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INPUT-TO-STATE CONVERGENCE OF NETWORKS WITH DISTRIBUTED DELAYS ON TIME SCALES

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Abstract. *This paper studies the input-to-state convergence (ISC) on time scales for neural networks with distributed delays. By using the time scale calculus theory and constructing appropriate Lyapunov functions, new sufficient conditions on input-to-state convergence of such neural networks on time scales are derived. At last illustrative examples demonstrate the effectiveness of the input-to-state convergence criteria. The new results given are general which unify continuous-time with corresponding discrete-time situations and extend the existing relevant input-to-state convergence results in the literature to cover more general neural networks.*

Keywords: Neural networks, Input-to-state convergence, On time scales, Lyapunov functions

1. **Introduction.** Recently, there has been increasing interest in the potential applications of neural networks in many areas. Hopfield-type neural networks [1, 2, 3] and their various generalizations can be described as

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} t_{ij} g_j(x_j(t)) + I_i, \quad i = 1, 2, \ldots, n,$$

where $t \geq 0$, the coefficient $d_i > 0$ represents the passive decay rate with which the unit self-regulates or resets its potential when isolated from other units and inputs, $x_i(t)$ corresponds to the state of the ith unit at time t; t_{ij} weighs the strength of the unit j on the unit i; $g_j(\cdot)$ denote activation functions of signal transmission; I_i is the input to the ith unit at time t from outside the networks. We refer for more detail to [4, 5] and references cited therein.

They have attracted the attention of many scientists due to their promising potential for tasks of classification, associative memory, signal processing, parallel computation and their ability to solve difficult optimization problems, see for example [6, 7] and the references therein. As we know, time delays may lead to an oscillation and possible instability of neural networks [8]. A lot of papers have studied the following delayed neural networks model [9]:

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} \left[t_{ij} g_j(x_j(t)) + t_{ij}^{\tau} g_j(x_j(t - \tau_{ij}(t))) \right] + I_i, \quad i = 1, 2, \ldots, n,$$

where t_{ij}^{τ} denotes the delayed feedback connection weight of the unit j on the unit i; τ_{ij} corresponds to the signal transmission delay along the axon of the jth unit which is nonnegative and bounded, and other notations are the same as above. Usually, bounded delays (either constant or time varying) in the models of delayed feedback systems serve as good approximation in simple circuits having a small number of cells. A neural network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and length. It is desirable to model them by introducing distributed delays [10, 11].

We studied the models above are continuous, but as we know sometimes we meet with the situation of neural network with time-varying delays being discontinuous in reality, and it is necessary for people to research it. The study of dynamic equations on time scales was initiated by S. Hilger [12] in order to unify the continuous and discrete analysis, and it allows a simultaneous treatment of differential and difference equations, extending those theories to so-called dynamic equations. It is a fairly new subject, and research in this area is rapidly growing. Many authors incorporate time scales into analysis of neural network models in all directions, and we can refer to [13-17]. In [13], existence and exponential stability of periodic solution for stochastic Hopfield neural networks on time scales were considered. In [14], the problem on the global exponential stability of neural networks on time scales was considered and got some nice results. In [15], based on contraction principle and Gronwall-Bellman's inequality some first results for the existence and exponential stability of almost periodic solution for a general type of delay neural networks with impulsive effects had established. In [16], authors pay attention to the periodic solutions for a class of neural networks delays on time scales. In [17], the scale-limited activating sets and multiperiodicity for threshold-linear networks on time scales were considered and got some nice results.

2. **Problem Statement and Preliminaries.**

Based on the above discussion, we also want to incorporate time scales $\mathbb{T}$ into the input-to-state convergence (ISC) analysis for a class of neural networks with distributed delays described by the following system of integro-differential equations on time scale $\mathbb{T}$:

$$x_i^{\Delta}(t) = \sum_{j=1}^{n} w_{ij} g_j\big(x_j(t)\big) + w_{ij}^{\tau} \int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s) g_j\big(x_j(t-s)\big)\Delta s + I_i(t), \quad i = 1, 2, \ldots, n, \tag{1}$$

where $t \geq 0$ and $t \in \mathbb{T}$, $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is the state vector; $W = (w_{ij})_{n\times n}$ and $W^{\tau} = (w_{ij}^{\tau})_{n\times n}$ are the feedback connection weight matrix and the delayed feedback connection weight matrix, respectively; $G(x(t)) = [g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t))]^T$ is a nonlinear vector-valued activation function from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $G(x(t))$ is called the output of the network (1); $K(s) = (k_{ij}(s))_{n\times n}$ in which the kernels $k_{ij} : [0, \infty]_{\mathbb{T}} \to [0, +\infty)$ are piecewise continuous functions, $I(t) = [I_1(t), I_2(t), \ldots, I_n(t)]^T \in \mathbb{R}^n$ is a locally Lipschitz continuous input vector function defined on $[0, \infty]_{\mathbb{T}} := [0, +\infty) \cap \mathbb{T}$.

We can rewrite (1) in a compact vector form

$$x^{\Delta}(t) = WG(x(t)) + \int_{[0,\infty]_{\mathbb{T}}} W^{\tau} K(s) G\big(x(t-s)\big)\Delta s + I(t). \tag{2}$$

Each activation function g_i will be assumed to be a sigmoid-type function and we assume that:

(H_1) There exist constants $l_i, L_i > 0$, such that

$$l_i \leq \frac{g_i(u) - g_i(v)}{u - v} \leq L_i,$$

for any $u, v \in \mathbb{R}$ and $u \neq v$, $i = 1, 2, \ldots, n$.

(H_2) There exists a constant vector $I \in \mathbb{R}^n$ such that

$$\lim_{t\to+\infty} I(t) = I.$$

(H_3) The functions k_{ij} satisfy

$$\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)\Delta s = 1, \quad i,j = 1,2,\dots,n.$$

The concept of ISC [18, 19] is similar to the widely recognized notion of input-to-state stability (ISS), which has been a useful concept in studying nonlinear control problems (see [20, 21]). In this paper, we continue the study of ISC for the network model (1) on time scales and obtain new ISC stability criteria which can include continuous and discrete-time cases. Illustrative examples demonstrate the effectiveness of the input-to-state convergence criteria.

2.1. **Several definitions and theorems.** Some basic definitions of dynamic equations and ISC convergence of networks on time scales are given in this section.

Definition 2.1. *A time scale $\mathbb{T}$ is arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$.*

Definition 2.2. *For any time scale $\mathbb{T}$, we define the forward and backward jump operators by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, we put $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$, where $\emptyset$ denotes the empty set. A point t is said to be left-dense if $t > \inf\mathbb{T}$ and $\rho(t) = t$, right-dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum m, then we defined $\mathbb{T}^k$ to be $\mathbb{T} - m$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.*

Definition 2.3. *For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative is defined by*

$$f^{\triangle} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered, then the derivative is defined by

$$f^{\triangle} = \lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

provided this limit exists.

Definition 2.4. *A function $F : \mathbb{T}^k \to \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\triangle} = f$ holds for all $t \in \mathbb{T}^k$. The integral of f is defined by $\int_{[a,t]_{\mathbb{T}}} f(s)\triangle s = F(t) - F(a)$ for $t \in \mathbb{T}$ and $\int_t^{\sigma(t)} f(s)\triangle s = \mu(t)f(t)$ for $t \in \mathbb{T}^k$.*

Definition 2.5. *For a given continuous vector function $I(t)$ and a constant vector I, $(I(t), I)$ will be called an input pair. The network (1), or equivalently (2), is said to be ISC with respect to an input pair $(I(t), I)$, if*

$$\Omega(I) \doteq \{x^* \in R^n | (W + W^{\tau})G(x^*) + I = 0\} \neq \emptyset$$

implies that for any $x(0) \in \mathbb{R}^n$, $\lim_{t\to+\infty} x(t) = x^*$ for some $x^* \in \Omega(I)$.

3. **Main Results.** In this section, we study the networks with distributed delays on time scale, and give new criteria for the ISC of the network model (1).

Theorem 3.1. *Assume that* (H_1)-(H_3) *hold and* $w_{jj} < 0$*, the input pair* $(I(t), I)$ *satisfies* $\int_{[0,\infty]_{\mathbb{T}}}(1+\mu(t))|I_i(t)-I_i|^2\Delta t < +\infty$ *and the following condition*

$$-2c_jw_{jj}l_j - \sum_{i=1,i\neq j}^{n} c_i\left(|w_{ji}|^{2-q_{ji}}L_i^{2-r_{ji}} + |w_{ij}|^{q_{ij}}L_j^{r_{ij}}\right) - 2c_j - \mu(t)3n\sum_{i=1}^{n} c_iw_{ij}^2L_j^2 - \left(n+\mu(t)3n\right)\sum_{i=1}^{n} c_iL_j^2\left(w_{ij}^{\tau}\right)^2 > 0, \quad j = 1,2,\ldots,n,$$

holds, then the network (1) is input-to-state convergent with respect to $(I(t), I)$.

Proof: By definition of ISC, let $\Omega(I) \neq \emptyset$. Then there exists a constant vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n}\left(w_{ij}+w_{ij}^{\tau}\right)g_j\left(x_j^*\right) + I_i = 0, \quad i = 1,2,\ldots,n. \tag{3}$$

Let $x(t)$ be the solution of (1) starting from $x(0) \in \mathbb{R}^n$. Then it follows from (1) that

$$\begin{aligned}(x_i(t)-x_i^*)^{\Delta} = &\sum_{j=1}^{n} w_{ij}\left[g_j\big(x_j(t)\big) - g_j\big(x_j^*\big)\right] \\ &+ w_{ij}^{\tau}\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)\left[g_j\big(x_j(t-s)\big) - g_j\big(x_j^*\big)\right]\Delta s + (I_i(t)-I_i).\end{aligned}$$

Letting $z(t) = x(t)-x^*$, $\hat{I}(t) = I(t)-I$, $f\big(z(t)\big) = g\big(z(t)+x^*\big)-g(x^*) = g\big(x(t)\big)-g(x^*)$, then we get:

$$z_i^{\Delta}(t) = \sum_{j=1}^{n}\left[w_{ij}f_j\big(z_j(t)\big) + w_{ij}^{\tau}\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)f_j\big(z_j(t-s)\big)\Delta s\right] + \hat{I}_i(t), \tag{4}$$

where $i = 1,2,\ldots,n$. Let $c_i > 0$ and $V(t) = \Sigma_{i=1}^{n}c_iz_i^2(t)$, we get

$$\begin{aligned}V^{\Delta}(t) &= \sum_{i=1}^{n} c_i\left[z_i^{\Delta}(t)z_i(\sigma(t)) + z_i(t)z_i^{\Delta}(t)\right] \\ &= \sum_{i=1}^{n} c_i\big[z_i(\sigma(t)) + z_i(t)\big]z_i^{\Delta}(t) \\ &= \sum_{i=1}^{n} c_i\left[\mu(t)z_i^{\Delta}(t) + 2z_i(t)\right]z_i^{\Delta}(t) \\ &= 2\sum_{i=1}^{n} c_iz_i(t)z_i^{\Delta}(t) + \mu(t)\sum_{i=1}^{n} c_i(z_i^{\Delta}(t))^2.\end{aligned}$$

It follows from (1) that

$$\begin{aligned}&2\sum_{i=1}^{n} c_iz_i(t)z_i^{\Delta}(t) \\ &\leq 2\sum_{i=1}^{n} c_iz_i(t)\left(\sum_{j=1}^{n}\left[w_{ij}f_j\big(z_j(t)\big) + w_{ij}^{\tau}\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)f_j\big(z_j(t-s)\big)\Delta s\right] + \hat{I}_i(t)\right)\end{aligned}$$

$$
\begin{aligned}
&\leq 2\sum_{i=1}^{n} c_i z_i(t) \sum_{j=1}^{n} w_{ij} f_j\big(z_j(t)\big) + 2\sum_{i=1}^{n} c_i z_i(t) \sum_{j=1}^{n} w_{ij}^{\tau} \int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s) f_j\big(z_j(t-s)\big)\Delta s \\
&\quad + 2\sum_{i=1}^{n} c_i z_i(t)\hat{I}_i(t) \\
&\leq \sum_{i=1}^{n} \left[2c_i w_{ii} z_i(t) f_i\big(z_i(t)\big) + 2\sum_{j=1, j\neq i}^{n} c_i |w_{ij}||z_i(t)| L_j |z_j(t)| \right] + \sum_{i=1}^{n} c_i z_i^2(t) \\
&\quad + \sum_{i=1}^{n} c_i \left(\sum_{j=1}^{n} |w_{ij}^{\tau}| L_j \int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)|z_j(t-s)|\Delta s \right)^2 + \sum_{i=1}^{n} c_i \left(z_i^2(t) + \hat{I}_i^2(t) \right) \\
&\leq \sum_{i=1}^{n} \left[2c_i w_{ii} z_i(t) f_i\big(z_i(t)\big) + 2\sum_{j=1, j\neq i}^{n} c_i \left[|z_i(t)||w_{ij}|^{\frac{2-q_{ij}}{2}} L_j^{\frac{2-r_{ij}}{2}} \right] \times \left[|w_{ij}|^{\frac{q_{ij}}{2}} L_j^{\frac{r_{ij}}{2}} |z_j(t)| \right] \right] \\
&\quad + 2\sum_{i=1}^{n} c_i z_i^2(t) + n\sum_{i=1}^{n} c_i \sum_{j=1}^{n} \left(w_{ij}^{\tau}\right)^2 L_j^2 \left(\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)|z_j(t-s)|\Delta s \right)^2 + \sum_{i=1}^{n} c_i \hat{I}_i^2(t) \\
&\leq \sum_{j=1}^{n} \left[2c_j w_{jj} z_j(t) f_j\big(z_j(t)\big) + \sum_{i=1, i\neq j}^{n} c_i \left[z_i^2(t)|w_{ij}|^{2-q_{ij}} L_j^{2-r_{ij}} + |w_{ij}|^{q_{ij}} L_j^{r_{ij}} z_j^2(t) \right] \right] \\
&\quad + 2\sum_{i=1}^{n} c_i z_i^2(t) + n\sum_{i=1}^{n} c_i \sum_{j=1}^{n} \left(w_{ij}^{\tau}\right)^2 L_j^2 \left(\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)|z_j(t-s)|\Delta s \right)^2 + \sum_{i=1}^{n} c_i \hat{I}_i^2(t) \\
&\leq \sum_{j=1}^{n} \left(2c_j w_{jj} l_j + \sum_{i=1, i\neq j}^{n} c_i \left(|w_{ji}|^{2-q_{ji}} L_i^{2-r_{ji}} + |w_{ij}|^{q_{ij}} L_j^{r_{ij}} \right) + 2c_j \right) z_j^2(t) \\
&\quad + n\sum_{i=1}^{n} c_i \sum_{j=1}^{n} \left(w_{ij}^{\tau}\right)^2 L_j^2 \left(\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)|z_j(t-s)|\Delta s \right)^2 + \sum_{i=1}^{n} c_i \hat{I}_i^2(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{n} c_i (z_i^{\Delta}(t))^2 \\
&\leq \sum_{i=1}^{n} c_i \left(\sum_{j=1}^{n} \left[w_{ij} f_j\big(z_j(t)\big) + w_{ij}^{\tau} \int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s) f_j\big(z_j(t-s)\big)\Delta s \right] + \hat{I}_i(t) \right)^2 \\
&\leq 3\sum_{i=1}^{n} c_i \left[\left(\sum_{j=1}^{n} w_{ij} f_j(z_j(t)) \right)^2 + \left(\sum_{j=1}^{n} w_{ij}^{\tau} \int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s) f_j\big(z_j(t-s)\big)\Delta s \right)^2 + \left(\hat{I}_i(t)\right)^2 \right] \\
&\leq 3\sum_{i=1}^{n} c_i \left(\sum_{j=1}^{n} |w_{ij}| L_j |z_j(t)| \right)^2 + 3\sum_{i=1}^{n} c_i \left(\sum_{j=1}^{n} |w_{ij}^{\tau}| \int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s) L_j |z_j(t-s)|\Delta s \right)^2 \\
&\quad + 3\sum_{i=1}^{n} c_i \big(\hat{I}_i(t)\big)^2 \\
&\leq 3n\sum_{i=1}^{n} c_i \sum_{j=1}^{n} w_{ij}^2 L_j^2 z_j^2(t) + 3n\sum_{i=1}^{n} c_i \sum_{j=1}^{n} \left(w_{ij}^{\tau}\right)^2 L_j^2 \left(\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)|z_j(t-s)|\Delta s \right)^2
\end{aligned}
$$

$$+ 3\sum_{i=1}^{n} c_i \left(\hat{I}_i(t)\right)^2.$$

Hence, let $u_i(t) = \sup_{s\in(-\infty,t]_{\mathbb{T}}} z_i^2(s)$, we can get

$$\begin{aligned}
&V^{\Delta}(t)\\
\le\ & \sum_{j=1}^{n}\left(2c_j w_{jj} l_j + \sum_{i=1,i\neq j}^{n} c_i \left(|w_{ji}|^{2-q_{ji}} L_i^{2-r_{ji}} + |w_{ij}|^{q_{ij}} L_j^{r_{ij}}\right) + 2c_j\right) z_j^2(t)\\
&+ n\sum_{i=1}^{n} c_i \sum_{j=1}^{n} \left(w_{ij}^{\tau}\right)^2 L_j^2 \left(\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)|z_j(t-s)|\Delta s\right)^2 + \sum_{i=1}^{n} c_i \hat{I}_i^2(t)\\
&+ \mu(t)\Bigg(3n\sum_{i=1}^{n} c_i \sum_{j=1}^{n} w_{ij}^2 L_j^2 z_j^2(t)\\
&+ 3n\sum_{i=1}^{n} c_i \sum_{j=1}^{n} L_j^2 \left(w_{ij}^{\tau}\right)^2 \left(\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)|z_j(t-s)|\Delta s\right)^2 + 3\sum_{i=1}^{n} c_i \big(\hat{I}_i(t)\big)^2\Bigg)\\
=\ & \sum_{j=1}^{n}\Bigg(2c_j w_{jj} l_j + \sum_{i=1,i\neq j}^{n} c_i \left(|w_{ji}|^{2-q_{ji}} L_i^{2-r_{ji}} + |w_{ij}|^{q_{ij}} L_j^{r_{ij}}\right) + 2c_j\\
&+ \mu(t)3n\sum_{i=1}^{n} c_i w_{ij}^2 L_j^2\Bigg) z_j^2(t)\\
&+ \big(n + \mu(t)3n\big)\sum_{i=1}^{n} c_i \sum_{j=1}^{n} L_j^2 \left(w_{ij}^{\tau}\right)^2 \left(\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)|z_j(t-s)|\Delta s\right)^2\\
&+ \sum_{i=1}^{n} c_i (1+3\mu(t))\big(\hat{I}_i(t)\big)^2\\
\le\ & \sum_{j=1}^{n}\Bigg(2c_j w_{jj} l_j + \sum_{i=1,i\neq j}^{n} c_i \left(|w_{ji}|^{2-q_{ji}} L_i^{2-r_{ji}} + |w_{ij}|^{q_{ij}} L_j^{r_{ij}}\right) + 2c_j\\
&+ \mu(t)3n\sum_{i=1}^{n} c_i w_{ij}^2 L_j^2 + \big(n + \mu(t)3n\big)\sum_{i=1}^{n} c_i L_j^2 \left(w_{ij}^{\tau}\right)^2 \left(\int_{[0,\infty]_{\mathbb{T}}} k_{ij}(s)\Delta s\right)^2\Bigg) u_j(t)\\
&+ \sum_{i=1}^{n} c_i (1+3\mu(t))\big(\hat{I}_i(t)\big)^2\\
=\ & \sum_{j=1}^{n}\Bigg(2c_j w_{jj} l_j + \sum_{i=1,i\neq j}^{n} c_i \left(|w_{ji}|^{2-q_{ji}} L_i^{2-r_{ji}} + |w_{ij}|^{q_{ij}} L_j^{r_{ij}}\right) + 2c_j\\
&+ \mu(t)3n\sum_{i=1}^{n} c_i w_{ij}^2 L_j^2 + \big(n + \mu(t)3n\big)\sum_{i=1}^{n} c_i L_j^2 \left(w_{ij}^{\tau}\right)^2\Bigg) u_j(t)\\
&+ \sum_{i=1}^{n} c_i (1+3\mu(t))\big(\hat{I}_i(t)\big)^2\\
=\ & -\sum_{j=1}^{n} \delta_j u_j(t) + \sum_{j=1}^{n} c_j (1+3\mu(t))\big(\hat{I}_j(t)\big)^2,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}\delta_j = &-2c_j w_{jj} l_j - \sum_{i=1, i\neq j}^{n} c_i \left(|w_{ji}|^{2-q_{ji}} L_i^{2-r_{ji}} + |w_{ij}|^{q_{ij}} L_j^{r_{ij}} \right) - 2c_j \\ &- \mu(t)3n \sum_{i=1}^{n} c_i w_{ij}^2 L_j^2 - \left(n + \mu(t)3n\right) \sum_{i=1}^{n} c_i L_j^2 \left(w_{ij}^{\tau}\right)^2 > 0, \quad j = 1, 2, \ldots, n.\end{aligned} \tag{6}$$

In what follows, we show that $\lim_{t\to\infty} x(t) = x^*$. On the contrary, we suppose that $x(t) \to x_i^*$ is not established, as $t \to \infty$. Then there exist $\xi > 0$ and a sequence $\{t_m\}_{t=1}^{\infty} \to \infty$ such that

$$\left|g_j\left(x_j(t_m)\right) - g_j\left(x_j^*\right)\right| = |f_j(z_j(t_m))| \geq l_j |z_j(t_m)| \geq \xi > 0, \quad j = 1, 2, \ldots, n.$$

Since $x(t)$ is continuous and each $g_i(x_i(t))$ is bounded, it follows from (4) and $\int_{[0,\infty]_{\mathbb{T}}}(1 + 3\mu(t))|I_i(t) - I_i|^2 \Delta t < +\infty$ that $x(t)$ is absolutely continuous. This implies that each $g_j(x_j(t))$ is also absolutely continuous. Therefore, for any $0 < \varepsilon < \xi$, there exists $\sigma > 0$ such that for all m, whenever $|t - t_m| < \sigma$,

$$|g_j\left(x_j(t)\right) - g_j\left(x_j^*\right)| \geq l_j |z_j(t)| \geq \xi - \varepsilon > 0, \quad j = 1, 2, \ldots, n,$$

$$|z_j(t)| \geq \frac{\xi - \varepsilon}{l_j} \doteq \eta, \quad j = 1, 2 \ldots, n.$$

Let

$$M = \int_{[t_0,\infty]_{\mathbb{T}}} \sum_{j=1}^{n} c_j (1 + 3\mu(t)) |I_j(t) - I_j|^2 \Delta t = \int_{[t_0,\infty]_{\mathbb{T}}} \sum_{j=1}^{n} c_j (1 + 3\mu(t)) |\hat{I}_j(t)|^2 \Delta t.$$

Integrating (5) on both sides from t_0 to t_N, we obtain

$$\begin{aligned} V(t_N) - V(t_0) &\leq -\int_{[t_0,t_N]_{\mathbb{T}}} \sum_{j=1}^{n} \delta_j z_j^2(t) \Delta t + \int_{[t_0,t_N]_{\mathbb{T}}} \sum_{j=1}^{n} c_j (1 + 3\mu(t)) |\hat{I}_j(t)| \Delta t \\ &\leq -\sum_{j=1}^{n} \delta_j \sum_{m=1}^{N} \int_{[t_m - \sigma, t_m]_{\mathbb{T}}} z_j^2(t) \Delta t + M \\ &\leq -\sum_{j=1}^{n} \delta_j \sum_{m=1}^{N} \sigma \eta^2 + M \\ &= -N\sigma\eta^2 \sum_{j=1}^{n} \delta_j + M, \end{aligned} \tag{7}$$

which leads to

$$V(t_N) \leq V(t_0) + M - N\sigma\eta^2 \sum_{j=1}^{n} \delta_j \to -\infty,$$

as $N \to \infty$. This contradicts the positivity of the function $V(t)$. Therefore, $\lim_{t\to\infty} x(t) = x^*$, and the proof is completed. □

By choosing all $c_i = q_{ij} = r_{ij} = 1$ $(i, j = 1, 2)$ in Theorem 3.1, we immediately obtain the following corollaries.

Corollary 3.1. *Assume that* (H_1)-(H_3) *hold and* $w_{jj} < 0$, *the input pair* $(I(t), I)$ *satisfies* $\int_{[0,\infty]_{\mathbb{T}}} (1 + 3\mu(t))|I_i(t) - I_i|^2 \Delta t < +\infty$ *and the following condition*

$$\begin{aligned} &- 2w_{jj}l_j - \sum_{i=1, i\neq j}^{n} \big(|w_{ji}|L_i + |w_{ij}|L_j\big) - 2 - \mu(t)3n \sum_{i=1}^{n} w_{ij}^2 L_j^2 \\ &\qquad\qquad - \big(n + \mu(t)3n\big) \sum_{i=1}^{n} L_j^2 \left(w_{ij}^{\tau}\right)^2 > 0, \quad j = 1, 2, \dots, n, \end{aligned}$$

holds, then the network (1) is input-to-state convergent with respect to $(I(t), I)$.

Suppose $\mathbb{T} = \mathbb{R}$, the network model (1) becomes

$$\frac{dx_i(t)}{dt} = WG(x(t)) + \int_0^{+\infty} W^{\tau} K(s) G\big(x(t-s)\big) ds + I(t), \quad t \geq 0, \tag{8}$$

then $\mu(t) = 0$, and we let $q_{ij} = r_{ij} = 1$ $(i, j = 1, 2)$. Then we obtain corollary as follows.

Corollary 3.2. *Assume that* (H_1)-(H_3) *hold and* $w_{jj} < 0$, $\int_0^{\infty} |I_i(t) - I_i|^2 dt < +\infty$ *and the following condition*

$$- 2c_j w_{jj} l_j > \sum_{i=1, i\neq j}^{n} c_i \big(|w_{ji}|L_i + |w_{ij}|L_j\big) + 2c_j + n \sum_{i=1}^{n} c_i L_j^2 \left(w_{ij}^{\tau}\right)^2, \quad j = 1, 2, \dots, n,$$

holds, and then the network (8) is input-to-state convergent with respect to $(I(t), I)$.

Suppose $\mathbb{T} = \mathbb{Z}$, the network model (1) becomes

$$x_i(n+1) = x_i(n) + WG(x(n)) + \sum_{j=0}^{+\infty} W^{\tau} K(j) G\big(x(n-j)\big) + I(n), \quad n \in \mathbb{Z}, \tag{9}$$

then $\mu(t) = 1$, and we also let $q_{ij} = r_{ij} = 1$ $(i, j = 1, 2)$. Then we obtain corollary as follows.

Corollary 3.3. *Assume that* (H_1)-(H_3) *hold and* $w_{jj} < 0$, $\sum_{n=0}^{+\infty} |I_i(n) - I_i| < +\infty$ *and the following condition*

$$\begin{aligned} -2c_j w_{jj} l_j > &\sum_{i=1, i\neq j}^{n} c_i \big(|w_{ji}|L_i + |w_{ij}|L_j\big) + 2c_j \\ &+ 3n \sum_{i=1}^{n} c_i w_{ij}^2 L_j^2 + 4n \sum_{i=1}^{n} c_i L_j^2 \left(w_{ij}^{\tau}\right)^2, \quad j = 1, 2, \dots, n, \end{aligned}$$

holds, and then the network (9) is input-to-state convergent with respect to $(I(n), I)$.

4. **Numerical Example.**

In this section, two examples are shown to verify the effectiveness of our results.

Example 4.1. *Consider the following network with delays on time scale* $\mathbb{T} = \mathbb{R}$ *with*

$$W = \begin{pmatrix} -10 & 1 \\ -1 & -10 \end{pmatrix}, \quad W^{\tau} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad I(t) = \begin{pmatrix} 2 + \frac{1}{1+t} \\ 1 + \frac{1}{1+t^2} \end{pmatrix},$$

choose $c_i = 1$, $l_i = \frac{1}{2}$, $L_i = q_{ij} = r_{ij} = 1$ $(i, j = 1, 2)$. *From Corollary 3.2 we can get*

$$- 2c_1 w_{11} l_1 = 10 > c_2 \big(|w_{12}|L_2 + |w_{21}|L_1\big) + 2c_1 + 2\left[c_1 L_1^2 \left(w_{11}^{\tau}\right)^2 + c_2 L_1^2 \left(w_{21}^{\tau}\right)^2\right] = 8,$$

$$- 2c_2 w_{22} l_2 = 10 > c_1 \big(|w_{21}|L_1 + |w_{12}|L_2\big) + 2c_2 + 2\left[c_1 L_2^2 \left(w_{12}^{\tau}\right)^2 + c_2 L_2^2 \left(w_{22}^{\tau}\right)^2\right] = 8,$$

and

$$\int_0^{+\infty} (I_1(t) - I_1)^2 \Delta t = \int_0^{+\infty} \left(\frac{1}{1+t}\right)^2 \Delta t = 1 < +\infty,$$

$$\int_0^{+\infty} (I_2(t) - I_2)^2 \Delta t = \int_0^{+\infty} \left(\frac{1}{1+t^2}\right)^2 \Delta t = \frac{\pi}{4} < +\infty.$$

We can easily verify that the conditions of Corollary 3.2 are all satisfied, respectively, then the network (1) is input-to-state convergent with respect to $(I(t), I)$.

Example 4.2. *Consider the following network with delays on time scale* $\mathbb{T} = \frac{7}{100}\mathbb{Z}$ *with*

$$W = \begin{pmatrix} -0.25 & 0.01 \\ 0.01 & -0.25 \end{pmatrix}, \quad W^\tau = \begin{pmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{pmatrix}, \quad I(t) = \begin{pmatrix} 2 + \frac{1}{1+t} \\ 1 + \frac{1}{1+t^2} \end{pmatrix}.$$

Then $\mu(t) = 0.07$. *Choose* $c_i = 1$, $l_i = 7$, $L_i = 7.1$, $q_{ij} = r_{ij} = 1$ $(i, j = 1, 2)$. *From Corollary 3.1, we can get*

$$\begin{aligned} -2c_1 w_{11} l_1 = 3.5 > c_2\big(|w_{12}|L_2 + |w_{21}|L_1\big) + 2c_1 + 3 \times 2\mu(t) \left[c_1 L_1^2 w_{11}^2 + c_2 L_1^2 w_{21}^2\right] \\ + \big(2 + 3 \times 2\mu(t)\big) \left[c_1 L_1^2 \left(w_{11}^\tau\right)^2 + c_2 L_1^2 \left(w_{21}^\tau\right)^2\right] = 3.4918, \end{aligned}$$

and

$$\begin{aligned} -2c_2 w_{22} l_2 = 3.5 > c_1\big(|w_{21}|L_1 + |w_{12}|L_2\big) + 2c_2 + 3 \times 2\mu(t) \left[c_1 L_2^2 w_{12}^2 + c_2 L_2^2 w_{22}^2\right] \\ + \big(2 + 3 \times 2\mu(t)\big) \left[c_1 L_2^2 (w_{12}^\tau)^2 + c_2 L_2^2 (w_{22}^\tau)^2\right] = 3.4918, \end{aligned}$$

and

$$\int_0^{+\infty} (I_1(t) - I_1)^2 \Delta t = \int_0^{+\infty} \left(\frac{1}{1+t}\right)^2 \Delta t = 1 < +\infty,$$

$$\int_0^{+\infty} (I_2(t) - I_2)^2 \Delta t = \int_0^{+\infty} \left(\frac{1}{1+t^2}\right)^2 \Delta t = \frac{\pi}{4} < +\infty.$$

We can easily verify that the conditions of Corollary 3.1 are all satisfied, respectively, then the network (1) is input-to-state convergent with respect to $(I(t), I)$.

5. **Conclusions.** This paper incorporates time scales $\mathbb{T}$ into the input-to-state convergence (ISC) analysis for a class of neural networks with distributed delays. By using the time scale calculus theory and constructing appropriate Lyapunov functions, some new criteria are given to the input-to-state convergence of the neural networks (1) on time scales. Our new results are general which unify continuous-time with corresponding discrete-time situations and extend the existing relevant input-to-state convergence results in the literature to cover more general neural networks, and they can also be easily checked by simple algebraic method. Several examples are also given to illustrate the effectiveness of the input-to-state convergence criteria.

We would like to point out that it is possible to generalize our main results to more complex neural networks, such as neural networks with time-varying delays [22], dynamical neural networks [23], and neural networks with unbounded time-varying delays [24]. The results will be carried out in the near future.

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