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STABILIZING CONTROL OF NONLINEAR SWITCHED SYSTEMS IN $\mathbb{R}^3$ WITH A GEOMETRIC APPROACH

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Abstract. *This paper proposes a new constructive method to stabilize a dynamical system in $\mathbb{R}^3$ around a desired point x_d. The system under consideration in $\mathbb{R}^3$ can be an extension of another in dimension two. We drive a sufficient condition for the existence and stability of a hybrid limit cycle consisting of a sequence of two operating new modes in $\mathbb{R}^3$. This cycle limit lies in an invariant plane by the flow of the dynamical system. This method is based on a geometric approach and on the relaxation of a condition on part of the control signal, u. This is then illustrated on a multicellular electrical energy converter and a nonlinear switched system in $\mathbb{R}^3$.*

Keywords: Switched dynamical system, Hybrid limit cycle, Vectors fields, Trajectories, Geometric approach, Invariant plane

1. **Introduction.** Power electronics knew important technological developments thanks to the improvements of semiconductors, power components and systems of energy conversion. Among these systems, multicellular converters, which are built upon a series-association of elementary commutation cells, are more and more used in industrial applications. Indeed, they are characterized by their modularity and high efficiency. However, the major drawback of this kind of converters is their control complexity. Modeling is a very important step to control law design and synthesis. In the literature, several approaches have been considered to develop models for multicellular converter. Initially, models have been developed to describe their instantaneous, harmonic or averaging behaviors [2, 3, 4]. The converter model must be enough simple to allow the control synthesis and enough precise to achieve the desired behavior. Because it is based on continuous and discrete variables, multicellular converter modeling is claimed to be difficult [5]. The aim of this paper is to propose a sequence of control to drive our system from an initial condition x_0 to a desired point x_d when the technological constraints impose a minimum and a maximum duration between two successive switchings. The model for multicellular converter considered lies in the area of switched dynamical systems.

Switched dynamical systems (SDS) have a long history in the control literature but along with hybrid systems, more generally, they have enjoyed a particular growth in interest since the 1990s. The large amount of data and ideas thus generated have, until now, lacked a co-ordinating framework to focus them effectively on some of the fundamental issues such as the problems of stabilizing switching design, feedback stabilization and optimal switching [13]. This class of dynamic systems typically consists of a process with autonomous switching (caused by shocks or using diodes, etc.) or controlled switching (using transistors, relays, valves, etc.). The controlled switching sequence is the discrete control. The SDS are found in many fields of application: transport, embedded systems, electronics power, aeronautics, chemical engineering, pharmaceutical, etc.

In the study of stability of equilibrium points of differential systems, specific results for switched and hybrid systems have been developed: see [6, 7] for multiple Lyapunov based approach, [10] for Lie Algebra based results, [11] for an approach based on dynamical systems techniques, and [12] for a survey of stability criteria for switched and hybrid systems.

This paper proposes a new constructive method for synthesizing a hybrid limit cycle for stabilizing control of a class of switched dynamical systems. This is an extension of the paper [1]. The main objective is to define the set of all points around which a hybrid limit cycle can be established. The hybrid limit cycle lies in an invariant plane obtained by a relaxation of a condition on part of a chosen control signal u_{1c} (supposed continuous and dependent of the state of the system). The invariant plane P depends essentially on the choice of the desired point. This method is illustrated on the multicellular electrical energy converter and a nonlinear dynamical system. Let us consider the time invariant dynamic system in $\mathbb{R}^3$:

$$\dot{x} = \sum_{i=1}^{3} u_i X^i(x) = f(x) \tag{1}$$

with X^i vector fields of class C^1, the control $u_i \geq 0$ for $i \in \{1, 2, 3\}$ and the state $x = (x_1, x_2, x_3)$ is in $\mathbb{R}^3$. If we choose N, a normal vector of plane P such that $\frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle} < 0$ and $\frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle} < 0$ and the control

$$u_1 = -u_2 \frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle} - u_3 \frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle}$$

the system (1) can be written as

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x) \tag{2}$$

where

$$f_1(x) = X^2(x) - \frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle} X^1(x) \tag{3}$$

and

$$f_2(x) = X^3(x) - \frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle} X^1(x) \tag{4}$$

In the first section, we define the set of all points around which a hybrid limit cycle $CC(x_{c1}, x_{c2})$ (for more detail see [1]) can be established for the planar SDS (2) and we give a method to construct a vector normal of the invariant plane P.

In the second part we define

$$A_0 = B_0 = C_0 = \delta \bigcup_{i=1}^{3} \{X_t^i(x);\ t \leq 0 \text{ and } x \in int(CC(x_{c1}, x_{c2}))\}$$

The subsets A_i, B_i and C_i are defined as the following

$$\begin{aligned} A_{i+1} &= \{X_t^1(x);\ t \leq 0 \text{ and } x \in B_i \textstyle\bigcup C_i\} \\ B_{i+1} &= \{X_t^2(x);\ t \leq 0 \text{ and } x \in A_i \textstyle\bigcup C_i\} \\ C_{i+1} &= \{X_t^3(x);\ t \leq 0 \text{ and } x \in A_i \textstyle\bigcup B_i\} \end{aligned}$$

These are the sets of trajectories that define all backward inferences from the hybrid limit cycle $CC(x_{c1}, x_{c2})$ following the three vectors fields flows. For x_0 contained in A_i or B_i or C_i we can drive our system from initial condition x_0 to the hybrid limit cycle $CC(x_{c1}, x_{c2})$ and we can stay around x_d because the $CC(x_{c1}, x_{c2})$ is in a neighborhood of x_d. This method is illustrated on the Buck electrical converter control and a nonlinear dynamical system.

2. **Preliminary.** We consider the time invariant switched non linear dynamic system (SDS) (1) in $\mathbb{R}^3$. The (Carathéodory) solution of the differential equation $\dot{x} = X^i(x)$ after elapsed time t with initial condition $x(0) = x_0$ is denoted as $X^i_t(x_0)$.

Remark 2.1. *If ϕ is a function we denote*

$$\phi^{(n)}(x) = \underbrace{\phi \circ \cdots \circ \phi}_{n\ times}(x)$$

It is well known that for a constant $u_i \geq 0$ with $i \in \{1,2,3\}$ one has (for more detail see [9]*):*

$$\left(u_1X^1 + u_2X^2 + u_3X^3\right)_t(x) = \lim_{n\to+\infty}\left(X^1_{\frac{u_1t}{n}} \circ X^2_{\frac{u_2t}{n}} \circ X^3_{\frac{u_3t}{n}}\right)^{(n)}(x) \tag{5}$$

Our goal is to construct a sequence with a finite number of switching, which leads the state of the system (1) from the initial point x_{in} to the area located around the desired point x_d.

Remark 2.2. *We denote $\delta_{\min}$ as the minimum duration time between two successive switchings (dwell time). If we supposed $\delta_{\min} = 0$, then the controllability of system (1) is equivalent to the controllability of switched dynamical system*

$$\dot{x} = \sum_{i=1}^{3} u_iX^i(x) \tag{6}$$

the control $u_i \in \{0,1\}$, for $i \in \{1,2,3\}$ with $\sum_{i=1}^{3} u_i = 1$.

Definition 2.1. *(Invariant subset)*
The subset $\mathcal{A} \subset \mathbb{R}^3$ is invariant by the trajectory of the open loop system (1) if for $x \in \mathcal{A}$, one has $X^i_t(x) \in \mathcal{A}$ for $i \in \{1,2,3\}$ and $t \in \mathbb{R}$.

3. **Sufficient Condition of Existence and Stability of a Hybrid Limit Cycle.** In this section we localized some points in the set of the equilibrium point of system (1) in which there exists a hybrid limit cycle in a neighborhood of these points.

Notation.

We denote: $\det(u,v,w)$ the determinant of the matrix in $\mathbb{R}^{3\times3}$ with column vectors u, v, w respectively in $\mathbb{R}^3$.

We consider $E_1 = \{x \in \mathbb{R}^3\text{: and there exists a unique } \gamma_1,\gamma_2 < 0 \text{ such that } X^3(x) = \gamma_1X^1(x) + \gamma_2X^2(x)\}$. E_1 is the set of all points x such that the vector $X^3(x)$ is uniquely represented as a linear combination of $(X^1(x); X^2(x))$ with negative coefficient.

$\langle .|. \rangle$ denotes the standard inner product on $\mathbb{R}^3$.

$B(x_d,\varepsilon) = \{x \in \mathbb{R}^3; \|x - x_d\| < \varepsilon\}$ is the open ball with center x_d and radius ε.

Let $N = (\alpha_1,\alpha_2,\alpha_3)$ be a fixed vector in $\mathbb{R}^3$ and $x_d \in E_1$ is a desired operating point of system (1), and we define the plane P as

$$P = \left\{x \in \mathbb{R}^3 \text{ such that } \langle x - x_d|N\rangle = 0\right\}$$

Remark 3.1. *P is invariant by the trajectory of system (1) if and only if for all $x \in P$ one has $\langle f(x)|N\rangle = 0$.*

The open set $\mathcal{O} = \{x \in \mathbb{R}^3 | \langle X^1(x)|N\rangle \neq 0\}$ *is supposed not empty. Then, for a choice of the first control* u_1

$$u_{1c} = -u_2\frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle} - u_3\frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle} \tag{7}$$

one has, for $x \in \mathcal{O}$ *the set* $P \cap \mathcal{O}$ *is invariant by the trajectory of the system (1).*

The conditions $\frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle} < 0$ and $\frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle} < 0$ yield $u_{1c} \geq 0$.

According to the form (7) of u_{1c}, the system (1) becomes:

$$\begin{aligned}\dot{x} &= \left(-u_2\frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle} - u_3\frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle}\right)X^1(x) + u_2X^2(x) + u_3X^3(x)\\ &= u_2\left(X^2(x) - \frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle}X^1(x)\right) + u_3\left(X^3(x) - \frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle}X^1(x)\right)\end{aligned}$$

Let

$$f_1(x) = X^2(x) - \frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle}X^1(x) \tag{8}$$

and

$$f_2(x) = X^3(x) - \frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle}X^1(x) \tag{9}$$

The system (1) can be written as:

$$\dot{x} = u_2f_1(x) + u_3f_2(x) \tag{10}$$

with $u_k \in \{0,1\}$, $\forall k \in \{2,3\}$. The solution of the differential equation $\dot{x} = f_j(x)$ after elapsed time t with initial condition $x(0) = x_0 \in \mathcal{O}\bigcap P$ is denoted as $\phi_j(t, x_0)$, $j \in \{1,2\}$. with the so-called switching function (or discrete control) $u(.) : [0\ \infty[\mapsto \{0,1\}$ and $x \in \mathcal{O}\bigcap P$. It is made of two locations (modes, subsystems), with respective Lipchitz vector fields $f_1(x)$ and $f_2(x)$ in $\mathcal{O}\bigcap P$. There is no state discontinuity at the moment of switching.

Definition 3.1. *Let us consider* x_{c_1} *and* x_{c_2} *two points in* $\mathcal{O}\bigcap P$*, with* $x_{c_1} \neq x_{c_2}$*.* $CC(x_{c_1}, x_{c_2})$ *is the hybrid limit cycle of the SDS* $\dot{x} = f_i(x)$*,* $i \in \{1,2\}$*, between the switching points* x_{c_1} *and* x_{c_2}*, if and only if* $(\delta_{c_1}, \delta_{c_2}) \in \mathbb{R}^2_+$ *exists such that:* $x_{c_1} = \Phi_1(\delta_{c_1}, x_{c_2})$ *and* $x_{c_2} = \Phi_2(\delta_{c_2}, x_{c_1})$*. Then*

$$CC(x_{c_1}, x_{c_2}) = \{\Phi_1(\delta, x_{c_2})/0 \leq \delta \leq \delta_{c_1}\} \cup \{\Phi_2(\delta, x_{c_1})/0 \leq \delta \leq \delta_{c_2}\}$$

We recall necessary and sufficient condition for existence and stability of a hybrid limit cycle.

Let us consider two maps enough smooth γ_j: $I \subset \mathbb{R} \longrightarrow \mathbb{R}^2$, $j \in \{1,2\}$. Suppose that $\gamma_1(t_0) = \gamma_2(t_0)$ (the two trajectories intersect at t_0) with $\gamma_1'(t_0) \neq 0$ and $\gamma_2'(t_0) \neq 0$.

Definition 3.2. [1] *We call that curves* γ_1 *and* γ_2 *are transverse if and only if* γ_1 *crosses the curve* γ_2 *in* $x_0 = \gamma_1(t_0)$*. We explain this property in the following figure:*

case when curves γ_1 and γ_2 are transverse — case when γ_1 and γ_2 not transverse

Notation.

Let us denote as $d^{p-1}f_i(x) = \frac{d^p\Phi_i(t,x)}{dt^p}_{|t=0}$ with $p \geq 1$. We also denote $v = f_1(x) \neq 0$ and $p(x)$ is the smallest positive integer such that:

$$\langle f_2(x)|v\rangle^p \left\langle d^{p-1}f_1(x)|v^\perp\right\rangle \neq \langle f_1(x)|v\rangle^p \left\langle d^{p-1}f_2(x)|v^\perp\right\rangle$$

The vector $v^\perp$ is orthogonal vector of v in the vector space

$$VP = \{x \in \mathbb{R}^3 \text{ such that } \langle x|N\rangle = 0\}$$

Remark 3.2. *Since the plane P is invariant by the trajectory of the system (10) then $\Phi_i(t, x_d) \in P$ for $i \in \{1, 2\}$ and all derivative $d^{p-1}f_i(x) = \frac{d^p\Phi_i(t,x)}{dt^p}_{|t=0}$ is in the vector space VP. A simple computation gives*

$$\left\langle d^{p-1}f_i(x)|v^\perp\right\rangle = \det\left(d^{p-1}f_i(x), v, N\right)$$

Definition 3.3. *$E = \{z \in E_1$ such that $p(z)$ is even$\}$ is the set of points in P with collinear and opposite vector fields $f_i(x)$, $i \in \{1, 2\}$, and with non transverse trajectories.*

We recall a result concerning the existence of such hybrid limit cycle in the plane.

Theorem 3.1. [1] *Let us consider the SDS (10) with $x \in \mathcal{O} \bigcap P$. For each point $z \in E$, for any $\varepsilon > 0$ there exists a hybrid limit cycle $CC(x_{c1}, x_{c2})$ in $P \bigcap B(z, \varepsilon)$ such that $z \in \overline{Int(CC(x_{c1}, x_{c2}))}$.*

Theorem 3.2. *Let us consider the SDS (1) with $x_d \in E_1$. If there exists N, a vector in $\mathbb{R}^3$ such that*

$$\frac{\langle X^2(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle} < 0 \text{ and } \frac{\langle X^3(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle} < 0$$

with $p(x_d)$ being even, then, for any $\varepsilon > 0$ there exists a hybrid limit cycle

$$CC(x_{c1}, x_{c2}) \subset P \bigcap B(x_d, \varepsilon)$$

Proof: Since $x_d \in E_1$ then there exists $\gamma_1 < 0$ and $\gamma_2 < 0$ such that

$$X^3(x_d) = \gamma_1 X^1(x_d) + \gamma_2 X^2(x_d)$$

The vector N satisfies conditions $\frac{\langle X^2(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle} < 0$ and $\frac{\langle X^3(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle} < 0$.

Since $x \mapsto \frac{\langle X_k(x)|N\rangle}{\langle X^1(x)|N\rangle}$, and $k \in \{2, 3\}$ are continuous, then the subset

$$\mathfrak{V} = \left\{x \in P, \frac{\langle X_k(x)|N\rangle}{\langle X^1(x)|N\rangle} < 0,\ k \in \{2, 3\}\right\}$$

is open under the induced topology on the plane P and contains x_d. The trajectories of the vector field $\sum_{i=1}^{3} u_i X^i(x) = f(x)$ are in $\mathfrak{V}$ (when the initial condition is in $\mathfrak{V}$) if and only if

$$\langle f(x)|N\rangle = 0$$

Then

$$u_1\langle X^1(x)|N\rangle + u_2\langle X^2(x)|N\rangle + u_3\langle X^3(x)|N\rangle = 0$$

It follows that

$$u_1 = u_{1c} = -u_2\frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle} - u_3\frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle} \tag{11}$$

Since $u_2 \geq 0$, $u_3 \geq 0$, $\frac{\langle X^2(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle} < 0$ and $\frac{\langle X^3(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle} < 0$, then control u_{1c} is positive. The set $P \cap \mathfrak{V}$ is invariant by the trajectory of the system (10), with $f_1(x_d) = X^2(x_d) - \frac{\langle X^2(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle}X^1(x_d)$ and $f_2(x_d) = X^3(x_d) - \frac{\langle X^3(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle}X^1(x_d)$. It is clear that

$$\langle X^3(x_d)|N\rangle = \gamma_1\langle X^1(x_d)|N\rangle + \gamma_2\langle X^2(x_d)|N\rangle$$

then

$$\begin{aligned} f_2(x_d) &= (\gamma_1 X^1(x_d) + \gamma_2 X^2(x_d)) - \frac{\gamma_1\langle X^1(x_d)|N\rangle + \gamma_2\langle X^2(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle}X^1(x_d) \\ &= \gamma_2 X^2(x_d) - \gamma_2\frac{\langle X^3(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle}X^1(x_d) = \gamma_2 f_1(x_d) \end{aligned}$$

and the condition of Theorem 3.1 is satisfied. □

Construction of the vector N: For $x_d \in E_1$, we can construct the vector N which satisfies conditions of Theorem 3.2 as follows: since $x_d \in E_1$ then $rank\langle X^1(x_d), X^2(x_d), X^3(x_d)\rangle = 2$.

Without loss of generality we can suppose that $\{X^1(x_d), X^2(x_d)\}$ are linearly independent. We can construct an orthonormal basis of $\mathbb{R}^3$ as follows.

Let $v_1 = \frac{X^1(x_d)}{\|X^1(x_d)\|}$ and v_2 is a linear combination of the vectors $\{X^1(x_d), X^2(x_d)\}$ and satisfies $\langle v_2, X^1(x_d)\rangle = 0$, $\|v_2\| = 1$. It is clear that $B = \left\{v_1, \frac{v_2}{\|v_2\|}, v_1 \wedge \frac{v_2}{\|v_2\|}\right\}$ is an orthonormal basis of $\mathbb{R}^3$ ($v_1 \wedge v_2$ is the vector product of v_1 and v_2) and the new coordinates of $X^i(x_d)$, $i \in \{1,2,3\}$ and the vector N are:

$X^1(x_d) = (c,0,0)^T$, $X^2(x_d) = (a,b,0)^T$, $X^3(x_d) = (c_1,c_2,0)^T$ and $N = (1,s,s')$ with $c = \|X^1(x_d)\| > 0$.

Since x_d is in the set E_1, it is clear that $\frac{c_2}{b} < 0$ and $c_1 - c_2\frac{a}{b} < 0$. The vector N must be chosen such that $\frac{\langle X^2(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle} < 0$ and $\frac{\langle X^3(x_d)|N\rangle}{\langle X^1(x_d)|N\rangle} < 0$ then $a + bs < 0$ and $c_1 + c_2 s < 0$.

For $b > 0$ the real s is chosen such that

$$-\frac{c_1}{c_2} < s < -\frac{a}{b} \tag{12}$$

and if $b < 0$ the constant s satisfies

$$-\frac{a}{b} < s < -\frac{c_1}{c_2} \tag{13}$$

To simplify the construction of N, it is sufficient to choose $p(x_d) = 2$. We define the function

$$\begin{aligned} \psi(s,s') = &\langle f_2(x_d)|f_1(x_d)\rangle^2\langle d_{x_d}f_1(f_1(x_d))|(f_1(x_d))^{\perp}\rangle \\ &-\langle f_1(x)|f_1(x_d)\rangle^2\langle d_{x_d}f_2(f_2(x_d))|(f_1(x_d))^{\perp}\rangle \end{aligned}$$

where $d_{x_d}f_1$ is the differentiable function of f_1 at the point x_d. Under the form of the function f_1 (8) and f_2 (9) one has $f_1(x_d) = (-sb,b,0)$ and $f_2(x_d) = (-sc_2,c_2,0)$, the function ψ can be simplified as

$$\psi(s,s') = b^2c_2^2\left(s^2+1\right)^2\det\left(d_{x_d}f_1(f_1(x_d)), f_1(x_d), N\right)$$

$$-b^4\left(s^2+1\right)^2 \det\left(d_{x_d}f_2(f_2(x_d)), f_1(x_d), N\right)$$

Finally, it is clear that N satisfies conditions of Theorem 3.2 if we choose s and s' such that $\psi(s,s')\neq 0$ and s verifies (12) or (13).

Remark 3.3. *We construct a hybrid limit cycle for the system (2) with the control*

$$u_2, u_3 \in \{0,1\} \text{ and } u_{1c} = -u_2\frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle} - u_3\frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle}$$

Since the control u_{1c} is smooth enough then we can construct a sequence of a constant piecewise function $u_{1n}\geq 0$ such that

$$u_{1c} = \lim_{n\to+\infty} u_{1n}(x)$$

For p large enough we can take $u_{1c}\cong u_{1p}$ and from Remark 2.1 we can deduce some sequences of control for switched system (1) which makes the state around the desired point.

4. **Reachability Domain.**

Let us now recall the classic method for reaching the hybrid limit cycle determined in the previous section, from an initial state, and stabilizing it, with respect to the constraints on the continuous variables. Let us formulate this reachability problem in a generic way.

A trajectory (or solution) of an SDS from a hybrid initial state (x_0) to a hybrid final state $(x_c)\in CC(x_{c1},x_{c2})$ is defined as follows: $x_0 \to X^1 = X^{i_1}_{\delta_1}(x_0) \to x_2 = X^{i_2}_{\delta_2}(x_1) \to x_3 = X^{i_3}_{\delta_3}(x_2) \to \cdots \to x_c = X^{i_c}_{\delta_c}(x_{c-1})$ with $(i_j)\in\{1,2,3\}$ and $\forall\, j\in\{1,\ldots,c\}$, $\delta_j\in\mathbb{R}_+$. This continuous trajectory is made up of a succession of continuous trajectories with different dynamics. The concatenation of dynamics (discrete states) defines the switching sequence $(X_{i_1}, X_{i_2},\ldots,X_{i_c})$ with $i_k\in\{1,2,3\}$ and $1\leq k\leq c$. The switching points are $(x_1,x_2,\ldots,x_c)$. The SDS vector field may be discontinuous at the moment of switching, but there is no state discontinuity. The total duration of time the SDS takes to reach the hybrid limit cycle $CC(x_{c1},x_{c2})$ from the initial point is therefore: $t_c=\delta_1+\delta_2+\cdots+\delta_c$.

The global constraints of the system define the global operating domain which is denoted as Ω. It can be represented using some following linear inequalities in the state space.

Definition 4.1. *Let us consider a hybrid limit cycle $CC(x_{c1},x_{c2})$ a subset of the plane P. The $int(CC(x_{c1},x_{c2}))$ is the interior of the hybrid limit cycle under the induced topology on P. We define:*

$$A_0 = B_0 = C_0 = \delta\bigcup_{i=1}^{3}\{X^i_t(x);\ t\leq 0 \text{ and } x\in int(CC(x_{c1},x_{c2}))\}$$

The subsets A_i, B_i and C_i are defined as the following

$$\begin{array}{l} A_{i+1} = \{X^1_t(x);\ t\leq 0 \text{ and } x\in B_i\bigcup C_i\} \\ B_{i+1} = \{X^2_t(x);\ t\leq 0 \text{ and } x\in A_i\bigcup C_i\} \\ C_{i+1} = \{X^3_t(x);\ t\leq 0 \text{ and } x\in A_i\bigcup B_i\} \end{array}$$

These are the sets of trajectories that define all backward inferences from the hybrid limit cycle $CC(x_{c1},x_{c2})$ following the three vectors fields flows. It should be noted that these A_i, B_i and C_i do not necessarily define a partition of the state space.

A necessary and sufficient condition for SDS reachability by switching between three vector fields in $\mathbb{R}^3$ is given in the following Theorem 4.1.

Theorem 4.1. *Suppose that $int(CC(x_{c1}, x_{c2}))$ is a subset of $\mathfrak{V} \bigcap P$ with*

$$\mathfrak{V} = \left\{ x \in \mathbb{R}^3, \frac{\langle X_k(x)|N\rangle}{\langle X^1(x)|N\rangle} < 0,\ k \in \{2,3\} \right\}$$

Let us consider that $D = \left(\bigcup_{i\geq 1} A_i\right) \bigcup \left(\bigcup_{i\geq 1} B_i\right) \bigcup \left(\bigcup_{i\geq 1} C_i\right) \subseteq \mathbb{R}^3$ is an open set containing x_d and is the global reachability domain of the SDS. If $x_0 \in D$, there exists at least a sequence with a finite number of switchings, which leads the state of the system (1) from the point x_0 to the interior of hybrid limit cycle $CC(x_{c1}, x_{c2})$.

Proof: If we consider that $x_0 \in D$, then there exists an integer $J \geq 1$ such that for example $x_0 \in A_J$. Then, from Definition 4.1 and because $D = \left(\bigcup_{i\geq 1} A_i\right) \bigcup \left(\bigcup_{i\geq 1} B_i\right) \bigcup \left(\bigcup_{i\geq 1} C_i\right)$, $\exists \delta_1 \in \mathbb{R}_+$ and $x_1 \in B_{J-1} \bigcup C_{J-1}$ such that $x_1 = X^{i_1}_{\delta_1}(x_0)$ with $i_1 \in \{2,3\}$. We repeat this construction from x_1 to x_2. By inference, and after J steps we arrive to constructing $x_J \in int(CC(x_{c1}, x_{c2}))$.

The interior of $CC(x_{c1}, x_{c2})$ is supposed to be a subset of $\mathfrak{V} \bigcap P$, then for any $x \in int(CC(x_{c1}, x_{c2}))$ one has $\frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle} < 0$ and $\frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle} < 0$, then these sets

$$\{X^1_t(x);\ t \leq 0 \text{ and } x \in int(CC(x_{c1}, x_{c2}))\} \bigcup \{X^2_t(x);\ t \leq 0 \text{ and } x \in int(CC(x_{c1}, x_{c2}))\}$$

and

$$\{X^1_t(x);\ t \leq 0 \text{ and } x \in int(CC(x_{c1}, x_{c2}))\} \bigcup \{X^3_t(x);\ t \leq 0 \text{ and } x \in int(CC(x_{c1}, x_{c2}))\}$$

are open sets containing x_d. It follows that all sets A_i, B_i and C_i are open.

After the selection of a hybrid limit cycle $CC(x_{c1}, x_{c2})$, all the possible trajectories obtained by backward inferences from this cycle and meeting the global operating domain Ω can be determined, according to Definition 4.1. If D covers the totality of Ω, then, the hybrid limit cycle can be reached by switching from all point x_0 in Ω. Otherwise, the initial point x_0 must be in $D \subset \Omega$, so that there is at least one possible switching sequence that leads the SDS from this point x_0 to the hybrid limit cycle $CC(x_{c1}, x_{c2})$. This analysis involves calculating all the state space regions from which the final hybrid limit cycle can be reached. □

If one is interested only in finding a hybrid switching sequence that drives the system from the initial state x_0 to the final hybrid limit cycle, the analysis is completed as soon as the initial point is included into one of these regions. This analysis is done backwards in the continuous state space.

5. **Application to the Multicellular Converter.**

Multilevel converters can deliver a higher voltage and better than conventional converters. Their field of application is the field of medium and high voltage to high frequency pulse.

Noting the state $x = [v_{c_1}, v_{c_2}, i_L]^T \in \mathbb{R}^3$ and respecting the conventions of figure previous, the state of the converter equations is

$$\dot{x} = A_0 x(t) + \sum_{i=1}^{3} u_i(t)(A_i(x(t)) + B_i) \tag{14}$$

Matrices A_i, $i \in \{0, 1, 2, 3\}$ are defined by:

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{R}{L} \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 & -\frac{1}{C_1} \\ 0 & 0 & 0 \\ \frac{1}{L} & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & \frac{1}{C_1} \\ 0 & 0 & -\frac{1}{C_2} \\ -\frac{1}{L} & \frac{1}{L} & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{pmatrix}$$

the vectors $B_1 = B_2 = 0$ and $B_3 = \left(0, 0, \frac{E}{L}\right)^T$. $u = (u_1, u_2, u_3)^T \in \{0,1\}^3$ corresponds to the control position of the various switches: $u_i = 0$ means that the switch is open from the top and the bottom of the cell is closed, $u_j = 1$ means that the switch is closed from the top and the bottom of the cell is open.

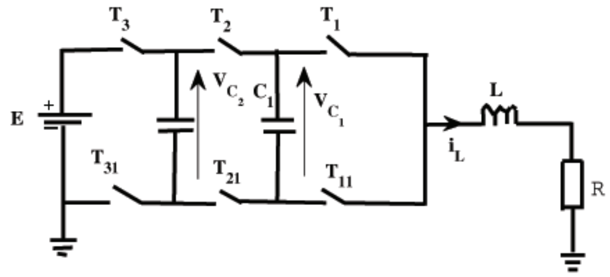

FIGURE 1. Multicellular converter

The system (14) is equivalent to (1) with

$$X^1(x) = (A_0 + A_1)x,\ X^2(x) = (A_0 + A_2)x$$

and

$$X^3(x) = (A_0 + A_3)x + B_3$$

The parameter values are as follows: $R = 30\Omega$, $C_1 = 45\mu$F, $C_2 = 45\mu$F, $L = 0.1$H, $E = 30$V. The minimum time between two switching is $T_{\min} = 5 \times 10^{-6}$s.

The set of possible points of equilibrium for this system (14) is the same as all the equilibrium points of the system (1). It is such that $u_{1moy} = u_{2moy} = u_{3moy}$, $0 < x_{d_1} <$ 30V, $0 < x_{d_2} < 30$V and $0 < x_{d_3} < 1$A because $x_{d_3} = \frac{Eu_{3moy}}{R}$.

The desired operating point x_d is:

$$x_d = (x_{d_1}, x_{d_2}, x_{d_3})^T$$

with $x_{d_1} = 15$V, $x_{d_2} = 5$V and $x_{d_3} = \frac{1}{3}$A. Note that x_d lies in E_1, because it satisfies

$$X^3(x_d) = \gamma_1 X^1(x_d) + \gamma_2 X^2(x_d)$$

with $\gamma_1 = \gamma_2 = -1$.

Let $B = \left\{\frac{v_1}{\|v1\|}, v_2, \frac{v_1}{\|v1\|} \wedge v_2\right\}$ a basis of $\mathbb{R}^3$ with v_1 be chosen in the plane spanned by $\{X^1(x_d), X^2(x_d)\}$ and satisfy $\langle v_1, X^2(x_d)\rangle = 0$ and $v_2 = \frac{X^2(x_d)}{\|X^2(x_d)\|}$. In this basis we can write: $X^1(x_d) = 10^3(5.2381, -5.2378, 0)^T$, $X^2(x_d) = 10^3(0, 1.0478, 0)^T$ and $X^3(x_d) = 10^3(-5.2381, -5.2397, 0)^T$. According to remark (6), the normal vector is given by $N = (3, -\frac{1}{7}, -3)^T$. Furthermore, this normal satisfies $\frac{\langle X^1(x_d), N\rangle}{\langle X^2(x_d), N\rangle} < 0$, $\frac{\langle X^3(x_d), N\rangle}{\langle X^2(x_d), N\rangle} < 0$ with $\langle X^1(x_d), N\rangle = -2.2372 \times 10^4$, $\langle X^2(x_d), N\rangle = 2.3880 \times 10^4$ and $\langle X^3(x_d), N\rangle = -1.5082 \times 10^3$. In this case, the both control u_1, u_3 are in $\{0,1\}$ and u_{2c} is positive in the neighborhood of x_d (Figure 5). The both vectors in the plane P, f_1 and f_2 are given by

$$f_1(x) = X^1(x) - \frac{\langle X^1(x)|N\rangle}{\langle X^2(x)|N\rangle} X^2(x)$$

and

$$f_2(x) = X^3(x) - \frac{\langle X^3(x)|N\rangle}{\langle X^2(x)|N\rangle} X^2(x)$$

Moreover, we can remark that $f_1(x_d) = -f_2(x_d)$.

Furthermore, for $p = 2$, the condition

$$\langle f_2(x), v\rangle^2 \det(d_{x_d} f_1(f_1(x_d)), f_1(x_d), N) \neq \langle f_1(x_d), f_1(x_d)\rangle^2 \det(d_{x_d} f_2(f_2(x_d)), f_1(x_d), N)$$

is checked. Therefore, using this condition, a limit cycle exists in the invariant plane P (Figure 3).

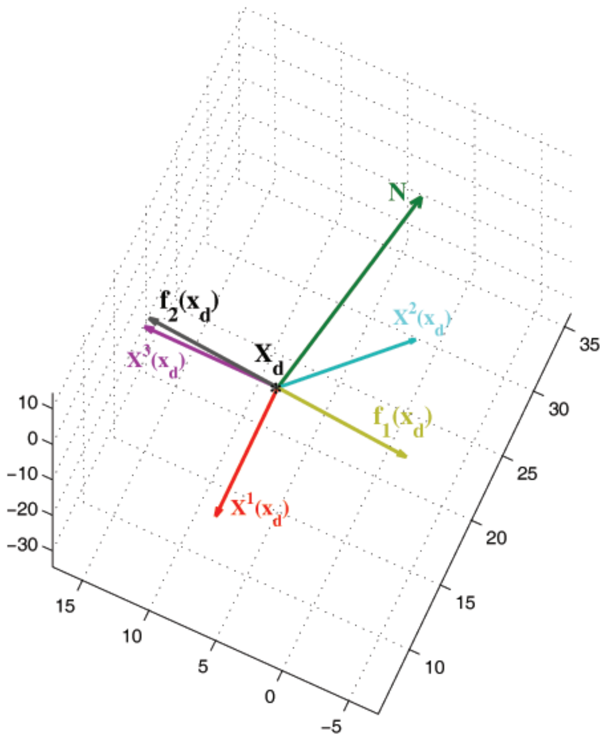

FIGURE 2. Position of vectors around x_d

The two switching points are given by $x_{c_1} = (14.74\ 1.432\ 0.25)^T$ and $x_{c_2} = (15.27\ 9.324\ 0.4)^T$. The limit cycle around x_d has a duration of 23×10^{-4} seconds. In the proposed cycle, one has: $\triangle v_{c_1} = 3.5\%$, $\triangle v_{c_2} = 87.6\%$ and $\triangle i_L = 37.5\%$. We can remark that the limit cycle (Figure 3) is far from x_d. Then we can have the cycle closest to x_d (Figure 4) with the two switching points $x_{c_1} = (14.99, 4.906, 0.331)^T$ and $x_{c_2} = (15.01, 5.092, 0.335)^T$. With this new cycle $\triangle v_{c_1} = 0.13\%$, $\triangle v_{c_2} = 3.72\%$ and $\triangle i_L = 1.2\%$. The cycle around x_d has a duration of 10^{-4} seconds.

6. **Nonlinear Example in $\mathbb{R}^3$.** Let us consider the nonlinear system:

$$\dot{x} = u_1 X^1(x) + u_2 X^2(x) + u_3 X^3(x) \tag{15}$$

with

$$X^1(x) = \begin{pmatrix} x_1^3 - x_2 + x_3 \\ -x_1^3 - x_2 \\ x_3 - x_2 \end{pmatrix}$$

$$X^2(x) = \begin{pmatrix} -x_2^3 + x_1 - x_3^2 - x_3 \\ x_1 + x_2 + x_3 \\ x_1^3 - x_3 \end{pmatrix}$$

and

$$X^3(x) = \begin{pmatrix} x_2 - x_1 + x_3^2 \\ -x_1 + 2x_2 \\ x_2 - x_3 - 2 \end{pmatrix}$$

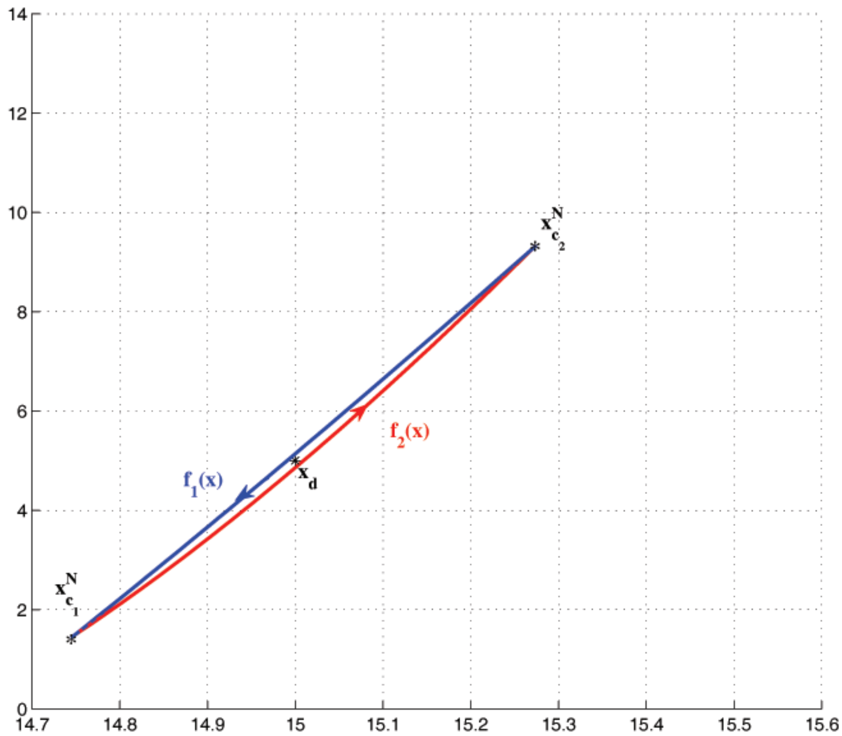

FIGURE 3. Limit cycle around x_d

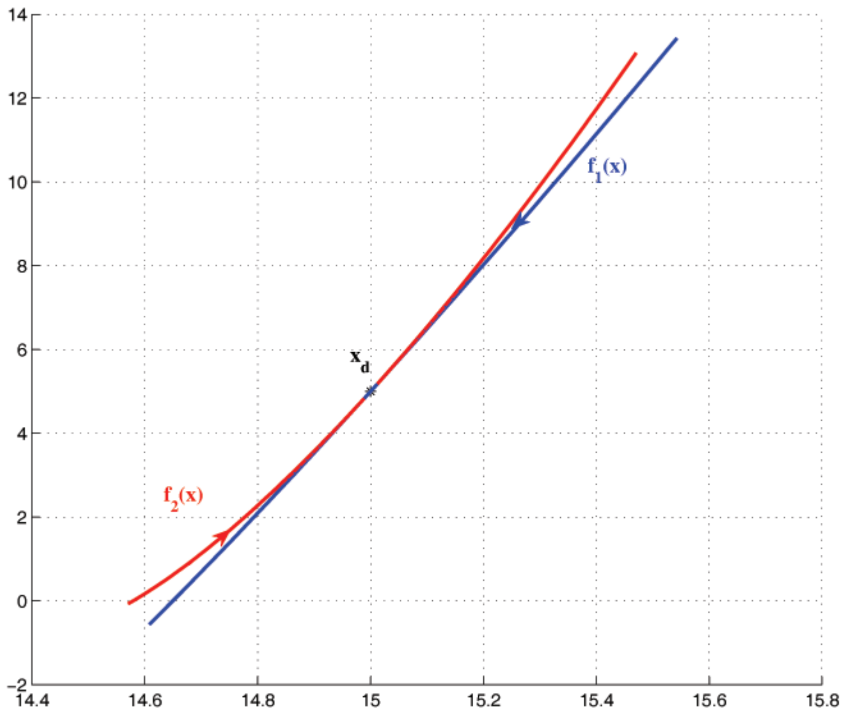

FIGURE 4. Limit cycle closest to x_d

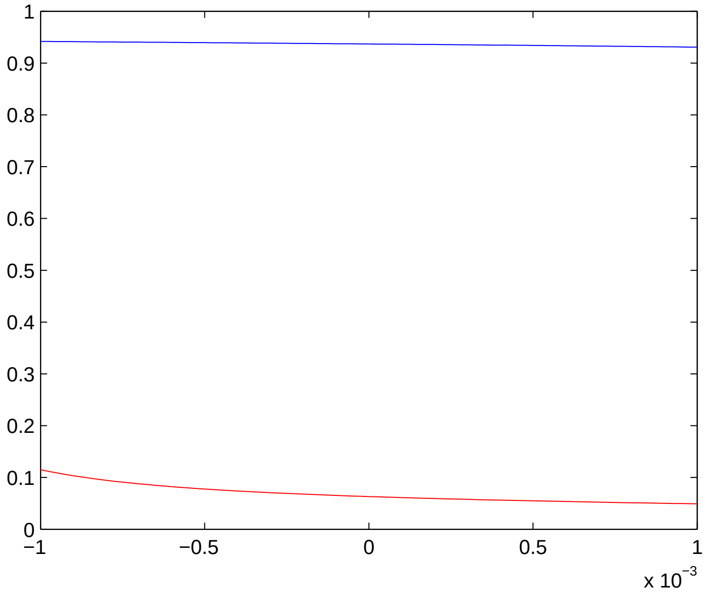

FIGURE 5. Graph of function u_{2c}

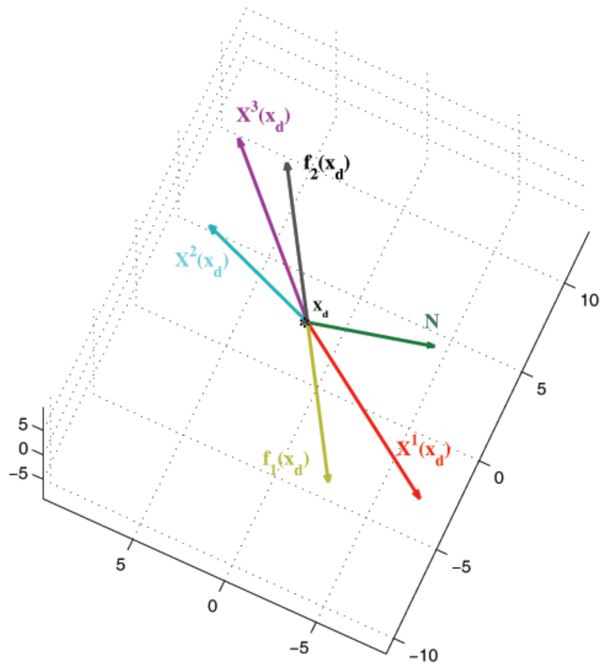

FIGURE 6. Position of vectors in the point x_d

Our goal is to construct a hybrid limit cycle around a desired point. For $x_d = (1, 1, -1)^T$ one has $X^1(x_d) + X^2(x_d) + X^3(x_d) = 0$. Then $x_d \in E_1$.

In the new orthonormal basis $B = \{v_1, v_2, v_1 \wedge v_2\}$ we can write $X^1(x_d) = (3, 0, 0)^T$, $X^2(x_d) = (-2, 1, 0)^T$ and $X^3(x_d) = (-1, -1, 0)^T$. A simple computation gives $N = (1, -3, -1)^T$ which satisfies $\frac{\langle X^2(x_d),N\rangle}{\langle X^1(x_d),N\rangle} < 0$ and $\frac{\langle X^3(x_d),N\rangle}{\langle X^1(x_d),N\rangle} < 0$ with $\langle X^1(x_d), N\rangle = 7$, $\langle X^2(x_d), N\rangle = -5$ and $\langle X^3(x_d), N\rangle = -2$. It follows that u_{1c} is positive in the point x_d, and therefore, in the neighborhood of x_d (Figure 8). The vectors f_1 and f_2 in the plane

P are given by

$$f_1(x) = X^2(x) - \frac{\langle X^2(x)|N\rangle}{\langle X^1(x)|N\rangle} X^1(x)$$

and

$$f_2(x) = X^3(x) - \frac{\langle X^3(x)|N\rangle}{\langle X^1(x)|N\rangle} X^1(x)$$

The condition $p = 2$ such that the trajectories of both f_1 and f_2 are not transverse is checked, and then there exists a hybrid limit cycle around x_d in the invariant plan P (Figure 7).

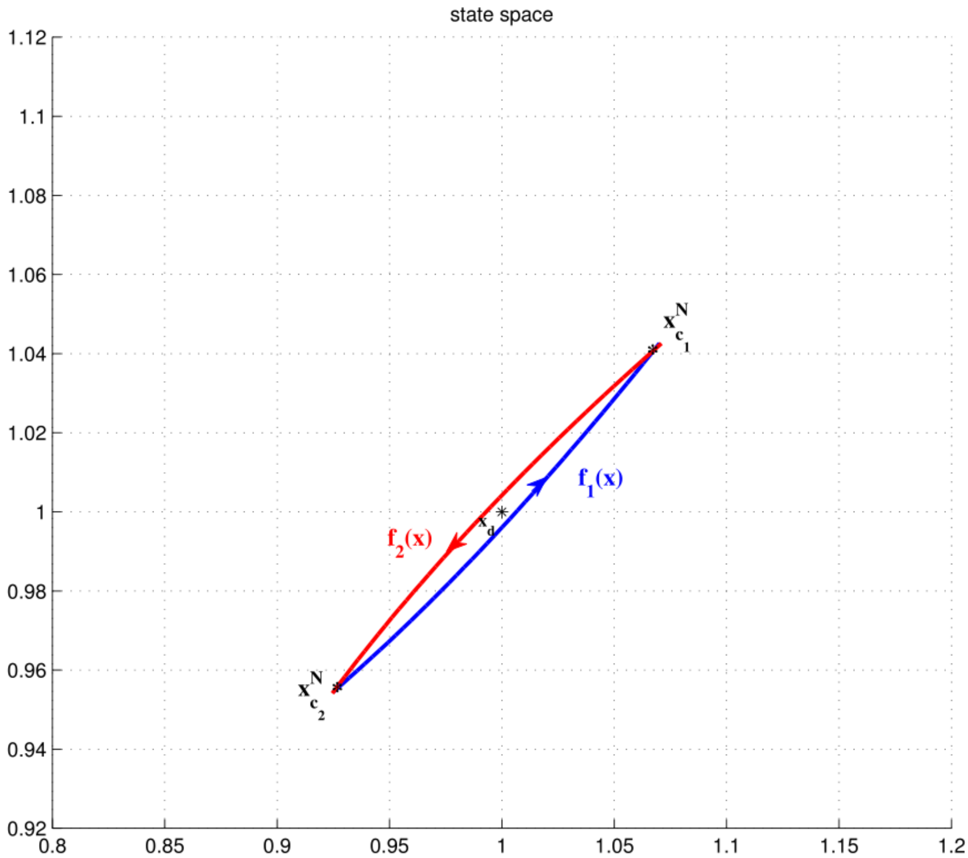

FIGURE 7. Limit cycle around x_d

The duration between two successive switching points of the extremity of the hybrid limit cycle $x_{c_1} = (0.925\ 0.953\ \ -0.938)^T$ and $x_{c_2} = (1.067\ 1.038\ \ -1.046)^T$ is 48×10^{-3} seconds. It may be noted that the duration of this cycle is very long. It may be closer to the operating point x_d with a duration 2×10^{-2} seconds between two successive switching points $x_{c_1} = (0.998\ 0.999\ \ -0.998)^T$ and $x_{c_2} = (1.002\ 1.001\ \ -1.001)^T$. The hybrid limit cycle is defined by $\triangle x_1 = 0.36\%$, $\triangle x_2 = 0.2\%$ and $\triangle x_3 = 0.23\%$.

7. **Conclusion.** This paper presents a new constructive method for the synthesis of a stabilizing control for a class of switched dynamical systems in $\mathbb{R}^3$, switching between three discrete modes, without state discontinuity and which respect the technological constraints (minimum duration between two successive switchings, boundedness of the real valued state variables). The hybrid limit cycle is constructed in a theory section with $\delta_{\min} = 0$. Furthermore, in the multilevel converters there exists a real hybrid limit cycle around the desired point x_d.

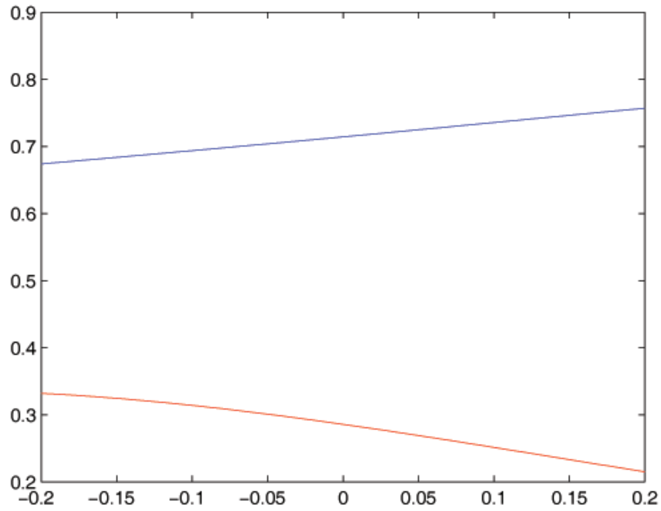

FIGURE 8. Graph of the function u_{1c}

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