International Journal of Innovative
Computing, Information and Control
Volume **13**, Number **2**, April **2017**

ICIC International ©2017 ISSN 1349-4198
pp. 485–507

VARIABLE STRUCTURE ADAPTIVE POLE PLACEMENT CONTROL FOR UNCERTAIN SYSTEMS: AN INTERVAL APPROACH

Francisco Silva Junior[1], Josenalde Oliveira[2,4] and Aldayr Araújo[3]

[1]Campus Natal Zona Norte
Federal Institute of Education, Science and Technology of Rio Grande do Norte
Natal, RN, 59112-490, Brazil
francisco.junior@ifrn.edu.br

[2]Agricultural School of Jundiaí
Federal University of Rio Grande do Norte
Macaíba, RN, 59280-000, Brazil
josenalde@eaj.ufrn.br

[3]Department of Electrical Engineering
Federal University of Rio Grande do Norte
Natal, RN, 59078-970, Brazil
aldayr@dca.ufrn.br

[4]INESC TEC – INESC Technology and Science
Campus da FEUP, 4200-465 Porto, Portugal

Received October 2016; revised February 2017

Abstract. *This paper presents the design of a robust pole placement controller for linear and time invariant systems described by models whose parameters are unknown but with known bounds, using only plant input and output signals. The main objective is to locate the closed-loop poles within a region specified by an interval characteristic polynomial, chosen based on performance specifications and whose stability is guaranteed by Kharitonov's theorem. The procedure to find a closed interval for each controller parameter is formulated as a nonlinear programming (NLP) problem and switching laws based on the variable structure adaptive pole placement control (VS-APPC) are used to estimate the unknown plant parameters. This innovative strategy gives a fast and non-oscillatory transient, smooth control signal and robustness to large model parameter variations. Moreover, the constraints imposed by the interval controller meet the closed-loop performance requirements. Simulation results are presented for second order nonminimum phase plants, to illustrate the properties of the proposed technique.*
Keywords: Pole placement, Adaptive control, Robust control, Interval plants, Variable structure systems, Sliding mode, Kharitonov's theorem, Nonminimum phase plants

1. **Introduction.** Pole placement control (PPC) is a well-known and accepted control strategy in the literature, since it provides satisfactory results in closed-loop, both on transient and at steady state, when applied to linear time invariant (LTI) plants. Several performance requirements can be satisfied by using dynamic output feedback to properly place the closed-loop poles in the complex plane. As presented in [1, 2], the solution for the classical PPC can be reduced under appropriate conditions to the solution of a *Diophantine Equation*, whose matricial formulation takes the form of a linear system, $Ax = b$, where A is the Sylvester matrix associated with a particular system, x is the vector of controller coefficients to be designed and b is the vector of the coefficients of a characteristic polynomial. On the other hand, PPC robustness features depend on the accuracy of the available plant model, and therefore, the selection of the closed-loop poles

[3]. For uncertain systems, the arbitrary choice of the closed-loop poles may result in a poor controller design [4].

A common strategy is the inclusion of an estimation stage for the plant parameters, usually with integral-based adaptive laws (gradient method, least-squares, etc.). This combination leads to the scheme so-called *adaptive pole placement control* (APPC), which is mainly developed in the indirect approach, where the control signal is a function of the plant parameters estimates [5, 6]. Unlike *model reference adaptive control* (MRAC), which is restricted to minimum phase plants (MPP), APPC is also suitable to nonminimum phase plants (NPP), since no cancellation of zeros and poles is involved. Zeros in the right-half (ZRH) of the complex s-plane cause a particular behavior in closed-loop and may generate practical implications. For example, the attraction of poles to zeros limits the magnitude of the feedback gain, in the root locus analysis [7], which implies a limitation on the robustness of the closed-loop system. In an asymptotically stable transfer function, each zero has a specific effect on the asymptotic response for certain inputs. Some impacts of ZRH are evident in the step response, as initial undershoot, zero crossings and direction reversals [8]. Uncertain ZRH in adaptive control increases the difficulty to get robustness, since zero-pole cancellation is ineffective and may lead to unbounded signals. On the other hand, the consideration of only minimum phase models makes the control design rather conservative, since a simple first order Padé approximation of a time delay results in ZRH for actual systems, such as a first order plus time delay (FOPDT) model. This fact is even more highlighted for large parameter variations, for example in water level control along irrigation channels [9]. For this problem, a second order plus time delay $\frac{K}{(T_1s+1)(T_2s+1)}e^{-\tau s}$ was considered, with each parameter constrained in a large interval: $0.01 \leq K \leq 0.1$; $500s \leq T_1 \leq 15000s$; $10s \leq T_2 \leq 300s$ and $300s \leq \tau \leq 360s$.

Usually LTI models treated in practical situations are approximations which are made to simplify the design and analysis of the control system, since most real systems are non-linear. From a nonlinear system, an LTI model can be obtained by fixing the operating point and linearizing the system equations on it. As the operating point changes, the parameters of the linear approximation also change. Therefore, there is significant uncertainty about the "true" model of the plant and the controller must be designed to stabilize the system for the entire range of expected variations in the plant parameters. In many real situations, uncertainties in plant parameters are related to the physical properties of the components that comprise the system. In addition, other perturbations of the plant model must also be tolerated without disturbing the stability of the closed-loop. These unstructured perturbations arise typically from truncating a complex model by retaining only some of the dominant modes, which usually lie in the low frequency range [10]. The tolerance of these cases of uncertainty is the problem of robust stability. The term *robust parametric stability* refers to the ability of a control system to maintain stability despite large variations of the plant parameters. Therefore, the main practical motivation of this paper is to propose a robust adaptive controller to treat single-input single-output (SISO) LTI systems with known parametric uncertainties, including nonminimum phase systems. With this focus, the technique presented in this paper can be implemented in a wide range of real practical situations.

With the advent of the celebrated Kharitonov's theorem [11], which showed that the Hurwitz stability of an entire family of polynomials of fixed but arbitrary degree corresponding to an entire box in the coefficient space could be verified by checking the stability of four prescribed vertex polynomials [10], a large number of significant works on the study of robust stability under real parametric uncertainties have been presented [12, 13, 14, 15, 16, 17, 18]. Beside this, efficient computational solutions and results of the theory of linear interval equations [19, 20, 21, 22, 23, 24] have enabled the construction of robust controllers for interval plants without too much difficulty. The *interval analysis* has become a useful tool for dealing with many important problems in control systems area, which has resulted in the increasing number of publications on its applications in the last decades [25, 26, 27, 28, 29, 30].

When an uncertain model is considered, with each parameter constrained within a known interval, the APPC scheme can be extended to locate the closed-loop poles in a suitable region of the complex s-plane, and not to specific points. This behavior is specified by the roots of an interval characteristic polynomial, and therefore the required performance is more insensitive to parameter variations and disturbances affecting the actual system.

This paper is motivated by some previous works that combined traditional adaptive control schemes with a nonlinear control technique called *variable structure control* (VSC), which is based on the relay system theory [31], in order to improve the transient performance and robustness, thus exploiting the best of each technique. Following this concept, it was proposed the *variable structure model reference adaptive control* (VS-MRAC) [32], where only the plant input and output are measurable and switching laws are used to replace the integral adaptive laws of MRAC scheme [33], which in its original description presents slow and oscillatory transient. The VS-MRAC scheme provides interesting features of robustness as well as a good transient performance [34]. Recently, other studies were based on the original VS-MRAC scheme to propose improvements to the project, mainly modifications to get a smooth control signal and in such a way that chattering can be reduced or eliminated. The *dual mode adaptive robust controller* (DMARC) [35] proposes a smooth transition between VS-MRAC (on transient stage) and MRAC (at steady-state stage) schemes. These controllers are based on direct adaptive control [5, 6]. Both controllers were also designed based on *indirect* adaptive control approach, IVS-MRAC [36] and IDMRAC [37]. The basic idea is the possibility of an easier design process, since bounds for the relays amplitudes and initial conditions are easily calculated due to their direct relation with the physical plant parameters, reflected in the model.

This paper presents a VS-APPC [38, 39, 40] strategy for SISO LTI uncertain systems modeled by proper transfer functions with unknown coefficients, but constrained to known real intervals, called *interval variable structure adaptive pole placement control* (IVS-APPC). Similar conditions were treated in [4, 41], where the main idea is to design a controller which attempts to keep the closed-loop poles within a convex region specified by an interval characteristic polynomial, whose stability is guaranteed by Kharitonov's theorem. Likewise VS-APPC scheme, here the estimates of the unknown plant parameters are generated by switching laws, instead of the adaptive laws used in the APPC design. Moreover, the VS-APPC design is extended in the sense that, given an interval plant and an interval characteristic polynomial, it is possible to obtain an interval controller as well, which meets the control objective, i.e., we can obtain upper and lower limits for each coefficient of the control law, by solving an NLP problem, which ensures that the closed-loop poles are placed in a desired region, even in the presence of large model parameter variations.

The rest of this paper is organized as follows. Section 2 presents the problem formulations and main assumptions. Sections 3 and 4 present the necessary steps to design the adaptive laws to estimate the plant parameters. Section 5 is dedicated to the pole placement control design for interval plants, using the polynomial approach. Section 6 presents the entire IVS-APPC design, and numerical examples to illustrate the properties of the proposal are presented in Section 7. Finally, Section 8 concludes this paper.

Notation: A closed real interval is represented as $[a] = [a^-, a^+]$, where the endpoints a^- and a^+ are known real quantities. The center or nominal value of $[a]$ is represented as $a^c = \frac{1}{2}(a^+ + a^-)$ and the radius is represented as $a^r = \frac{1}{2}(a^+ - a^-)$. For a polynomial $A(s)$ of degree n, $[A(s)]$ represents the correspondent interval polynomial of $A(s)$, i.e., each interval coefficient $[a_i]$ of $[A(s)]$ contains the correspondent coefficient a_i of $A(s)$, and therefore we can say that $a_i \in [a_i]$ for $i = 0, 1, \ldots, n$, and $A(s) \in [A(s)]$. An interval matrix is defined as $[A] := \{[a_{ij}]\}$, where $[a_{ij}] = \left[a_{ij}^-, a_{ij}^+\right]$, for each i, j. Another representation for $[A]$ is $[A] = [A^-, A^+] = \{A : A^- \leq A \leq A^+\}$, where $A^- := \left\{\left[a_{ij}^-\right]\right\}$ and $A^+ := \left\{\left[a_{ij}^+\right]\right\}$. The center and radius matrices of $[A]$ are given by $A_c := \frac{1}{2}(A^+ + A^-)$ and $A_r := \frac{1}{2}(A^+ - A^-)$, and consequently $[A] = [A_c - A_r, A_c + A_r]$. Given an interval vector $[\sigma] \in \mathbb{R}^n$, $\mathcal{V}([\sigma])$ denotes the set of all 2^n vertices of $[\sigma]$, then each element i of the vertice $v_j \in \mathcal{V}([\sigma])$ is the lower or upper limit of $[\sigma_i]$, for $i = 1, \ldots, n$ and $j = 1, \ldots, 2^n$.

2. **Problem Statement and Preliminaries.**

Let us consider the SISO LTI plant described by the transfer function

$$y = G(s)u, \quad G(s) = \frac{Z(s)}{R(s)} \tag{1}$$

where $Z(s)$ is a polynomial of degree m and $R(s)$ is a monic polynomial of degree n, with $m < n$, which are represented as

$$Z(s) = \beta_{n-1}s^{n-1} + \ldots + \beta_1 s + \beta_0 \tag{2}$$

$$R(s) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0 \tag{3}$$

where the coefficients $\beta_{n-1}, \ldots, \beta_1, \beta_0$ and $\alpha_{n-1}, \ldots, \alpha_1, \alpha_0$ are unknown. If $Z(s)$ has degree $m < n-1$, then the coefficients β_i, for $i = n-1, n-2, \ldots, m+1$ are equal to zero.

Let us assume that the unknown coefficients of $G(s)$ have bounded uncertainties, i.e., are constrained to known closed intervals, such that $\beta_i \in [\beta_i] = \left[\beta_i^-, \beta_i^+\right]$ and $\alpha_i \in [\alpha_i] = \left[\alpha_i^-, \alpha_i^+\right]$, for $i = 0, 1, \ldots, n-1$. Therefore, we can represent the interval plant $[G(s)]$, such that $G(s) \in [G(s)]$, as follows:

$$[G(s)] = \frac{[Z(s)]}{[R(s)]} = \frac{[\beta_{n-1}]\, s^{n-1} + \ldots + [\beta_1]s + [\beta_0]}{s^n + [\alpha_{n-1}]s^{n-1} + \ldots + [\alpha_1]s + [\alpha_0]} \tag{4}$$

The objective of this paper is to design a robust adaptive control law that keeps the closed-loop poles within a convex region specified by the roots of a monic Hurwitz interval characteristic polynomial $[B(s)]$, chosen based on the closed-loop performance requirements and whose stability is guaranteed by Kharitonov's theorem [11], despite the presence of certain parameter variations, and the plant output y follows a reference signal $r \in \mathcal{L}_\infty$ that is assumed to satisfy:

$$Q_m(s)r = 0 \tag{5}$$

where $Q_m(s)$ is a known monic polynomial of degree q, called *internal model* of r [42]. For example, when $r =$ constant, $Q_m(s) = s$. When $r = t$, $Q_m(s) = s^2$ and when $r = A\sin(\omega_0 t)$ for some constants A and ω_0, then $Q_m(s) = s^2 + \omega_0^2$.

The properties of the system associated with $G(s)$ depend very much on the properties of $Z(s)$ and $R(s)$. Therefore, to meet the control objective, the following assumptions about the plant are necessary, in a manner similar to that of traditional APPC [5]:

A1: $[R(s)]$ is a monic interval polynomial whose degree n is known.
A2: $[Z(s)]$ is an interval polynomial whose degree $m < n$ is known.
A3: $[Z(s)]$ and $[R(s)]$ are coprime.
A4: $Q_m(s)$ and $[Z(s)]$ are coprime.

The fact that $[R(s)]$ is monic and has known degree, described in assumption **A1**, together with assumption **A2** indicates that at most $2n$ parameters are required to uniquely specify the I/O properties of (1). The parameterization of the plant considered in this paper and presented in Section 3 is based on assumption **A1**. Moreover, the knowledge of n is used to choose the order of the controller polynomials and, consequently, the order of the characteristic polynomial, as will be presented in Section 5.

Assumptions **A3** and **A4** are related to the coprimeness of polynomials. Two polynomials are said to be coprime (or relatively prime) if they have no common factors other than a constant [5]. This property is necessary for $[Z(s)]$ and $[R(s)]$, and consequently for $Z(s)$ and $R(s)$, so that the calculation of the adaptive controller parameters at each instant of time t by using Diophantine equation has a unique solution, as shown by the following properties.

Lemma 2.1. *(Bezout Identity) Two polynomials $a(s)$ and $b(s)$ are coprime if and only if there exist polynomials $c(s)$ and $d(s)$ such that*

$$c(s)a(s) + d(s)b(s) = 1$$

Proof: For a proof of Lemma 2.1, see [43, 44].

Theorem 2.1. *If $a(s)$ and $b(s)$ are coprime and of degree n_a and n_b, respectively, where $n_a > n_b$, then for any given arbitrary polynomial $a^*(s)$ of degree $n_{a^*} \geq n_a$, the polynomial equation*

$$a(s)l(s) + b(s)p(s) = a^*(s)$$

known as Diophantine equation, has a unique solution $l(s)$ and $p(s)$ whose degrees n_l and n_p, respectively, satisfy the constraints $n_p < n_a$, $n_l \leq \max(n_{a^*} - n_a, n_b - 1)$.

Proof: The proof of Theorem 2.1 can be found in [5].

Therefore, the fact that $[Z(s)]$ and $[R(s)]$ are coprime described in assumption **A3**, together with Theorem 2.1 makes it possible to choose an arbitrary pole placement in the complex s-plane and implies that the solution of the adaptive controller polynomials at each instant of time t is unique, provided that restrictions are imposed on the degrees of controller polynomials.

Assumption **A4** aims to guarantee that $[Z(s)]$ and $Q_m(s)[R(s)]$ are coprime so that the property presented in Theorem 2.1 is satisfied. For example, if y is required to track the reference signal $r = 2 + \sin(2t)$, then $Q_m(s) = s\,(s^2 + 4)$ and, therefore, $[Z(s)]$ should not have s or $s^2 + 4$ as a factor. The idea behind the internal model principle is that by including the factor $1/Q_m(s)$ in the compensator $C(s)$, we can null the effect of r on the tracking error $e_1 = y - r$ [5]. This principle will be considered in the control design presented in Section 5.

Let us now consider two interval polynomials given by $[C(s)]$ and $[D(s)]$, such that

$$[C(s)] = \sum_{i=0}^{n} \left[c_i^-, c_i^+\right] s^i, \quad [D(s)] = \sum_{i=0}^{n} \left[d_i^-, d_i^+\right] s^i$$

whose *exposed edges* [10] are represented as

$$C^-(s) = \sum_{i=0}^{n} c_i^- s^i, \quad C^+(s) = \sum_{i=0}^{n} c_i^+ s^i$$

$$D^-(s) = \sum_{i=0}^{n} d_i^- s^i, \quad D^+(s) = \sum_{i=0}^{n} d_i^+ s^i$$

with coefficients lumped in the vectors

$$c^- = \left[c_n^-, c_{n-1}^-, \ldots, c_1^-, c_0^-\right]^\top, \quad c^+ = \left[c_n^+, c_{n-1}^+, \ldots, c_1^+, c_0^+\right]^\top$$
$$d^- = \left[d_n^-, d_{n-1}^-, \ldots, d_1^-, d_0^-\right]^\top, \quad d^+ = \left[d_n^+, d_{n-1}^+, \ldots, d_1^+, d_0^+\right]^\top$$

Next, we have some important properties of robust nonsingularity of interval matrices and robust coprimeness of interval polynomials.

Definition 2.1. [45] *The spectral set of* $[C(s)]$*, denoted as* $\mathcal{S}([C(s)])$*, is defined as the set of all roots of* $[C(s)]$*, when the coefficients of* $[C(s)]$ *assume values in* $[c] = [c^-, c^+]$.

Theorem 2.2. *(Edge Theorem) The spectral set* $\mathcal{S}[C(s)]$ *of an interval polynomial* $[C(s)]$ *is limited by the roots of their edges* $C^-(s)$ *and* $C^+(s)$.

Proof: Readers are referred to [10].

Lemma 2.2. [45] *Two interval polynomials* $[C(s)]$ *and* $[D(s)]$ *are robustly coprime if* $\mathcal{S}([C(s)]) \cap \mathcal{S}([D(s)]) = \varnothing$.

A trial-and-error visual approach based on the edge theorem to determine the robust coprimeness of interval polynomials is suggested in [10]. A simple sufficient condition for robust coprimeness based on interval analysis results is presented below.

Lemma 2.3. [45] *Two interval polynomials* $[C(s)]$ *and* $[D(s)]$ *are robustly coprime if their interval Sylvester resultant* $[A]$ *associated is robustly nonsingular.*

Proof: Suppose that $[A]$ is robustly nonsingular. Then all Sylvester resultants in $[A^-, A^+]$ are nonsingular, implying that $[C(s)]$ and $[D(s)]$ are robustly coprime.

Lemma 2.4. [45] *A square interval matrix* $[A]$ *is said to be robustly nonsingular (or regular) if all matrices* $A \in [A]$ *are nonsingular.*

A sufficient condition for checking the robust nonsingularity of an interval matrix $[A]$, with center matrix A_c and radius matrix A_r, described in [46], is given by

$$\rho\left(|A_c^{-1}|A_r\right) < 1 \tag{6}$$

where $\rho(A) := \max\{|\lambda| : det(\lambda I - A) = 0\}$.

The problem of determining the robust nonsingularity of an interval matrix $[A] \in \mathbb{R}^{2n\times 2n}$ has been extensively addressed in [47]. Unfortunately, all known necessary and sufficient conditions for robust nonsingularity present exponential behavior, i.e., it is necessary to solve at least 2^{2n} problems of some sort. Some considerations and algorithms on necessary and sufficient conditions that deal with solving the problem of determining whether an interval matrix is robustly nonsingular or if two polynomials are robustly coprime are presented in [45].

3. **Parametric Model.** The first step for designing on-line parameter estimators is to select an appropriate parameterization of the plant model. The plant model is parameterized with respect to some unknown parameter vector θ^*. In this section, we consider a very useful plant parameterization where parameters to be estimated are lumped together and separated from I/O signals of the plant [5]. In this case, θ^* represents the unknown coefficients of the numerator and denominator of the plant model transfer function (1). For the estimation and control problem treated in this paper, this type of plant parameterization is more convenient than others.

We can express (1) as an nth-order differential equation given by

$$y^{(n)} = -\alpha_{n-1}y^{(n-1)} - \ldots - \alpha_1 y^{(1)} - \alpha_0 y + \beta_{n-1}u^{(n-1)} + \ldots + \beta_1 u^{(1)} + \beta_0 u \tag{7}$$

where $y^{(i)}(t) \triangleq \frac{d^i}{dt^i}y(t)$ and $u^{(i)}(t) \triangleq \frac{d^i}{dt^i}u(t)$.

Equation (7) may be rewritten in the following compact form:

$$y^{(n)} = \theta_\beta^{*\top}\psi_u + \theta_\alpha^{*\top}\psi_y \tag{8}$$

where

$$\theta_\beta^* = [\beta_{n-1}, \dots, \beta_1, \beta_0]^\top, \quad \theta_\alpha^* = [\alpha_{n-1}, \dots, \alpha_1, \alpha_0]^\top$$

and

$$\psi_u = \left[u^{(n-1)}, u^{(n-2)}, \dots, u^{(1)}, u\right]^\top, \quad \psi_y = \left[-y^{(n-1)}, -y^{(n-2)}, \dots, -y^{(1)}, -y\right]^\top$$

The coefficient vector of $G(s)$ is defined as

$$\theta^* = \left[\theta_\beta^{*\top}, \theta_\alpha^{*\top}\right]^\top = [\beta_{n-1}, \dots, \beta_0, \alpha_{n-1}, \dots, \alpha_0]^\top \in \mathbb{R}^{2n} \tag{9}$$

and the coefficient vector of $[G(s)]$ is defined as

$$[\theta^*] = \left[[\theta_\beta^*]^\top, [\theta_\alpha^*]^\top\right]^\top = [[\beta_{n-1}], \dots, [\beta_0], [\alpha_{n-1}], \dots, [\alpha_0]]^\top \in \mathbb{R}^{2n} \tag{10}$$

Let us now consider the monic Hurwitz polynomial of degree n

$$\Lambda(s) = s^n + \lambda_{n-1}s^{n-1} + \ldots + \lambda_1 s + \lambda_0$$

whose coefficients are the elements of the vector $\lambda = [\lambda_{n-1}, \dots, \lambda_1, \lambda_0]^\top$. Therefore, adding the term $(\Lambda(s) - s^n)y$ in both sides of (8), we obtain

$$y^{(n)} + \lambda_{n-1}y^{(n-1)} + \dots + \lambda_0 y = \theta_\beta^{*\top}\psi_u + \theta_\alpha^{*\top}\psi_y + \lambda_{n-1}y^{(n-1)} + \dots + \lambda_0 y$$

which can be rewritten as

$$y = W(s)\left(\theta_\beta^{*\top}\psi_u + \theta_\alpha^{*\top}\psi_y - \lambda^\top\psi_y\right) \tag{11}$$

where

$$W(s) = \frac{1}{\Lambda(s)} = \frac{1}{s^n + \lambda_{n-1}s^{n-1} + \ldots + \lambda_1 s + \lambda_0}$$

Let us assume that θ_β^*, θ_α^* and λ are constant vectors. Multiplying both sides of (11) by a polynomial $L(s)$ and rearranging the equation, we rewrite (11) in the form of the following linear parametric model, which will be used in the next sections:

$$y = W(s)L(s)\left(\theta_\beta^{*\top}\phi_u + \theta_\alpha^{*\top}\phi_y - \lambda^\top\phi_y\right) \tag{12}$$

where

$$\phi_u = L^{-1}(s)\psi_u, \quad \phi_y = L^{-1}(s)\psi_y$$

and the polynomial $L(s)$ has to be chosen so that $L^{-1}(s)$ is a proper stable transfer function and $W(s)L(s)$ is a proper and strictly positive real (SPR) transfer function. For example, when

$$W(s) = \frac{1}{(s+1)(s+3)(s+5)}$$

we can choose $L(s)$ as

$$L(s) = (s+2)(s+4)$$

which implies that $W(s)L(s)$ is an SPR transfer function. The motivation for introducing $L(s)$ in the parametric model is to be able to use the SPR-Lyapunov design approach to construct the adaptive laws for estimating θ_β^* and θ_α^* [5], as will be presented in the next section. Therefore, the next step of the parameter estimation design, fundamental part in the adaptive control design, starts from the parametric model built in (12).

4. **Parameter Estimation.** In this section, we present the second step of the on-line estimation procedure: to generate on-line estimates of the plant parameters by using appropriate adaptive laws, which are designed to minimize the error between the system response $y(t)$ and its estimate $\hat{y}(t)$ which is generated by a parameterized model whose format is the same as the parametric model in (12). The estimates vector $\theta(t)$ is adjusted continuously so that $\hat{y}(t)$ approaches $y(t)$ as t increases. The proposal presented in this paper generates the plant parameters estimates using switching laws [39], instead the traditional integral laws [33], in order to add the characteristics of VSC [31] to the adaptive control design.

The design approach known as SPR-Lyapunov is used in this section to design adaptive laws for estimating the plant parameters in the parametric model (12). This approach allows to treat the parameter estimation part independently of the control part, which allows to combine various estimation and control designs, splitting the complexity of each part and thus simplifying the analysis and the design of adaptive control schemes.

The parametric model in (1) may be rewritten in many different forms giving rise to different equations for generating an estimate for $y(t)$. By considering the parameterization of the plant given by (12), let us now consider an equation which has the same format of (12) to estimate $y(t)$, but with estimates instead of the true parameters, known as the *series-parallel model* and widely used for parameter estimation [48], given by

$$\hat{y} = W(s)L(s)\left(\hat{\theta}_\beta^\top \phi_u + \hat{\theta}_\alpha^\top \phi_y - \lambda^\top \phi_y\right) \tag{13}$$

where $\hat{\theta}_\beta$ and $\hat{\theta}_\alpha$ are the on-line estimates of θ_β^* and θ_α^*, respectively, and form the estimates vector $\hat{\theta} = \left[\hat{\theta}_\beta^\top, \hat{\theta}_\alpha^\top\right]^\top \in \mathbb{R}^{2n}$. The estimation error, which reflects the parametric uncertainty, is defined as

$$\epsilon_1 = y - \hat{y} \tag{14}$$

Therefore, substituting for y and $\hat{y}$ in (14), we obtain

$$\epsilon_1 = W(s)L(s)\left(\theta_\beta^{*\top} \phi_u + \theta_\alpha^{*\top} \phi_y - \lambda^\top \phi_y - \hat{\theta}_\beta^\top \phi_u - \hat{\theta}_\alpha^\top \phi_y + \lambda^\top \phi_y\right)$$

i.e.,

$$\epsilon_1 = W(s)L(s)\left(-\tilde{\theta}_\beta^\top \phi_u - \tilde{\theta}_\alpha^\top \phi_y\right) \tag{15}$$

where $\tilde{\theta}_\beta = \hat{\theta}_\beta - \theta_\beta^*$ and $\tilde{\theta}_\alpha = \hat{\theta}_\alpha - \theta_\alpha^*$, or the following compact model:

$$\epsilon_1 = W(s)L(s)\left(-\tilde{\theta}^\top \phi\right) \tag{16}$$

where $\tilde{\theta} = \left[\tilde{\theta}_\beta^\top, \tilde{\theta}_\alpha^\top\right]^\top$ and $\phi = \left[\phi_u^\top, \phi_y^\top\right]^\top$.

A state-space representation of (16) is given by

$$\begin{cases} \dot{\epsilon} = A_e \epsilon + B_e\left(-\tilde{\theta}^\top \phi\right) \\ \epsilon_1 = C_e^\top \epsilon \end{cases} \tag{17}$$

where A_e, B_e and C_e are the matrices associated with the transfer function $W(s)L(s) = C_e^\top (sI - A_e)^{-1} B_e$.

By using the Kalman-Yakubovich-Popov lemma [49, 50], the SPR property of $W(s)L(s)$ guarantees that

$$\begin{cases} A_e^\top P_e + P_e A_e = -2Q_e \\ P_e B_e = C_e \end{cases} \tag{18}$$

where $P_e = P_e^\top > 0$ for some matrix $Q_e = Q_e^\top > 0$.

Let us now consider the following Lyapunov-like function for the differential Equation (17):

$$V(\epsilon) = \frac{\epsilon^\top P_e \epsilon}{2} \tag{19}$$

The time derivative $\dot{V}$ along the solution of (17) is given by

$$\begin{aligned}
\dot{V}(\epsilon) &= \frac{1}{2}\left(\dot{\epsilon}^\top P_e \epsilon + \epsilon^\top P_e \dot{\epsilon}\right) \\
&= \frac{1}{2}\left(\left(A_e\epsilon + B_e\left(-\tilde{\theta}^\top \phi\right)\right)^\top P_e \epsilon + \epsilon^\top P_e \left(A_e \epsilon + B_e\left(-\tilde{\theta}^\top\phi\right)\right)\right) \\
&= \frac{1}{2}\left(\epsilon^\top\left(P_e A_e + A_e^\top P_e\right)\epsilon + 2\epsilon^\top P_e B_e\left(-\tilde{\theta}^\top\phi\right)\right) \\
&= -\epsilon^\top Q_e \epsilon + \epsilon^\top P_e B_e \left(-\tilde{\theta}^\top \phi\right)
\end{aligned} \tag{20}$$

It follows from (18) that $P_eB_e = C_e$, which implies that $\epsilon^\top P_e B_e = \epsilon^\top C_e = \epsilon_1$. Therefore, (20) can be written as

$$\begin{aligned}
\dot{V}(\epsilon) &= -\epsilon^\top Q_e \epsilon - \epsilon_1 \tilde{\theta}^\top \phi \\
&= -\epsilon^\top Q_e \epsilon - \sum_{i=1}^{2n}\left[\hat{\theta}_i - \theta_i^*\right]\epsilon_1 \phi_i \\
&= -\epsilon^\top Q_e \epsilon - \sum_{i=0}^{n-1}\left[\hat{\beta}_i - \beta_i\right]\epsilon_1\phi_{u,n-i} - \sum_{i=0}^{n-1}\left[\hat{\alpha}_i - \alpha_i\right]\epsilon_1\phi_{y,n-i}
\end{aligned} \tag{21}$$

Although the plant parameters are assumed to be unknown, the closed intervals in which each plant coefficient is contained are known. In traditional adaptive control schemes, the parameter estimates are generated by integral laws (gradient method, least-squares, etc.) [5, 6]. In this paper, we consider the following switching laws for $\hat{\theta}$ [39]:

$$\begin{cases} \hat{\beta}_i = \bar{\beta}_i \operatorname{sgn}\left(\epsilon_1 \phi_{u,n-i}\right) + \beta_i^c, & \bar{\beta}_i > |\beta_i^+ - \beta_i^c| \\ \hat{\alpha}_i = \bar{\alpha}_i \operatorname{sgn}\left(\epsilon_1\phi_{y,n-i}\right) + \alpha_i^c, & \bar{\alpha}_i > |\alpha_i^+ - \alpha_i^c| \end{cases} \tag{22}$$

for $i = 0, 1, \ldots, n-1$. Therefore, we have

$$\begin{aligned}
\dot{V}(\epsilon) = &-\epsilon^\top Q_e\epsilon - \sum_{i=0}^{n-1}\left[\bar{\beta}_i|\epsilon_1\phi_{u,n-i}| + \beta_i^c \epsilon_1 \phi_{u,n-i} - \beta_i \epsilon_1 \phi_{u,n-i}\right] \\
&- \sum_{i=0}^{n-1}\left[\bar{\alpha}_i |\epsilon_1 \phi_{y,n-i}| + \alpha_i^c \epsilon_1\phi_{y,n-i} - \alpha_i \epsilon_1 \phi_{y,n-i}\right]
\end{aligned}$$

As shown in (22), $\bar{\beta}_i > \left|\beta_i^+ - \beta_i^c\right|$ and $\bar{\alpha}_i > \left|\alpha_i^+ - \alpha_i^c\right|$ for $i = 0, 1, \ldots, n-1$. Therefore, we have that $\bar{\beta}_i > |\beta_i^c - \beta_i|$ and $\bar{\alpha}_i > |\alpha_i^c - \alpha_i|$, and we obtain

$$\dot{V}(\epsilon) \leq -\epsilon^\top Q_e \epsilon < 0 \tag{23}$$

that guarantees $\epsilon = 0$ as an asymptotically stable equilibrium state in the large, which implies that $\epsilon \in \mathcal{L}_\infty$ and $\epsilon \in \mathcal{L}_2$. Since $\epsilon_1 = C_e^\top \epsilon$, we have that $\epsilon_1 \in \mathcal{L}_\infty$ and $\epsilon_1 \in \mathcal{L}_2$.

Remark 4.1. *With the use of switching laws to generate the plant parameters estimates, we cannot guarantee the convergence to the true plant parameters. However, the convergence of $\hat{y}$ to y is guaranteed.*

With the analysis and equations presented in this section, we conclude the parameter estimation design, which can be combined with a wide variety of control schemes. In the control approach presented in this paper, referred to as *indirect adaptive control* [5, 6], the plant parameters are estimated on-line by using the switching laws in (22) and used to calculate the controller parameters of a pole placement control law, as will be presented in the next sections.

5. **Interval Pole Placement Control.** The purpose of this section is to design a control law that can meet the pole placement for SISO LTI plants whose parameters are unknown but with bounds constrained to known intervals. The form of the control law and the mapping between interval plant and controller parameters presented in this section are the same as those used in the known plant parameter case, when a PPC law is considered [5, 6]. Therefore, the control design presented in this section is the basis for the proposed control technique of this paper, since the control law is exactly the same for both designs, as will be presented in Section 6.

Let us consider the polynomial approach [5] to design the following pole placement control law:

$$u = -\frac{P_c(s)}{Q_m(s)L_c(s)}e_1 \tag{24}$$

where $e_1 = y - r$ is the tracking error; $Q_m(s)$ is the internal model of r, which has degree q and satisfies assumption **A4**; $P_c(s)$ and $L_c(s)$ are polynomials (with $L_c(s)$ monic) of degree $n + q - 1$ and $n - 1$, respectively.

Applying (24) to the interval plant (4), we obtain the closed-loop plant equation:

$$y = \frac{P_c(s)[Z(s)]}{Q_m(s)L_c(s)[R(s)] + P_c(s)[Z(s)]}r \tag{25}$$

The objective now is to choose $P_c(s)$ and $L_c(s)$ in the characteristic equation

$$Q_m(s)L_c(s)[R(s)] + P_c(s)[Z(s)] = [F(s)] \tag{26}$$

such that the roots of the closed-loop system characteristic polynomial $[F(s)]$ are contained in the region specified by the roots of a given monic Hurwitz interval characteristic polynomial $[B(s)]$ of degree $2n+q-1$, i.e., $\mathcal{S}([F(s)]) \subseteq \mathcal{S}([B(s)])$. The interval coefficients of $[B(s)]$ are lumped in the interval vector $[b] = [b^-, b^+]$, where

$$b^- = \left[b^-_{2n+q-2}, \ldots, b^-_1, b^-_0\right]^\top, \quad b^+ = \left[b^+_{2n+q-2}, \ldots, b^+_1, b^+_0\right]^\top \tag{27}$$

Therefore, in other words, if we denote $[f] = [f^{-\top}, f^{+\top}]^\top = [[f_{2n+q-2}], \ldots, [f_1], [f_0]]^\top$ as the vector of interval coefficients of $[F(s)]$ in (26), the following relationships must be satisfied:

$$b^-_i \leq f^-_i, \quad f^+_i \leq b^+_i \tag{28}$$

for $i = 0, 1, \ldots, 2n + q - 2$.

Equations of the form (26) are referred to as *Diophantine equations* [5, 6] and are widely used in the algebraic design of controllers for LTI plants. Of course the size of each interval coefficient of $[B(s)]$ would have to be adjusted to ensure that the system performance remains satisfactory [41].

The cofficients of $L_c(s)$ and $P_c(s)$ in Equation (26) can be found by the following inequality:

$$b^-_s \leq [A_s]x_s \leq b^+_s \tag{29}$$

where $[A_s] = [A^-_s, A^+_s]$ is the interval Sylvester matrix of dimension $2(n + q) \times 2(n + q)$ associated to the polynomials $Q_m(s)[R(s)]$ and $[Z(s)]$, whose lower and upper limiting matrices A^-_s and A^+_s are obtained when the coefficients of $[A_s]$ are replaced by their lower and upper values, respectively; the vectors b_s^- and b_s^+ are associated to the interval characteristic polynomial $[B(s)]$; and the vector x_s is associated to the controller polynomials $P_c(s)$ and $L_c(s)$, i.e.,

$$[A_s] = \begin{bmatrix} [\gamma_{n+q}] & & 0 & [\beta_{n+q}] & & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ [\gamma_1] & \ddots & 0 & [\beta_1] & \ddots & 0 \\ [\gamma_0] & \ddots & [\gamma_{n+q}] & [\beta_0] & \ddots & [\beta_{n+q}] \\ 0 & \ddots & \vdots & 0 & \ddots & \vdots \\ \vdots & \ddots & [\gamma_1] & \vdots & \ddots & [\beta_1] \\ 0 & & [\gamma_0] & 0 & & [\beta_0] \end{bmatrix} \tag{30}$$

$$b_s^- = [\underbrace{0, \ldots, 0}_{q}, 1, b^{-\top}]^\top = [\underbrace{0, \ldots, 0}_{q}, 1, b_{2n+q-2}^-, b_{2n+q-3}^-, \ldots, b_1^-, b_0^-]^\top \in \mathbb{R}^{2n+2q} \tag{31}$$

$$b_s^+ = [\underbrace{0, \ldots, 0}_{q}, 1, b^{+\top}]^\top = [\underbrace{0, \ldots, 0}_{q}, 1, b_{2n+q-2}^+, b_{2n+q-3}^+, \ldots, b_1^+, b_0^+]^\top \in \mathbb{R}^{2n+2q} \tag{32}$$

$$x_s = [\underbrace{0, \ldots, 0}_{q}, 1, x^\top]^\top \in \mathbb{R}^{2n+2q} \tag{33}$$

$$x = [l_{n-2}, l_{n-3}, \ldots, l_0, p_{n+q-1}, p_{n+q-2}, \ldots, p_1, p_0]^\top \in \mathbb{R}^{2n+q-1} \tag{34}$$

where $[\gamma_i] = \left[\gamma_i^-, \gamma_i^+\right]$, for $i = 0, 1, \ldots, n+q-1$, are the coefficients of $Q_m(s)[R(s)] - s^{n+q}$ and l_i and p_i are the coefficients of

$$P_c(s) = p_{n+q-1}s^{n+q-1} + \ldots + p_1 s + p_0$$
$$L_c(s) = s^{n-1} + l_{n-2}s^{n-2} + \ldots + l_1 s + l_0$$

As $Q_m(s)[R(s)]$ is a monic interval polynomial and $Z(s)$ has degree $m < n$, we have that $[\gamma_{n+q}] = 1$ and $[\beta_{n+q}] = [\beta_{n+q-1}] = \cdots = [\beta_{m+1}] = 0$ in (30).

The approach presented in this section is similar to that presented in [5], where the closed-loop poles are designed to be placed in fixed positions, which is not always possible due to the controller adaptation and the possible parametric variations of the plant. However, with the approach presented here, we conclude that any vector x_s, constant or not, which meets the condition in (29) can be selected as it ensures the pole placement in the region delimited by the roots of $[B(s)]$, even if there are parametric variations within the known ranges described in $[G(s)]$.

The pole placement control design presented in this section is used together with the parameter estimation design presented in Section 4 to form the IVS-APPC design, presented in the next section.

6. **IVS-APPC Design.** In this section, we present the interval VS-APPC design, which is the main contribution of this paper. This approach combines the PPC law for the known interval parameter case (Section 5) with adaptive laws that generate on-line estimates for the unknown plant parameters (Section 4). All steps of the design are summarized as follows: Subsection 6.1 presents Kharitonov's theorem, which is used to choose a monic Hurwitz interval characteristic polynomial and therefore, the closed-loop poles; in Subsection 6.2, an algorithm to find an interval controller that guarantees the desired pole placement is presented for SISO LTI interval plants; in Subsection 6.3, the parameter estimation design presented in Section 4 is reviewed; and in Subsection 6.4 the mapping between the parameter estimates and the controller parameters as well as the IVS-APPC law are presented.

6.1. **Interval characteristic polynomial.** Given an interval plant, the next step is to define an interval characteristic polynomial $[B(s)]$ that must be chosen so that all roots have negative real part. This guarantee can be obtained by Kharitonov's theorem [11], which says that an interval polynomial represented as

$$[\Delta(s)] = s^n + [\delta_{n-1}]s^{n-1} + \ldots + [\delta_1]s + [\delta_0]$$

where $[\delta_i] = \left[\delta_i^-, \delta_i^+\right]$, for $i = 0, 1, \ldots, n-1$, is Hurwitz if and only if the following four extreme polynomials are Hurwitz:

$$\Delta_1(s) = s^n + \ldots + \delta_4^- s^4 + \delta_3^+ s^3 + \delta_2^+ s^2 + \delta_1^- s + \delta_0^-$$

$$\Delta_2(s) = s^n + \ldots + \delta_4^- s^4 + \delta_3^- s^3 + \delta_2^+ s^2 + \delta_1^+ s + \delta_0^-$$

$$\Delta_3(s) = s^n + \ldots + \delta_4^+ s^4 + \delta_3^+ s^3 + \delta_2^- s^2 + \delta_1^- s + \delta_0^+$$

$$\Delta_4(s) = s^n + \ldots + \delta_4^+ s^4 + \delta_3^- s^3 + \delta_2^- s^2 + \delta_1^+ s + \delta_0^+$$

During the last few decades, Kharitonov's theorem has been one of the most popular approaches to investigate stability of interval systems, since the stability analysis of an entire family of infinite interval polynomials can be simplified to the stability verification of four extreme polynomials, regardless of the polynomials degree. Many important results derived from Kharitonov's theorem have been reported in the closed-loop system stability analysis [10, 51].

6.2. **Interval controller.** As we have shown in Section 5, an infinite number of solutions for x_s that satisfy the inequality in (29) can be found, i.e., we can obtain an infinite number of vectors x_s that guarantee the desired regional pole placement described by $[B(s)]$.

This section presents an algorithm that finds an interval controller $[x_s]$, given an interval plant and an interval characteristic polynomial. The interval controller defines lower and upper limits for each control law parameter, which may have constant value or not, so that the pole placement is guaranteed even with parametric uncertainties. To achieve this objective, an NLP problem must be constructed by using the coefficients of an interval plant and an interval characteristic polynomial and can be solved by using the Matlab® function called *fmincon*, which seeks to find a minimum of a nonlinear multivariable function subject to linear inequalities of type $Ax \leq b$.

If we denote $S(\xi)$ as the Sylvester matrix associated to the polynomials $Q_m(s)X(s)$ and $Y(s)$, where the coefficients of $Y(s)/X(s)$ are the elements of ξ, we have from (30) that $[A_s] = S\left([\theta^*]\right)$. This notation is important to describe the following theorem.

Theorem 6.1. [41] *Assuming that* $[A_s]x_s$ *are multilinear functions of* $[\theta^*]$ *then*

$$b_s^- \leq [A_s]x_s \leq b_s^+$$

holds for all $\theta^* \in [\theta^*]$ *if and only if*

$$b_s^- \leq S(v_i)x_s \leq b_s^+ \tag{35}$$

where $v_i \in \mathcal{V}([\theta^*])$, *for* $i = 0, 1, \ldots, 2^{2n}$.

Proof: Readers are referred to [41] or [10].

Remark 6.1. [41] *If* x_s *represents an interval controller and appears multilinearly in* $[A_s]x_s$*, the conditions given in (35) need be verified only at the vertices of* x_s*. Therefore, to verify if an interval controller described by* $[x_{int}]$ *meets the pole placement control objective, we check only if the vertices of* $[x_{int}]$ *meet the conditions given in (35).*

The *fmincon* function sintax requires that the constraints defined in Equation (29) be expressed in the form $Ax \leq b$. Therefore, based on Theorem 6.1 we have the inequality

$$A_f x_s \leq b_f \tag{36}$$

with

$$A_f = \begin{bmatrix} A_l \\ -A_l \end{bmatrix}, \quad b_f = \begin{bmatrix} b_l^+ \\ -b_l^- \end{bmatrix}$$

where the matrix A_l is constructed replacing the interval coefficients of $[A_s]$ by their lower and upper values, so that each line i of $[A_s]$ is replaced by 2^r lines that represent all the combinations of lower and upper values of the interval coefficients that appear in the line i of $[A_s]$, where r is the number of interval coefficients in the line i of $[A_s]$. If we denote $A_{l,i,j}$ as the line j derivated from the line i of $[A_s]$, the following constraints must be satisfied:

$$b_{s,i}^- \leq A_{l,i,j} x_s \leq b_{s,i}^+ \tag{37}$$

where $b_{s,i}^-$ and $b_{s,i}^+$ represent the elements i of the vectors b_s^- and b_s^+, respectively. Therefore, after all the constraints of (29) are contructed as in (37), the elements of vectors b_l^- and b_l^+ are the lower and upper limits of all constraints, respectively.

With Equation (36) constructed, we need to define the objective functions, whose minimum values will represent the lower and upper limits of a temporary interval controller described by $[x_{int}]$. To determine the interval vector $[x_{int}]$ through the *fmincon* function, we consider two objective functions: $f_{\min}$ to find the lower limit of $[x_{int}]$ and $f_{\max}$ to find the upper limit of $[x_{int}]$, which are given by:

$$x_{\min} = \min_{l_b \leq x \leq u_b} f_{\min}(x), \quad f_{\min}(x) = \sum_{i=1}^{2n+2q} \left(\frac{x_i - l_{b,i}}{u_{b,i} - l_{b,i}} \right)^2 \tag{38}$$

$$x_{\max} = \min_{l_b \leq x \leq u_b} f_{\max}(x), \quad f_{\max}(x) = \sum_{i=1}^{2n+2q} \left(\frac{x_i - u_{b,i}}{u_{b,i} - l_{b,i}} \right)^2 \tag{39}$$

where the vectors $l_b, u_b \in \mathbb{R}^{2n+2q}$ are the lower and upper bounds to fetch the vectors $x_{\min}$ and $x_{\max}$, so that the solutions are always in the ranges $l_b \leq x_{\min} \leq u_b$ and $l_b \leq x_{\max} \leq u_b$, and are used as initial points for $f_{\min}(x)$ and $f_{\max}(x)$, respectively. Since we want to find the largest possible intervals for the parameters of the interval controller, it is recommended that l_b and u_b be initialized so that the range $u_{b,i} - l_{b,i}$ is large, for $i = 1, 2, \ldots, 2n + 2q$.

After finding an interval vector $[x_{int}] = [x_{\min}^\top, x_{\max}^\top]^\top$ by *fmincon*, it is necessary to check its validity, i.e., in case of any vertice $x \in \mathcal{V}([x_{int}])$ that does not satisfy the inequality in (36), a new search must be performed, since in this case with $[x_{int}]$ we cannot guarantee the desired closed-loop pole placement. If a new search is necessary, the vectors l_b and u_b are updated to $x_{\min}$ and $x_{\max}$, respectively, reducing the search space. If $[x_{int}]$ represents a valid controller, we have that $[x_s] = [x_{int}] = [x_{\min}^\top, x_{\max}^\top]^\top$, which is described similarly to (33):

$$[x_s] = [\underbrace{0, \ldots, 0}_{q}, 1, [x]^\top]^\top \in \mathbb{R}^{2n+2q} \tag{40}$$

$$[x] = [[l_{n-2}], [l_{n-3}], \ldots, [l_0], [p_{n+q-1}], [p_{n+q-2}], \ldots, [p_0]]^\top \in \mathbb{R}^{2n+q-1} \tag{41}$$

The complete routine to find the interval controller is presented in Algorithm 1.

Algorithm 1 Search for $[x_s]$

$found = 0$
while $found = 0$ **do**
    $x_{\min} = \min\limits_{l_b \le x \le u_b} f_{\min}(x)$ such that $b_s^- \le [A_s]x \le b_s^+$
    $x_{\max} = \min\limits_{l_b \le x \le u_b} f_{\max}(x)$ such that $b_s^- \le [A_s]x \le b_s^+$
    $[x_{int}] = \left[x_{\min}^\top, x_{\max}^\top\right]^\top$
    for all $v_\theta \in \mathcal{V}([\theta^*])$ **do**
        for all $v_x \in \mathcal{V}([x_{int}])$ **do**
            if $(S(v_\theta)v_x < b_s^-)$ or $(S(v_\theta)v_x > b_s^+)$ **then**
                $l_b = x_{\min}$
                $u_b = x_{\max}$
                go to 2
            end if
        end for
    end for
    $found = 1$
    $[x_s] = \left[x_{\min}^\top, x_{\max}^\top\right]^\top$
end while

6.3. **Parametric model and adaptive laws.** As we have shown in Section 4, the adaptive laws design is based on a parametric model for θ^*, which is represented as

$$y = W(s)L(s)\left(\theta_\beta^{*\top}\phi_u + \theta_\alpha^{*\top}\phi_y - \lambda^\top\phi_y\right) \tag{42}$$

where $W(s)L(s)$ is an SPR transfer function constructed with the stable polynomials $\Lambda(s)$ and $L(s)$, and

$$W(s) = \frac{1}{\Lambda(s)} = \frac{1}{s^n + \lambda_{n-1}s^{n-1} + \ldots + \lambda_1 s + \lambda_0}$$

$$\lambda = [\lambda_{n-1}, \ldots, \lambda_1, \lambda_0]^\top$$

$$\phi_u = L^{-1}(s)\psi_u, \quad \phi_y = L^{-1}(s)\psi_y$$

$$\psi_u = \left[u^{(n-1)}, u^{(n-2)}, \ldots, u^{(1)}, u\right]^\top, \quad \psi_y = \left[-y^{(n-1)}, -y^{(n-2)}, \ldots, -y^{(1)}, -y\right]^\top$$

$$\theta_\beta^* = [\beta_{n-1}, \ldots, \beta_1, \beta_0]^\top, \quad \theta_\alpha^* = [\alpha_{n-1}, \ldots, \alpha_1, \alpha_0]^\top$$

To estimate the output signal y, a series-parallel model is constructed based on parametric model in (42), i.e., the estimate $\hat{y}$ is generated by a model whose true coefficients are substituted by the estimates as follows:

$$\hat{y} = W(s)L(s)\left(\hat{\theta}_\beta^\top\phi_u + \hat{\theta}_\alpha^\top\phi_y - \lambda^\top\phi_y\right) \tag{43}$$

where

$$\hat{\theta}_\beta = \left[\hat{\beta}_{n-1}, \ldots, \hat{\beta}_1, \hat{\beta}_0\right]^\top, \quad \hat{\theta}_\alpha = [\hat{\alpha}_{n-1}, \ldots, \hat{\alpha}_1, \hat{\alpha}_0]^\top$$

are the estimated plant parameter vectors that are calculated by the following switching laws [39]:

$$\begin{cases} \hat{\beta}_i = \bar{\beta}_i \operatorname{sgn}(\epsilon_1\phi_{u,n-i}) + \beta_i^c, & \bar{\beta}_i > |\beta_i^+ - \beta_i^c| \\ \hat{\alpha}_i = \bar{\alpha}_i \operatorname{sgn}(\epsilon_1\phi_{y,n-i}) + \alpha_i^c, & \bar{\alpha}_i > |\alpha_i^+ - \alpha_i^c| \end{cases} \tag{44}$$

where the estimation error is calculated as $\epsilon_1 = y - \hat{y}$.

The switching laws in (44) generate on-line estimates $\hat{\theta}_\beta$ and $\hat{\theta}_\alpha$ of the coefficient vectors, θ^*_β of $Z(s)$ and θ^*_α of $R(s)$, respectively, to form the estimated plant polynomials

$$\hat{Z}(s) = \hat{\beta}_{n-1}s^{n-1} + \ldots + \hat{\beta}_1 s + \hat{\beta}_0$$
$$\hat{R}(s) = s^{n-1} + \hat{\alpha}_{n-1}s^{n-1} + \ldots + \hat{\alpha}_1 s + \hat{\alpha}_0$$

6.4. **Calculation of controller parameters.** For the calculation of controller parameters to be performed, the plant parameters estimates and the characteristic polynomial are necessary. In the proposal presented in this paper, the characteristic polynomial used to solve the Diophantine equation in each instant of time t is the nominal (or center) polynomial of $[B(s)]$, whose coefficients are the center of the coefficients of $[B(s)]$, i.e.,

$$B_c(s) = s^{2n+q-1} + b^c_{2n+q-2}s^{2n+q-2} + \ldots + b^c_1 s + b^c_0 \tag{45}$$

where $b^c_i = \left(b^-_i + b^+_i\right)/2$, for $i = 0, 1, \ldots, 2n+q-2$, and the coefficient vector of $B_c(s)$ is defined as

$$b^c = \left[b^c_{2n+q-2}, \ldots, b^c_1, b^c_0\right]^\top \tag{46}$$

Therefore, the calculation of the controller parameters in each instant of time t is performed as follows:

$$\hat{x}_s = \hat{A}^{-1}_s b^c_s \tag{47}$$

where

$$b^c_s = [\underbrace{0, \ldots, 0}_{q}, 1, b^{c\top}]^\top \in \mathbb{R}^{2n+2q} \tag{48}$$

and $\hat{A}_s$ is the Sylvester matrix formed with the coefficients of the estimated plant polynomials ($\hat{Z}(s)$ and $Q_m(s)\hat{R}(s)$), i.e.,

$$\hat{A}_s = \begin{bmatrix} \hat{\gamma}_{n+q} & & 0 & \hat{\beta}_{n+q} & & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hat{\gamma}_1 & \ddots & 0 & \hat{\beta}_1 & \ddots & 0 \\ \hat{\gamma}_0 & \ddots & \hat{\gamma}_{n+q} & \hat{\beta}_0 & \ddots & \hat{\beta}_{n+q} \\ 0 & \ddots & \vdots & 0 & \ddots & \vdots \\ \vdots & \ddots & \hat{\gamma}_1 & \vdots & \ddots & \hat{\beta}_1 \\ 0 & & \hat{\gamma}_0 & 0 & & \hat{\beta}_0 \end{bmatrix} \tag{49}$$

where $\hat{\gamma}_{n+q-1}, \ldots, \hat{\gamma}_0$ are the coefficients of $Q_m(s)\hat{R}(s) - s^{n+q}$, $\hat{\gamma}_{n+q} = 1$ and $\hat{\beta}_{n+q} = \hat{\beta}_{n+q-1} = \cdots = \hat{\beta}_{m+1} = 0$.

Although the estimates of θ^* guarantee that $\hat{y}$ follows y, we do not have the guarantee that the controller parameters calculated as in (47) will lie in $[x_s]$, i.e., $\hat{x}_s \in [x_s]$. Therefore, the controller parameters are calculated as presented in Algorithm 2.

Algorithm 2 Calculation of $\hat{x}_s$

```
x̂_s = Â_s^{-1} b_s^c
if x̂_{s,i} < x^-_{s,i} then
    x̂_{s,i} = x^-_{s,i}
end if
if x̂_{s,i} > x^+_{s,i} then
    x̂_{s,i} = x^+_{s,i}
end if
```

Therefore, the IVS-APPC control law is formed at each instant of time t as

$$u = -\frac{\sum_{i=0}^{n+q-1} \hat{p}_i s^i}{Q_m(s)\left(s^{n-1} + \sum_{i=0}^{n-2} \hat{l}_i s^i\right)} e_1 \tag{50}$$

where $\hat{p}_i \in [p_i] = \left[p_i^-, p_i^+\right]$, for $i = 0, 1, \ldots, n + q - 1$, and $\hat{l}_j \in [l_j] = \left[l_j^-, l_j^+\right]$, for $j = 0, 1, \ldots, n - 2$, are the coefficients of

$$\hat{x}_s = [\underbrace{0, \ldots, 0}_{q}, 1, \hat{x}^\top]^\top \in \mathbb{R}^{2n+2q} \tag{51}$$

$$\hat{x} = \left[\hat{l}_{n-2}, \hat{l}_{n-3}, \ldots, \hat{l}_0, \hat{p}_{n+q-1}, \hat{p}_{n+q-2}, \ldots, \hat{p}_0\right]^\top \in \mathbb{R}^{2n+q-1} \tag{52}$$

7. **Numerical Examples.** This section presents simulation results for IVS-APPC and VS-APPC [38, 39, 40] schemes applied to two nonminimum phase second order plants. The IVS-APPC design presented in Section 6, as well as the VS-APPC design, can be applied without distinction to minimum and nonminimum phase plants, unlike VS-MRAC scheme that is restricted to minimum phase plants. Therefore, in the simulations presented next we focus on nonminimum phase plants since the control of this system class is one of the motivations of the proposed control design. Moreover, nonminimum phase systems have important features [8] that make the control design a great challenge.

While IVS-APPC design aims to place the closed-loop poles in a region defined by the interval polynomial $[B(s)]$, VS-APPC design uses the nominal polynomial of $[B(s)]$, i.e., a polynomial whose coefficients are the centers of the interval coefficients of $[B(s)]$. Both simulations were carried out with plant and model initial conditions equal to zero and with the same constants, an integration step $h = 10^{-3}$s, which is used in the Euler method for derivative approximations, and the reference signal $r = 1$ ($Q_m(s) = s$ and $q = 1$). The parametric model for the plant was constructed with the SPR transfer function

$$W(s)L(s) = \frac{L(s)}{\Lambda(s)} = \frac{s+2}{(s+1)(s+3)} \tag{53}$$

and therefore we have

$$\lambda = [4, 3]^\top, \quad \phi_u = L^{-1}(s)[\dot{u}, u]^\top, \quad \phi_y = L^{-1}(s)[-\dot{y}, -y]^\top$$

and the control law has the form

$$u = -\frac{\hat{p}_2 s^2 + \hat{p}_1 s + \hat{p}_0}{s\left(s + \hat{l}_0\right)} e_1$$

In Table 1 we have the average performance indexes used in this section to compare IVS-APPC and VS-APPC schemes.

TABLE 1. Performance indexes

Integral of Absolute Error (IAE)	$IAE = \int_{t=0}^{t_{\max}} \lvert e_1(t)\rvert dt$
Integral of Squared Error (ISE)	$ISE = \int_{t=0}^{t_{\max}} e_1^2 dt$
Integral of Time Multiply Absolute Error (ITAE)	$ITAE = \int_{t=0}^{t_{\max}} t\lvert e_1(t)\rvert dt$
Integral of Time Multiply Squared Error (ITSE)	$ITSE = \int_{t=0}^{t_{\max}} te_1^2(t)dt$
Energy Consumption (E_u)	$E_u = \int_{t=0}^{t_{\max}} \lvert u(t)\rvert dt$

7.1. **Nonminimum phase stable system.** Initially, let us consider the plant described by the transfer function

$$G(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0} = \frac{2s - 10}{s^2 + 8s + 16} \tag{54}$$

whose coefficients are assumed unknown but contained in known intervals that form the following interval plant:

$$[G(s)] = \frac{[Z(s)]}{[R(s)]} = \frac{[\beta_1]s + [\beta_0]}{s^2 + [\alpha_1]s + [\alpha_0]} = \frac{[1.6, 2.8]s + [-10.3, -9.1]}{s^2 + [7.7, 8.8]s + [15.5, 16.9]} \tag{55}$$

Besides tracking, the control objective is to place the closed-loop poles in the region specified by the roots of the following interval polynomial, whose stability is guaranteed by Kharitonov's theorem:

$$[B(s)] = s^4 + [18, 24]s^3 + [121.5, 216]s^2 + [364.5, 864]s + [410.0625, 1296] \tag{56}$$

Considering $[A]$ as the resultant Sylvester matrix associated to $[Z(s)]$ and $[R(s)]$, as mentioned in Section 2, we have that $\rho(|A_c^{-1}|A_r) = 0.2933 < 1$, which guarantees that $[A]$ is nonsingular and, consequently, $[R(s)]$ and $[Z(s)]$ are robustly coprime interval polynomials. Therefore, the interval plant and these findings are in agreement with assumptions **A1** to **A4**.

Given the interval plant (55) and the interval characteristic polynomial (56), by using Algorithm 1 with all elements of l_b equal to -10^3 and all elements of u_b equal to 10^3, we obtain the following interval controller, which defines the lower and upper limits for the controller coefficients:

$$[x] = \begin{bmatrix} l_0^-, & l_0^+ \\ p_2^-, & p_2^+ \\ p_1^-, & p_1^+ \\ p_0^-, & p_0^+ \end{bmatrix} = \begin{bmatrix} 21.6949, & 21.6999 \\ -4.0686, & -4.0629 \\ -34.9951, & -34.7561 \\ -94.0287, & -48.5338 \end{bmatrix}$$

Figures 1 and 2 show the system behavior for IVS-APPC and VS-APPC schemes applied to (54), respectively. In both simulations the system behavior is displayed for 30s, with a reference deviation to $r = 2$, from $t > 10$s, and a sudden change in all plant parameters, which are still held within the respective ranges, as follows: $\beta_1 = 2.8$, $\beta_0 = -9.1$, $\alpha_1 = 7.7$ and $\alpha_0 = 16.9$, from $t > 20$s. The relay amplitudes were selected in accordance with the restrictions imposed in (44): $\bar{\beta}_1 = 0.61$, $\bar{\beta}_0 = 0.61$, $\bar{\alpha}_1 = 0.56$ and $\bar{\alpha}_0 = 0.71$. Figure 3 presents the behavior of the closed-loop poles for both controllers.

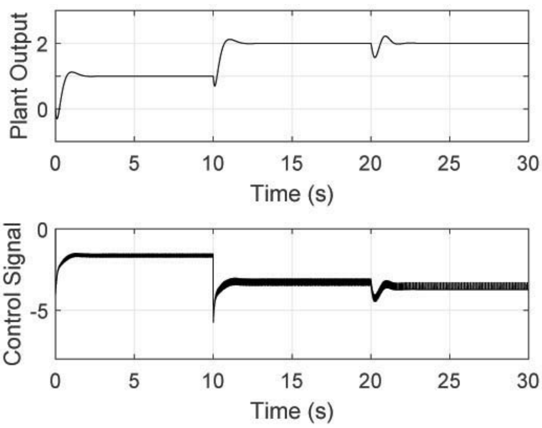

FIGURE 1. IVS-APPC behavior

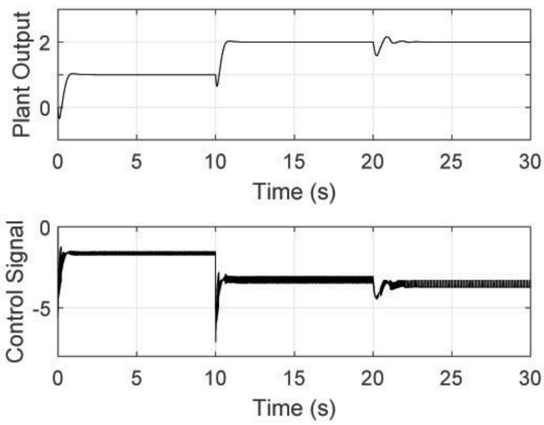

FIGURE 2. VS-APPC behavior

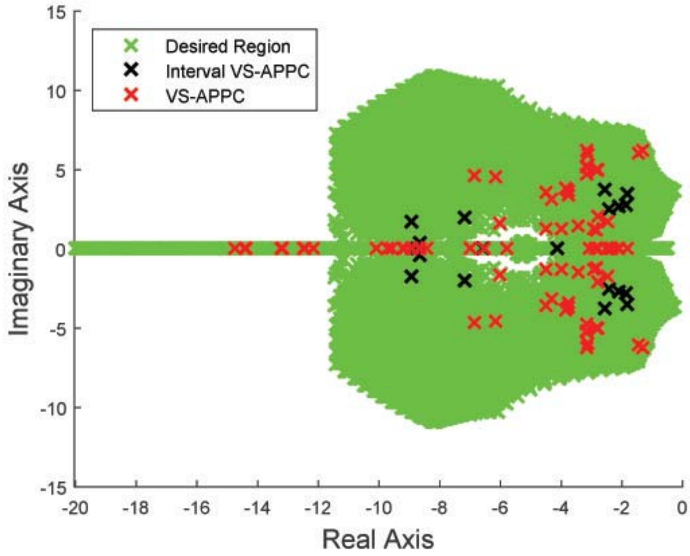

FIGURE 3. Closed-loop poles of IVS-APPC and VS-APPC – stable system

TABLE 2. Performance comparison – stable system

	Interval VS-APPC	VS-APPC
IAE	1.4142e+03	1.1843e+03
ISE	1.0153e+03	957.8595
ITAE	1.1976e+04	9.7247e+03
ITSE	6.3639e+03	5.7704e+03
E_u	8.6359e+04	8.6167e+04

As can be observed, IVS-APPC and VS-APPC schemes have similar transient and steady-state behaviors, with robustness to fast variations in the reference signal and the plant parameters. However, the closed-loop poles with VS-APPC scheme are very close to the boundary and are more scattered along the defined region. An important aspect of VS-APPC control law, presented in this simulation and in previous works [39, 40], is that the control signal switching has much lower amplitude than the controllers based on the VS-MRAC scheme [32, 34, 36]. This feature is very important because it allows the application of the VS-APPC (and IVS-APPC) scheme in various industrial processes. In Table 2 a comparison between IVS-APPC and VS-APPC schemes is presented based on common performance indexes based on tracking error (IAE, ISE, ITAE, ITSE) and on control signal (E_u), confirming similar behavior shown in Figures 1 and 2, although the VS-APPC scheme presents a slightly better performance than the IVS-APPC scheme in terms of control signal variation (less switching).

7.2. **Nonminimum phase unstable system.** Now, let us consider a more difficult system to control:

$$G(s) = \frac{3s-1}{s^2 - s + 9} \tag{57}$$

whose coefficients are assumed unknown but contained in the intervals coefficients of the plant:

$$[G(s)] = \frac{[Z(s)]}{[R(s)]} = \frac{[\beta_1]s + [\beta_0]}{s^2 + [\alpha_1]s + [\alpha_0]} = \frac{[2.8, 3.3]s + [-1.3, -0.9]}{s^2 + [-1.4, -0.7]s + [8.2, 9.3]} \tag{58}$$

The control objective is to place the closed-loop poles in the region specified by the roots of

$$[B(s)] = s^4 + [6, 8]s^3 + [13.5, 24]s^2 + [13.5, 32]s + [5.0625, 16] \tag{59}$$

Remark 7.1. *In the control design for nonminimum phase plants, the choice of an interval characteristic polynomial is a complicated task. In some cases, the choice of stable interval characteristic polynomials can lead the closed-loop system to instability. For example, when we use for the plant in (58) the same characteristic polynomial used for the plant in (55), the system becomes unstable, with VS-APPC or IVS-APPC schemes. Additionally, during the search for the interval controller, the fmincon function may converge to an infeasible point.*

Considering $[A]$ as the resultant Sylvester matrix associated to $[Z(s)]$ and $[R(s)]$, we have that $\rho(|A_c^{-1}|A_r) = 0.1696 < 1$, which guarantees that $[A]$ is nonsingular and, consequently, $[R(s)]$ and $[Z(s)]$ are robustly coprime interval polynomials.

Given the interval plant (58) and choosing the interval characteristic polynomial (59), by using Algorithm 1 with all elements of l_b equal to -10^3 and all elements of u_b equal to 10^3, we obtain the following interval controller:

$$[x] = \begin{bmatrix} l_0^-, & l_0^+ \\ p_2^-, & p_2^+ \\ p_1^-, & p_1^+ \\ p_0^-, & p_0^+ \end{bmatrix} = \begin{bmatrix} 6.3940, & 6.4526 \\ 0.3864, & 0.6549 \\ 5.6164, & 5.8584 \\ -9.4647, & -8.2438 \end{bmatrix}$$

Figures 4 and 5 show the system behavior when IVS-APPC and VS-APPC schemes are applied to (57), respectively. The reference signal is changed to $r = 2$, from $t > 20$s, and the plant parameters values are changed to $\beta_1 = 3.3$, $\beta_0 = -0.9$, $\alpha_1 = -1.4$ and $\alpha_0 = 8.2$, and thus remain contained in the intervals of (58), from $t > 40$s. The relay amplitudes were selected in accordance with the restrictions of (44): $\bar{\beta}_1 = 0.26$, $\bar{\beta}_0 = 0.21$, $\bar{\alpha}_1 = 0.36$ and $\bar{\alpha}_0 = 0.56$.

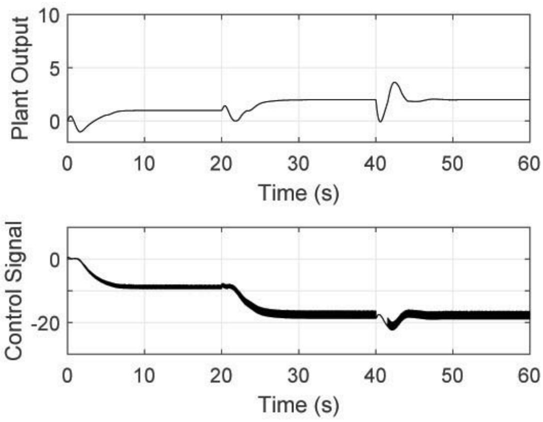

FIGURE 4. IVS-APPC behavior

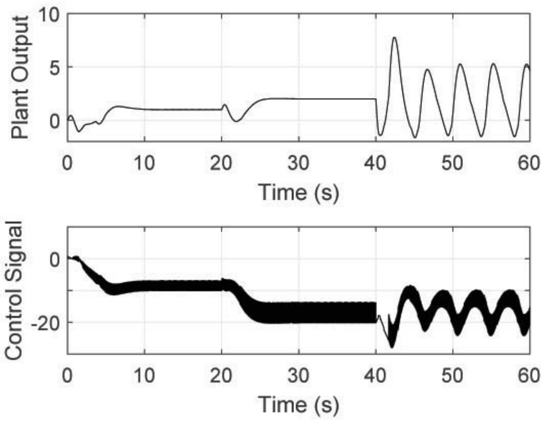

FIGURE 5. VS-APPC behavior

Both controllers have similar behavior during the first 40s, although the VS-APPC control signal amplitude is greater than in the IVS-APPC case (see E_u in Table 3). Furthermore, abrupt and wide parametric variations at $t = 40$s affect significantly the VS-APPC performance (see error indexes in Table 3), making the plant output signal quite oscillatory, and thus no longer following the reference signal. This fact can be verified in Figure 6, where some poles are outside the region and quite scattered. In the IVS-APPC case, the poles are more concentrated and thus keep the tracking objective

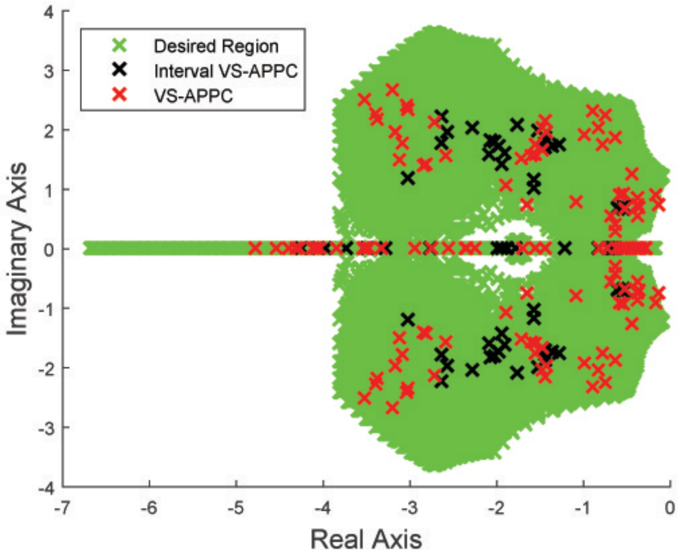

FIGURE 6. Closed-loop poles of IVS-APPC and VS-APPC – unstable system

TABLE 3. Performance comparison - unstable system

	Interval VS-APPC	VS-APPC
IAE	1.7509e+04	5.7773e+04
ISE	2.2385e+04	1.4738e+05
ITAE	3.6222e+05	2.3688e+06
ITSE	4.5487e+05	6.5545e+06
E_u	8.4735e+05	8.4075e+05

despite the parameter variation. Therefore, based on this simulation, where we have a very sensitive system to the adopted control design, we can conclude that the VS-APPC scheme may lead the system to instability in some situations such as wide known parametric variations (relays amplitude were chosen considering the range where the plant parameters were contained) and inappropriate choice of the characteristic polynomial, as the system is difficult to control.

8. **Conclusions.** This paper presents the design of a variable structure adaptive pole placement controller for interval SISO LTI plants, called IVS-APPC. This scheme holds some important characteristics of the original version of VS-APPC: is applicable to non-minimum phase plants; presents good transient performance; presents smaller amplitude control signal than the switching versions of VS-MRAC, without filtering; is robust to parametric variations; its design can be applied without distinction to plants with any order and relative degree. The previous computational effort of IVS-APPC design to find an interval controller is rewarded by the guarantee of closed-loop pole placement in a desired region, even if there are parametric variations within the known ranges of an interval plant. On the other hand, the VS-APPC design does not require the determination of thresholds for the control law parameters, but does not guarantee satisfactory performance in some cases if there are parametric variations, as shown by the simulation results. Therefore, this paper presents a robust control alternative for the case where the exact pole placement is necessary, thus fulfilling one of the VS-APPC deficiencies.

We can cite as future works for the control scheme presented in this paper: the robustness analysis to disturbances and unmodeled dynamics, optimization of controller parameters or plant parameters estimates, a systematic study based on performance criteria to design the interval characteristic polynomial, comparison with other adaptive control schemes and the practical implementation of the controller.

Acknowledgment. Second author thanks the funding by ERDF – European Regional Development Fund through the Operational Programme for Competitiveness and Internationalisation – COMPETE 2020 Programme, and by National Funds through the FCT – Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) within project <<POCI-01-0145-FEDER-006961>>. Third author thanks to CNPq – Conselho Nacional de Desenvolvimento Científico e Tecnológico (National Counsel of Technological and Scientific Development).

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