International Journal of Innovative Computing, Information and Control
Volume **13**, Number **3**, June **2017**

ICIC International ©2017 ISSN 1349-4198
pp. 741–750

DESIGN OF VARIABLE PHASE FILTER WITH MINIMAX CRITERION AND ITERATIVE METHOD

Hai Huyen Dam

Department of Mathematics and Statistics
Curtin University of Technology
Kent Street, Bentley, Perth 6102, Australia
H.Dam@curtin.edu.au

Received November 2016; revised March 2017

Abstract. *This paper investigates the design of a variable phase digital filter with a minimax criterion. The problem becomes a non-convex optimization problem with respect to the filter coefficients. An iterative optimization procedure is developed for solving the optimization problem. Design examples show that the proposed methods can obtain the optimal solution with a small number of iterations. In addition, the proposed methods can obtain a significant improvement in the maximum frequency response error and the maximum phase error deviation over other existing methods.*

Keywords: Variable phase filter design, Minimax criterion, Iterative method, Optimization

1. **Introduction.** Digital phase-compensation filters have been used in digital signal processing and digital communication systems as a means of compensating the phase characteristics of the original non-linear phase system [1-19]. Here, we are interested in designing tunable phase-compensation filters as the phase characteristics may frequently be changed from time to time.

In [4-7], digital filters with tunable phase delay or group delay, referred to as variable fractional delay filters are investigated. A range of applications have been considered, including timing offset recovery in digital receivers, comb filter design, sampling rate conversion, speech coding, time delay estimation, one-dimensional digital signal interpolation and image interpolation. For finite impulse response (FIR) based variable fractional delay filters, an appropriate optimization problem can be formulated. It is relatively easy to solve the problem in such a way that the desired characteristics are achieved [4-11].

The design of allpass variable fractional delay filters is more difficult and has been investigated in [12-14]. In [15], the design of allpass filters with equiripple group delay errors has been investigated. The key advantage of the allpass filters is that they can achieve higher design accuracy than FIR filters, yielding smaller frequency response errors for applications that require unity gain. The design of minimax allpass variable fractional delay filters is discussed in [16,17]. In [18], the design of an allpass variable fractional delay filter with minimum integral squared error is developed.

In [19], the design of variable phase allpass filter with minimax criterion is investigated. The optimization problem is transformed into a linear optimization problem by ignoring the denominator for the allpass frequency response. As such, the method may not result in a global solution as the denominator is not unity for all the frequencies. In this paper, we propose an iterative optimization procedure for solving the optimization problem directly without using approximation to improve the performance of the variable phase allpass filter over the method obtained in [19]. Design examples show that the proposed iterative method can obtain the optimal solution with low number of iterations. In addition, the proposed method can obtain a significant improvement in the maximum frequency response error and the maximum phase error deviation when compared with the existing methods.

The paper is organized as follows. The problem formulation is discussed in Section 2. The optimization procedure is developed in Section 3. Design examples are discussed in Section 4 and concluding remarks are in Section 5.

2. **Problem Formulation.** Consider the design of an allpass variable phase filter with coefficients $a_n(p)$, $1 \le n \le N$, depending on a tuning parameter p. Each coefficient $a_n(p)$ is expressed as a polynomial of p,

$$a_n(p) = \sum_{m=0}^{M} h_{n,m} p^m \tag{1}$$

where $h_{n,m}$ are real coefficients and M is the order of the polynomial, $M > 0$. Here, the parameter p is varied in the range $\mathcal{P} = [p_l, p_l + 1]$, where p_l denotes the lower bound. Denote by $\mathbf{h}$ the vector of all the allpass filter coefficients,

$$\mathbf{h} = [h_{1,0}, \dots, h_{N,M}] .$$

The frequency response of the allpass filter for $\omega \in \Omega = [0, \omega_p]$ where ω_p is a constant and $p \in \mathcal{P}$ is given by

$$H\left(\mathbf{h}, e^{j\omega}, p\right) = \frac{a_N(p) + \cdots + a_1(p)e^{-j(N-1)\omega} + e^{-jN\omega}}{1 + a_1(p)e^{-j\omega} + \cdots + a_N(p)e^{-jN\omega}} = e^{-jN\omega} \frac{A\left(\mathbf{h}, e^{-j\omega}, p\right)}{A\left(\mathbf{h}, e^{j\omega}, p\right)} \tag{2}$$

where the denominator $A\left(\mathbf{h}, e^{-j\omega}, p\right)$ is given as

$$A\left(\mathbf{h}, e^{j\omega}, p\right) = 1 + \sum_{n=1}^{N} a_n(p) e^{-jn\omega} = 1 + \sum_{n=1}^{N} \sum_{m=0}^{M} h_{n,m} p^m e^{-jn\omega} \tag{3}$$

Denote by $\theta(\mathbf{h}, \omega, p)$ the phase of the allpass variable phase filter. The filter frequency response (2) can be expressed as

$$H\left(\mathbf{h}, e^{j\omega}, p\right) = e^{j\theta(\mathbf{h},\omega,p)}$$

where $\theta(\mathbf{h}, \omega, p) = -N\omega - 2\phi(\mathbf{h}, \omega, p)$ and $\phi(\mathbf{h}, \omega, p)$ is the phase of the denominator $A(\mathbf{h}, e^{j\omega}, p)$. The goal for the design of an allpass variable phase filter is to approximate $\theta(\mathbf{h}, \omega, p)$ by a given variable phase specification $\theta_d(\omega, p)$ for all $\omega \in \Omega$ and $p \in \mathcal{P}$. The minimax design of the allpass variable phase filter can be formulated as

$$\begin{cases} \min\limits_{\mathbf{h},\gamma} \gamma \\ |\theta(\mathbf{h}, \omega, p) - \theta_d(\omega, p)| \le \gamma, \ \forall \omega \in \Omega, \ p \in \mathcal{P} \end{cases} \tag{4}$$

For a small value of γ, the constraints in (4) are equivalent to

$$\left| \frac{\theta(\mathbf{h}, \omega, p) - \theta_d(\omega, p)}{2} \right| \le \frac{\gamma}{2}, \ \forall \omega \in \Omega, \ p \in \mathcal{P}.$$

These constraints can be rewritten as:

$$\left| \tan\left(\frac{\theta(\mathbf{h}, \omega, p) - \theta_d(\omega, p)}{2} \right) \right| \le \tan\left(\frac{\gamma}{2} \right), \ \forall \omega \in \Omega, \ p \in \mathcal{P}.$$

As such, the optimization problem (4) is equivalent to

$$\begin{cases} \min\limits_{\mathbf{h},\gamma} \gamma \\ \left|\tan\left(\frac{\theta(\mathbf{h},\omega,p)-\theta_d(\omega,p)}{2}\right)\right| \le \tan\left(\frac{\gamma}{2}\right), \ \forall\omega\in\Omega,\ p\in\mathcal{P} \end{cases} \tag{5}$$

The problem (5) is a non-linear optimization problem with respect to $\mathbf{h}$ and γ. In the following, we will develop an approach for solving (5) by deriving an expression for $\tan\left(\frac{\theta(\mathbf{h},\omega,p)-\theta_d(\omega,p)}{2}\right)$ in terms of the allpass variable phase filter coefficients.

3. **Optimization Procedure.** We now derive the expression for the tangent constraint in Equation (5). Following from (2), we have

$$\begin{aligned} e^{j(\theta(\mathbf{h},\omega,p)-\theta_d(\omega,p))} &= e^{-j(N\omega+\theta_d(\omega,p))}\frac{1+\sum\limits_{n=1}^{N}a_n(p)e^{jn\omega}}{1+\sum\limits_{n=1}^{N}a_n(p)e^{-jn\omega}} \\ &= \frac{e^{-j\frac{N\omega+\theta_d(\omega,p)}{2}}+\sum\limits_{n=1}^{N}a_n(p)e^{j\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)}}{e^{j\frac{N\omega+\theta_d(\omega,p)}{2}}+\sum\limits_{n=1}^{N}a_n(p)e^{j\left(-n\omega+\frac{N\omega+\theta_d(\omega,p)}{2}\right)}} \end{aligned} \tag{6}$$

As the numerator of the last expression in (6) is the conjugate of the denominator, we have the following expression for the phase error:

$$\frac{\theta(\mathbf{h},\omega,p)-\theta_d(\omega,p)}{2} = \angle\left\{e^{-j\frac{N\omega+\theta_d(\omega,p)}{2}}+\sum_{n=1}^{N}a_n(p)e^{j\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)}\right\} \tag{7}$$

where $\angle\{\cdot\}$ denotes the phase angle (in radians) for the complex number $\{\cdot\}$.

Following from the expression in (7), we have

$$\begin{aligned} \tan\left(\frac{\theta(\mathbf{h},\omega,p)-\theta_d(\omega,p)}{2}\right) &= \frac{\mathcal{I}\left\{e^{-j\frac{N\omega+\theta_d(\omega,p)}{2}}+\sum\limits_{n=1}^{N}a_n(p)e^{j\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)}\right\}}{\mathcal{R}\left\{e^{-j\frac{N\omega+\theta_d(\omega,p)}{2}}+\sum\limits_{n=1}^{N}a_n(p)e^{j\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)}\right\}} \\ &= \frac{-\sin\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right)+\sum\limits_{n=1}^{N}a_n(p)\sin\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)}{\cos\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right)+\sum\limits_{n=1}^{N}a_n(p)\cos\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)} \end{aligned} \tag{8}$$

The optimization problem (5) can be rewritten in terms of the allpass variable phase filter coefficients as

$$\begin{cases} \min\limits_{\mathbf{h},\gamma} \gamma \\ \left|\frac{-\sin\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right)+\sum\limits_{n=1}^{N}a_n(p)\sin\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)}{\cos\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right)+\sum\limits_{n=1}^{N}a_n(p)\cos\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)}\right| \le \tan\left(\frac{\gamma}{2}\right), \ \forall\omega\in\Omega,\ p\in\mathcal{P} \end{cases} \tag{9}$$

We have the following property.

Property 3.1. *Assume that we have the following constraint on the variable phase error deviation*

$$|\theta(\omega,p)-\theta_d(\omega,p)| < \pi, \ \forall\omega\in\Omega,\ p\in\mathcal{P} \tag{10}$$

Then, the sign of the denominator in (8) is always positive. The above assumption is generally valid for a good design as variable phase error deviation is generally small.

Proof: Following from the assumption (10), we have

$$\left|\frac{\theta(\omega,p)-\theta_d(\omega,p)}{2}\right| < \frac{\pi}{2}, \quad \forall\omega\in\Omega,\ p\in\mathcal{P} \tag{11}$$

As such, $\cos\left(\frac{\theta(\omega,p)-\theta_d(\omega,p)}{2}\right) > 0$. It follows from (7) that

$$\cos\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right)+\sum_{n=1}^{N}a_n(p)\cos\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)>0 \tag{12}$$

for all $\omega\in\Omega$ and $p\in\mathcal{P}$. Thus, the property is proved. □

In the following, we will develop an iterative method for solving the problem (9).

3.1. **Iterative method.** For a small value of γ, the assumption in Property 3.1 is valid. As such, the denominator in (8) has a positive sign and the optimization problem (9) can be rewritten equivalently as

$$\begin{cases}\min\limits_{\mathbf{h},\gamma}\gamma\\ \sum\limits_{m=0}^{M}\sum\limits_{n=1}^{N}\left(\sin\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)-\tan\left(\frac{\gamma}{2}\right)\times\cos\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)\right)p^m h_{n,m}\\ \le\sin\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right)+\tan\left(\frac{\gamma}{2}\right)\cos\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right),\ \forall\omega\in\Omega,\ p\in\mathcal{P}\\ \sum\limits_{m=0}^{M}\sum\limits_{n=1}^{N}\left(-\sin\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)-\tan\left(\frac{\gamma}{2}\right)\times\cos\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)\right)p^m h_{n,m}\\ \le-\sin\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right)+\tan\left(\frac{\gamma}{2}\right)\cos\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right),\ \forall\omega\in\Omega,\ p\in\mathcal{P}\end{cases} \tag{13}$$

For a fixed value of γ, the problem (13) becomes finding a feasible solution $\mathbf{h}$ for a semi-infinite linear optimization problem in terms of $\mathbf{h}$. One possible method to obtain a feasible solution is by using discretization and a linear programming technique. The question becomes how to obtain a minimum value of γ, so that a feasible solution exists for the problem (13). In the following, we propose an effective iterative procedure to obtain the minimum value γ, so that the sequence γ reduces at each iteration until no further improvement is obtained. The advantage of the proposed procedure is that the optimal solution from one iteration can be used as an initial solution for the next iteration.

Initially, we start with a large value of γ_0 so that the problem (13) has a feasible solution and set $k=0$. For an iteration k with a fixed γ_k, $k\ge 0$, we introduce a new parameter $\eta>0$ and solve the following optimization problem:

$$\begin{cases}\max\limits_{\mathbf{h},\eta}\ \eta\\ \sum\limits_{m=0}^{M}\sum\limits_{n=1}^{N}\left(\sin\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)-\tan\left(\frac{\gamma_k}{2}\right)\times\cos\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)\right)p^m h_{n,m}+\eta\\ \le\sin\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right)+\tan\left(\frac{\gamma_k}{2}\right)\cos\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right),\ \forall\omega\in\Omega,\ p\in\mathcal{P}\\ \sum\limits_{m=0}^{M}\sum\limits_{n=1}^{N}\left(-\sin\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)-\tan\left(\frac{\gamma_k}{2}\right)\times\cos\left(n\omega-\frac{N\omega+\theta_d(\omega,p)}{2}\right)\right)p^m h_{n,m}+\eta\\ \le-\sin\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right)+\tan\left(\frac{\gamma_k}{2}\right)\cos\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right),\ \forall\omega\in\Omega,\ p\in\mathcal{P}\end{cases} \tag{14}$$

Denote by $\left(\mathbf{h}^{(k)}, \eta^{(k)}\right)$ the optimal solution for (14) with $\eta^{(k)} \geq 0$. Following from (9), the update γ_{k+1} for the $(k+1)^{\text{th}}$ iteration can be obtained based on $\mathbf{h}^{(k)}$ as

$$\gamma_{k+1} = 2\arctan(\beta) \tag{15}$$

where

$$\beta = \max_{\omega,p}\left|\frac{-\sin\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right) + \sum_{m=0}^{M}\sum_{n=1}^{N} p^m h_{n,m}^{(k)} \sin\left(n\omega - \frac{N\omega+\theta_d(\omega,p)}{2}\right)}{\cos\left(\frac{N\omega+\theta_d(\omega,p)}{2}\right) + \sum_{m=0}^{M}\sum_{n=1}^{N} p^m h_{n,m}^{(k)} \cos\left(n\omega - \frac{N\omega+\theta_d(\omega,p)}{2}\right)}\right|.$$

We have the following property.

Property 3.2. *The sequence of $\{\gamma_k\}$ reduces from one iteration to the next with $\gamma_k \geq \gamma_{k+1}$, $k \geq 0$.*

Proof: For the kth iteration, the problem (13) has a feasible solution with γ_k. Thus, the problem (14) has a feasible solution with γ_k and $\eta = 0$. Also, the optimal solution for (14) satisfies $\eta^o \geq 0$. As the constraints in (14) are satisfied, we have

$$\beta \leq \tan\left(\frac{\gamma_k}{2}\right) \tag{16}$$

or $\gamma_{k+1} \leq \gamma_k$. □

Remark 3.1. *As can be seen from Property 3.2, $\{\gamma_k\}$ is a reducing sequence. Since the sequence is bounded, $\gamma_k \geq 0$, $\forall k$, the limit of the sequence exists and it is the same as the minimum for the sequence,*

$$\lim_{k\to\infty} \gamma_k = \gamma_{opt} \tag{17}$$

3.2. **Sufficient stability constraints.** We now investigate the sufficient stability condition for the allpass variable phase filter $H\left(\mathbf{h}, e^{j\omega}, p\right)$. The sufficient stability constraints for the allpass filters [20] are given as

$$\mathcal{R}\left\{A\left(\mathbf{h}, e^{j\omega}, p\right)\right\} > 0, \quad \omega \in [0, \pi] \tag{18}$$

These constraints can be formulated as linear constraints with respect to the allpass filter coefficients $\mathbf{h}$ and can be added into the optimization problem (14) if the optimal allpass filters are unstable. Here, we consider the allpass variable phase filters specifications as given in [19]. If the constraints in (4) are satisfied, then we have

$$\begin{aligned}|\phi(\mathbf{h},\omega,p)| &= \left|\frac{\theta(\mathbf{h},\omega,p)+N\omega}{2}\right| \\ &\leq \left|\frac{\theta_d(\omega,p)+N\omega}{2}\right| + \left|\frac{\theta(\mathbf{h},\omega,p)-\theta_d(\omega,p)}{2}\right| \\ &\leq \left|\frac{\theta_d(\omega,p)+N\omega}{2}\right| + \frac{\gamma}{2}, \quad \forall\omega\in\Omega,\ p\in\mathcal{P}\end{aligned} \tag{19}$$

We now consider two design examples as given in [19].

Example 3.1. *We consider*

$$\theta_d(\omega,p) = -N\omega + p\omega^2/\pi \tag{20}$$

where $\omega_p = \alpha\pi$, $\alpha = 0.9$ and $p_l = -1/2$. As such

$$|\phi(\mathbf{h},\omega,p)| \leq \left|\frac{p\omega^2}{2\pi}\right| + \frac{\gamma}{2} \leq \frac{\alpha^2\pi^2}{4\pi} + \frac{\gamma}{2} = \frac{\alpha^2\pi}{4} + \frac{\gamma}{2} < \frac{\pi}{2}, \quad \forall\omega\in[0, 0.9\pi],\ p\in[-1/2, 1/2].$$

As such, $\cos(\phi(\mathbf{h},\omega,p)) > 0$ *for all* $\omega \in [0, 0.9\pi]$, $p \in [-1/2, 1/2]$ *and the condition (18) is satisfied for all* $\omega \in [0, 0.9\pi]$ *and* $p \in [-1/2, 1/2]$. *As such, if the allpass filters are found to be unstable, we only need to enforce the constraints (18) for* $\omega \in [0.9\pi, \pi]$.

Example 3.2. *We consider the design in which the desired phase changes linearly with respect to the tuning parameter* p,

$$\theta_d(\omega,p) = -N\omega + p\omega \tag{21}$$

with $\omega_p = \alpha\pi$, $\alpha = 0.9$, *and* $p_l = -1/2$. *Similar to the previous example, we have*

$$|\phi(\mathbf{h},\omega,p)| \le |p\omega| + \frac{\gamma}{2} \le \frac{\alpha\pi}{2} + \frac{\gamma}{2} < \frac{\pi}{2}, \;\; \forall\omega \in [0, 0.9\pi], \; p \in [-1/2, 1/2].$$

Thus, if the allpass filters are found to be unstable, we only need to enforce the constraints (18) for $\omega \in [0.9\pi, \pi]$.

The algorithm for finding the minimax solution can be summarized as:
Procedure 3.1:

- Step 1: Initialize a small tolerance ε. For the design examples, ε is chosen as $\varepsilon = 10^{-4}$.
- Step 2: Initialize γ_0 so that the problem (13) has a feasible solution or the problem (14) has an optimal solution with $\eta^o \ge 0$. If $\eta^o < 0$ then increase $\gamma_0 := 2\gamma_0$ and return to the beginning of Step 2. Otherwise, set $k = 0$ and go to Step 3.
- Step 3: Solve the problem (14) for $\left(\mathbf{h}^{(k)}, \eta^{(k)}\right)$ with γ_k. Update γ_{k+1} as in (15). Check the pole positions of the allpass filters to ensure stability. If the filters are unstable, then add the stability constraint into the design problem [16] and go back to the beginning of Step 3. Otherwise, check for the convergence of the procedure. If

$$|\gamma_{k+1} - \gamma_k| \le \varepsilon \tag{22}$$

  then stop the procedure and go to Step 4. Otherwise, set $k := k + 1$ and return to the beginning of Step 3.
- Step 4: The optimal solution to the problem is $\gamma_{\text{opt}} = \gamma_{k+1}$ with the variable allpass filter coefficient $\mathbf{h}^{(k)}$.

In the following, we will investigate the effectiveness of Procedure 3.1 in different cases.

4. Design Examples.

We consider two examples given in [19] for comparison purpose. The proposed method, however, is general and applicable for any design specification.

Example 4.1. *Consider the design of a variable phase allpass filter as in* [19] *with the designed phase response* $\theta_d(\omega,p)$ *as in (20). The optimization problem (13) is solved using discretization with* $L = 201$ *points for* ω *and* $K = 21$ *points for* p. *Design examples are run using an Intel Core i7-4790 CPU with 3.60GHz.*

Table 1 shows the number of iterations, the maximum frequency response error

$$e_{f,\max} = \max_{\omega,p} |H(\omega,p) - H_d(\omega,p)| \, [\text{dB}], \tag{23}$$

the maximum phase error

$$e_{P,\max} = \max_{\omega,p} |\theta(\omega,p) - \theta_d(\omega,p)| \tag{24}$$

TABLE 1. Comparison for the first case with $\theta_d(\omega,p) = -N\omega + p\omega^2/\pi$

Methods	Number of iterations	Error $e_{f,\max}$ [dB]	Error $e_{P,\max}$	Max pole mag.
Iterative method	8	-39.36 dB	0.0108	0.891
Method [19]	–	-35.87 dB	0.0161	0.885

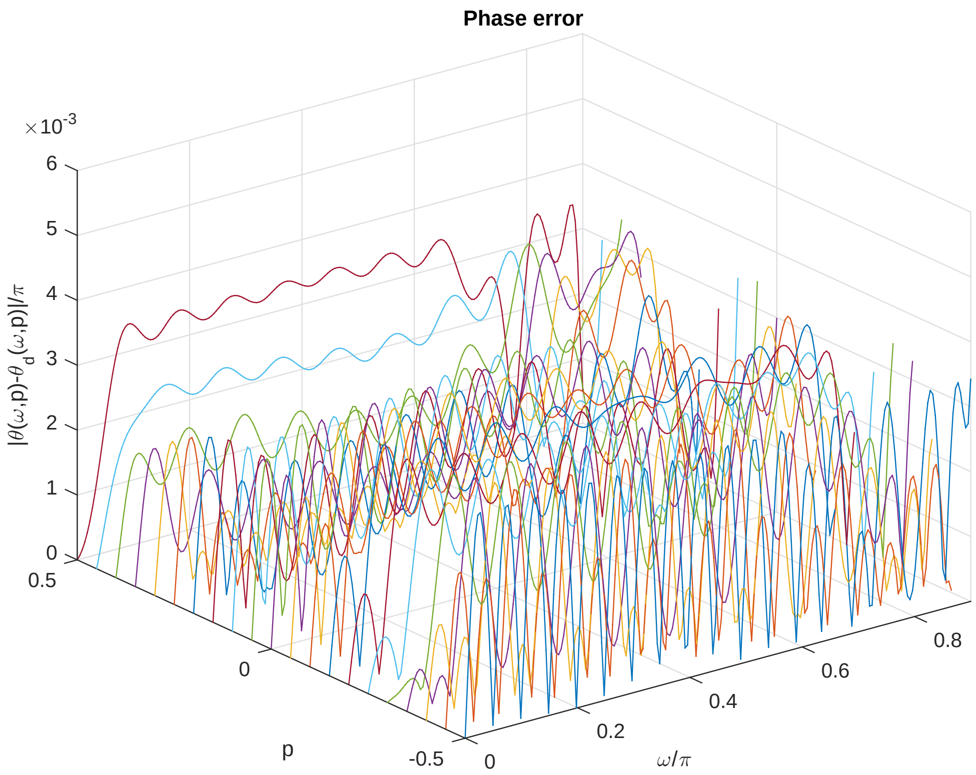

FIGURE 1. Phase error with $\theta_d(\omega, p) = -N\omega + p\omega^2/\pi$

and the maximum pole magnitude for the iterative method and the method obtained using [19]*. For the iterative method, Procedure 3.1 converges with* $\varepsilon = 10^{-4}$ *and* $\gamma_0 = 0.1$.

It can be seen from the table that maximum frequency response error for the proposed method is 3.49 dB better than the method obtained in [19]*. In addition, the maximum phase error for the optimal solution using the iterative method is 0.0108, which is lower than 0.0161 obtained as in* [19] *as the method in* [19] *ignores the denominator in (13) and as such the solution obtained might not be the global solution. For this case, the allpass filters obtained in all the iterations are stable, as such the stability constraints are not required to be included. Furthermore, the table shows the maximum pole magnitudes for the optimal allpass filters, which are all smaller than one. As such, all the variable phase allpass filters are stable with poles inside the unit circle.*

Figure 1 plots the phase error $(\theta(\omega, p) - \theta_d(\omega, p)|)/\pi$ *for* $\omega \in \Omega$ *and* $p \in \mathcal{P}$ *as in* [19]*. The maximum phase error for the solution obtained is smaller than the method obtained in* [19].

Example 4.2. *We now consider another example with the desired phase given as in (21) with* $\gamma_0 = 0.1$ *and* $\epsilon = 10^{-4}$*. Table 2 shows the number of iterations, the maximum frequency response error, the maximum phase error and also the maximum pole magnitude*

TABLE 2. Comparison for the second case with $\theta_d(\omega, p) = -N\omega + p\omega$

Methods	Number of iterations	Error $e_{f,\max}$ [dB]	Error $e_{P,\max}$	Max pole mag.
Iterative	10	−33.35 dB	0.0215	0.9341
Method [19]	–	−26.01 dB	0.0500	0.9434

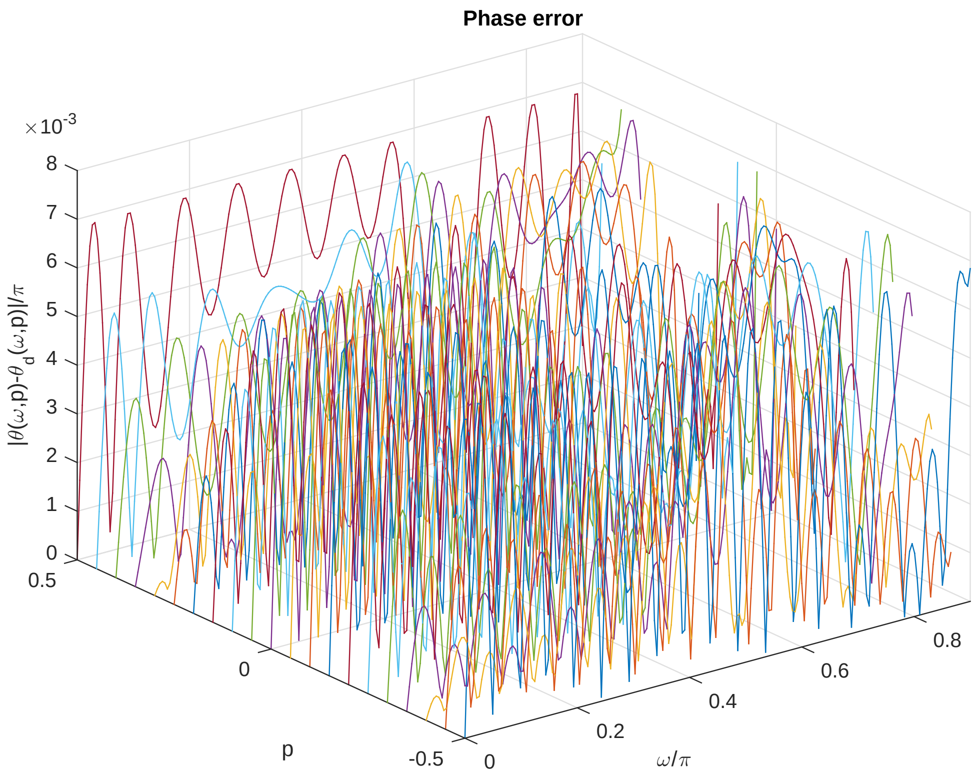

FIGURE 2. Phase error with $\theta_d(\omega, p) = -N\omega + p\omega$

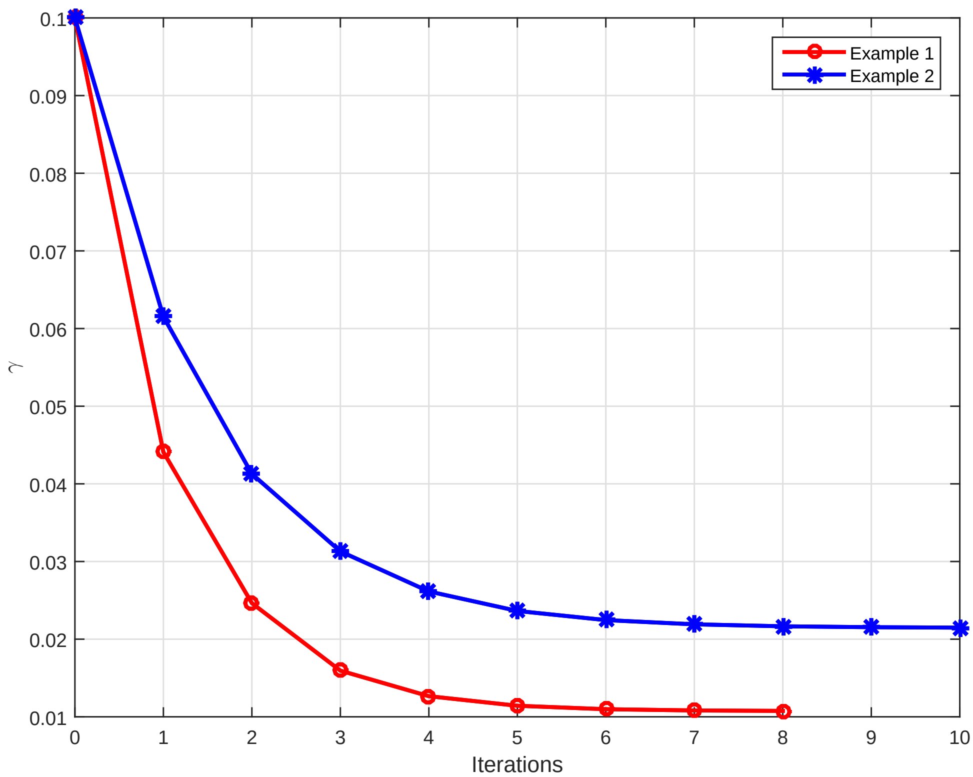

FIGURE 3. Convergence of γ for different iterations with an initial $\gamma_0 = 0.1$

for the two methods. Similar to Example 4.1, the maximum frequency response for the proposed method is 7.34 dB better than the error obtained using the method in [19]. *In addition the maximum phase error for the iterative method is 0.0215, which is significantly smaller than 0.050 obtained using the method in* [19]. *Figure 2 shows the phase error for the optimal allpass filters. The maximum phase error is significantly lower than the errors obtained in* [19]. *Similar to the previous example, all the poles for all the iterations are inside the unit circle. As such it is not necessary to include the extra sufficient stability constraints.*

Figure 3 shows the convergence plot for the iterative method for the two examples with the initial $\gamma = 0.1$. *The iterative method converges fast for both cases, which requires only a few iterations for convergence. Note that the algorithm converges much faster if the accuracy* ε *is chosen less than* 10^{-4}.

5. **Conclusions.** This paper investigates the design of a variable phase allpass filter with a minimax criterion. The problem results in a non-linear optimization problem with respect to the allpass filter coefficients. An iterative optimization approach is developed for solving the problem. Design examples show that the iterative method requires only a few iterations for convergence. In addition, the iterative method can obtain a significant improvement in the maximum frequency response error and maximum phase error deviation when compared with an existing method.

REFERENCES

[1] N. Ito, Iterative SOCP scheme for designing digital phase-networks, *Proc. of IEEE ICIT*, pp.600-603, 2013.

[2] W. Qin and N. Ito, Lp-norm based all-pass system design with improved convergence and robustness, *Proc. of IEEE ICCAS*, pp.1774-1781, 2013.

[3] J. J. Shyu, S. C. Pei and C. H. Chan, Minimax phase error design of allpass variable fractional-delay digital filters by iterative weighted least-squares method, *Signal Processing*, vol.89, no.9, pp.1774-1781, 2009.

[4] C. W. Farrow, A continuously variable digital delay element, *Proc. of IEEE Int. Symp. Circuits Syst.*, vol.3, pp.2641-2645, 1988.

[5] H. H. Dam, A. Cantoni, K. L. Teo and S. Nordholm, FIR variable digital filter with signed power-of-two coefficients, *IEEE Trans. Circuits Systems I*, vol.54, no.6, pp.1348-1357, 2007.

[6] H. H. Dam, A. Cantoni, K. L. Teo and S. Nordholm, Vatiable digital filter with least square criterion and peak gain constraints, *IEEE Trans. Circuits Systems II*, vol.54, no.1, pp.24-28, 2007.

[7] H. H. Dam, A. Cantoni, K. L. Teo and S. Nordholm, Variable digital filter with group delay flatness specification or phase constraints, *IEEE Trans. Circuits and Systems II*, vol.55, no.5, 2008.

[8] H. H. Dam, Design of variable fractional delay filter with fractional delay constraints, *IEEE Signal Processing Letters*, vol.21, no.11, pp.1361-1364, 2014.

[9] C. Yu, K. L. Teo and H. H. Dam, Design of allpass variable fractional delay filter with signed powers-of-two coefficients, *Signal Processing*, no.95, pp.32-42, 2014.

[10] H. H. Dam, Variable fractional delay FIR filter design with a bicriteria and coefficient relationship, *IEEE Trans. Circuits and Systems II: Express Briefs*, vol.61, no.1, pp.36-40, 2014.

[11] H. H. Dam, Variable fractional delay filter with sub-expression coefficients, *International Journal of Innovative Computing, Information and Control*, vol.9, no.7, pp.2995-3003, 2013.

[12] M. Makundi, T. I. Laakso and V. Valimaki, Efficient tunable IIR and allpass structures, *Electron. Lett.*, vol.37, pp.344-345, 2001.

[13] H. H. Dam and K. L. Teo, Allpass VFD filter design, *IEEE Trans. Signal Processing*, vol.58, no.8, pp.4432-4436, 2010.

[14] J. Y. Kaakinen and T. Saramaki, An algorithm for the optimization of adjustable fractional delay all-pass filters, *Proc. of IEEE ISCAS'04*, Vnacouver, QC, Canada, vol.3, pp.153-156, 2006.

[15] T. Hong, H. Bai, S. Li, C. Huang and L. Chang, A new method for designing allpass filters with equiripple group delay errors, *Proc. of IEEE Conference on Industrial Electronics and Applications*, pp.1024-1028, 2014.

[16] H. H. Dam, Design of allpass variable fractional delay filter, *IEEE Trans. Signal Processing*, vol.59, no.12, pp.6240-6244, 2011.
[17] T. B. Deng, Minimax design of low-complexity allpass variable fractional-delay digital filters, *IEEE Trans. Circuits Systems-I*, vol.57, no.8, pp.2075-2086, 2010.
[18] C. C. Tseng, Design of 1-D and 2-D variable fractional delay allpass filters using weighted least square methods, *IEEE Trans. Circuits Syst. I*, vol.49, no.10, pp.1413-1422, 2002.
[19] T. B. Deng, Variable-phase filter design using linear programming, *International Journal of Electronics Letters*, 2015.
[20] A. T. Chottera and G. A. Jullien, A linear programming approach to recursive digital filter design with linear phase, *IEEE Trans. Circuits Systems*, vol.CAS-29, no.3, pp.139-149, 1982.