \frac{6M_x}{L^3} - \frac{6M_y}{L^3} \tag{20}$$

$$0 \leq \left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} \right\} \leq \sigma_{\max} \tag{21}$$

2.3.2. *Constraint functions for edge footings.* Substituting the values of $a = c_2/2$ and $b = L/2$ into Equations (12) and (15), constraint functions are presented:

$$\sigma_1 = \frac{P}{L^2} + \frac{6M_x}{L^3} + \frac{6[M_y + P(L/2 - c_2/2)]}{L^3} \tag{22}$$

$$\sigma_2 = \frac{P}{L^2} + \frac{6M_x}{L^3} - \frac{6[M_y + P(L/2 - c_2/2)]}{L^3} \tag{23}$$

$$\sigma_3 = \frac{P}{L^2} - \frac{6M_x}{L^3} + \frac{6[M_y + P(L/2 - c_2/2)]}{L^3} \tag{24}$$

$$\sigma_4 = \frac{P}{L^2} - \frac{6M_x}{L^3} - \frac{6[M_y + P(L/2 - c_2/2)]}{L^3} \tag{25}$$

$$0 \leq \left\{ \begin{matrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{matrix} \right\} \leq \sigma_{\max} \tag{26}$$

2.3.3. *Constraint functions for corner footings.* Substituting the values of $a = c_2/2$ and $b = c_1/2$ into Equations (12) and (15), constraint functions are presented:

$$\sigma_1 = \frac{P}{L^2} + \frac{6[M_x + P(L/2 - c_1/2)]}{L^3} + \frac{6[M_y + P(L/2 - c_2/2)]}{L^3} \tag{27}$$

$$\sigma_2 = \frac{P}{L^2} + \frac{6[M_x + P(L/2 - c_1/2)]}{L^3} - \frac{6[M_y + P(L/2 - c_2/2)]}{L^3} \tag{28}$$

$$\sigma_3 = \frac{P}{L^2} - \frac{6[M_x + P(L/2 - c_1/2)]}{L^3} + \frac{6[M_y + P(L/2 - c_2/2)]}{L^3} \tag{29}$$

$$\sigma_4 = \frac{P}{L^2} - \frac{6[M_x + P(L/2 - c_1/2)]}{L^3} - \frac{6[M_y + P(L/2 - c_2/2)]}{L^3} \tag{30}$$

$$0 \leq \left\{ \begin{matrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{matrix} \right\} \leq \sigma_{\max} \tag{31}$$

3. **Numerical Examples.** The tables for the dimensioning of square footings are presented with three cases and each case with four problems different, where the soil load capacity varies of 250, 200, 150 and 100 kN/m^2. Table 1 shows the results for the dimensioning of concentric square footings using the new model with the optimization techniques; the objective function (minimum contact surface) by Equation (11) is obtained, and constraint functions by Equations (17) to (21) are found. Table 2 presents the results for the dimensioning of edge square footings using the new model with the optimization techniques; the objective function (minimum contact surface) by Equation (11) is found, and constraint functions by Equations (22) to (26) are obtained. Table 3 shows the results for the dimensioning of corner square footings using the new model with the optimization techniques; the objective function (minimum contact surface) by Equation (11) is obtained, and constraint functions by Equations (27) to (31) are found. The minimum areas and dimensions for square footings by the MAPLE-15 software are obtained.

4. **Results and Discussion.** Table 1 shows the results for the dimensioning of concentric square footings. Case 1 presents that the second condition prevails in all types; this means that the footing should be dimensioned on the basis of soil load capacity. To case 2 governs the second condition for the last three types and the first condition prevails in the first type. Finally, to case 3 is governed by the first condition for the first three types (soil load

TABLE 1. Dimensioning of concentric footings using the new model

Soil load capacity $\sigma_{\max}$ kN/m^2	Axial load P kN	Moments		Minimum area	Minimum dimensions	Pressure in the corners of the footing			
		M_y kN-m	M_x kN-m	A m^2	L m	σ_1 kN/m^2	σ_2 kN/m^2	σ_3 kN/m^2	σ_4 kN/m^2
Case 1									
250.00	700.00	70.00	100.00	4.6850	2.1645	250.0000	167.1642	131.6632	48.8274
200.00	700.00	70.00	100.00	5.6463	2.3762	200.0000	137.3913	110.5591	47.9504
150.00	700.00	70.00	100.00	7.2007	2.6834	150.0000	106.5277	87.8967	44.4243
100.00	700.00	70.00	100.00	10.1946	3.1929	100.0000	74.1938	63.1340	37.3277
Case 2									
250.00	500.00	70.00	100.00	4.1616	2.0400	240.2922	141.3484	98.9438	0.0000
200.00	500.00	70.00	100.00	4.8224	2.1960	200.0000	120.6798	86.6855	7.3653
150.00	500.00	70.00	100.00	6.0890	2.4676	150.0000	94.0944	70.1349	14.2293
100.00	500.00	70.00	100.00	8.4988	2.9153	100.0000	66.0967	51.5667	17.6635
Case 3									
250.00	500.00	100.00	150.00	9.0000	3.0000	111.1111	66.6666	44.4444	0.0000
200.00	500.00	100.00	150.00	9.0000	3.0000	111.1111	66.6666	44.4444	0.0000
150.00	500.00	100.00	150.00	9.0000	3.0000	111.1111	66.6666	44.4444	0.0000
100.00	500.00	100.00	150.00	9.7932	3.1294	100.0000	60.8445	41.2668	2.1114

TABLE 2. Dimensioning of edge footings using the new model

Soil load capacity $\sigma_{\max}$ kN/m^2	Axial load P kN	Moments		Minimum area	Minimum dimensions	Pressure in the corners of the footing			
		M_y kN-m	M_x kN-m	A m^2	L m	σ_1 kN/m^2	σ_2 kN/m^2	σ_3 kN/m^2	σ_4 kN/m^2
Case 1									
250.00	450.00	−211.87	0.00	1.8000	1.3416	250.0000	250.0000	250.0000	250.0000
200.00	450.00	−247.50	0.00	2.2500	1.5000	200.0000	200.0000	200.0000	200.0000
150.00	450.00	−299.71	0.00	3.0000	1.7321	150.0000	150.0000	150.0000	150.0000
100.00	450.00	−387.30	0.00	4.5000	2.1213	100.0000	100.0000	100.0000	100.0000
Case 2									
250.00	439.40	−300.00	100.00	3.1170	1.7655	250.0000	250.0000	31.9387	31.9387
200.00	401.39	−300.00	100.00	3.5902	1.8948	200.0000	200.0000	23.6017	23.6017
150.00	357.88	−300.00	100.00	4.3121	2.0766	150.0000	150.0000	15.9867	15.9867
100.00	305.40	−300.00	100.00	5.5914	2.3646	100.0000	100.0000	9.2395	9.2395
Case 3									
250.00	400.00	−300.00	0.00	2.5317	1.5911	65.9891	250.0000	65.9891	250.0000
200.00	400.00	−300.00	0.00	2.8058	1.6750	85.1256	200.0000	85.1256	200.0000
150.00	400.00	−300.00	0.00	3.1844	1.7845	101.2209	150.0000	101.2209	150.0000
100.00	400.00	−300.00	0.00	7.8755	2.8063	100.0000	1.5804	100.0000	1.5804

capacity), and the fourth type is dominated by the second condition (minimum pressure zero).

Table 2 presents the results for the dimensioning of edge square footings. This table shows numerical experiments. For the case 1 consider $P = 450$ kN, $M_x = 0$ and the maximum value of M_y is obtained to produce the optimal area. For the case 2 take into account $M_y = 100$ kN-m, $M_x = -300$ kN-m and the maximum value of P is found to produce the optimal area. For the case 3 consider $P = 400$ kN, $M_y = -300$ kN-m and

TABLE 3. Dimensioning of corner footings using the new model

Soil load capacity $\sigma_{\max}$ kN/m^2	Axial load P kN	Moments		Minimum area	Minimum dimensions	Pressure in the corners of the footing			
		M_y kN-m	M_x kN-m	A m^2	L m	σ_1 kN/m^2	σ_2 kN/m^2	σ_3 kN/m^2	σ_4 kN/m^2
Case 1									
250.00	300.00	−104.32	−104.32	1.2000	1.0954	250.0000	250.0000	250.0000	250.0000
200.00	300.00	−123.71	−123.71	1.5000	1.2247	200.0000	200.0000	200.0000	200.0000
150.00	300.00	−152.13	−152.13	2.0000	1.4142	150.0000	150.0000	150.0000	150.0000
100.00	300.00	−199.81	−199.81	3.0000	1.7321	100.0000	100.0000	100.0000	100.0000
Case 2									
250.00	400.00	−300.00	−238.23	2.5317	1.5911	65.9892	250.0000	65.9892	250.0000
200.00	400.00	−300.00	−255.01	2.8058	1.6750	85.1256	200.0000	85.1256	200.0000
150.00	400.00	−300.00	−276.90	3.1844	1.7845	101.2209	150.0000	101.2209	150.0000
100.00	400.00	−300.00	−481.27	7.8755	2.8063	100.0000	100.0000	1.5804	1.5804
Case 3									
250.00	400.00	−238.23	−300.00	2.5317	1.5911	65.9892	65.9892	250.0000	250.0000
200.00	400.00	−255.01	−300.00	2.8058	1.6750	85.1256	85.1256	200.0000	200.0000
150.00	400.00	−276.90	−300.00	3.1844	1.7845	101.2209	101.2209	150.0000	150.0000
100.00	400.00	−481.27	−300.00	7.8755	2.8063	100.0000	100.0000	1.5804	1.5804

the maximum value of M_x obtain to produce the optimal area. To all cases are dominant the second condition.

Table 3 shows the results for the dimensioning of corner square footings. This table presents numerical experiments: For the case 1 consider $P = 300$ kN, $M_x = M_y$ and the maximum values of M_y and M_x are obtained to produce the optimal area. For the case 2 take into account $P = 400$ kN, $M_y = -300$ kN-m and the maximum value of M_x is found to produce the optimal area. For the case 3 consider $P = 400$ kN, $M_x = -300$ kN-m and the maximum value of M_y is obtained to produce the optimal area. To all cases are dominant the second condition.

5. **Conclusions.** This paper presents a new mathematical model for dimensioning of squ-are footings using optimization techniques (general case), i.e., the column is localized anywhere of the footing to obtain the most economical contact surface on the soil (optimal area), which must meet the conditions as the following: 1) the minimum pressure should be equal to or greater than zero; 2) the maximum pressure must be equal to or less than the soil load capacity.

The study aims to show the minimum contact surface (minimum area) for square isolated footings, and present the types more known are concentric, edge, and corner footings.

Real examples for the dimensioning of square footings have been presented to demonstrate the efficiency of the optimization techniques.

The research reported in this paper concludes as follows.

1) The dimensioning of concentric square footings can be used for any load and moments.

2) The dimensioning of edge square footings can be used for any load and moment around the axis "X", but the moment around the axis "Y" must be negative to balance the moment that produces the eccentric load.

3) The dimensioning of corner square footings can be used for any load, but the moments around of the axes "X" and "Y" must be negatives to balance the moments that produces the eccentric load in the two directions.

The model developed in this paper applies only to rigid soils satisfying expression of the biaxial bending, i.e., the variation of the pressure is linear. The suggestions for future research are: when another type of soil is presented, by example in cohesive soils and granular soils, because the pressure diagram is not linear and should be treated differently to the problem presented in this paper.

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