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# PYTHAGOREAN FUZZY INDUCED GENERALIZED OWA OPERATOR AND ITS APPLICATION TO MULTI-ATTRIBUTE GROUP DECISION MAKING

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Abstract. *With respect to multi-attribute group decision making (MAGDM) problems in which the attribute values take form of Pythagorean fuzzy numbers, a new Pythagorean fuzzy aggregation operator called Pythagorean fuzzy induced generalized ordered weighted averaging (PFIGOWA) operator is proposed. Some desirable properties and families of the proposed operator are discussed. Furthermore, based on the proposed operator, a novel method is developed to solve MAGDM problems under Pythagorean fuzzy environment. Finally, an illustrative example of photovoltaic cell selection is provided to illustrate the applicability and effectiveness of the proposed method.*
**Keywords:** Pythagorean fuzzy set, IOWA operator, GOWA, Multi-attribute decision making

1. **Introduction.** Since fuzzy set (FS) theory was introduced by Zadeh [1], a number of extensions both in theoretical and practical areas have been presented. Among these extensions, the concept of intuitionistic fuzzy set (IFS) introduced by Atanassov [2], characterized by a membership function and a nonmembership function, has turned out to be a powerful tool in depicting the vagueness and uncertainty of things more comprehensively. In the past few decades, the intuitionistic fuzzy set theory has gained great attention and been applied to many practical areas such as decision-making, pattern recognition, medical diagnosis, and clustering analysis [3-9].

Recently, Yager [10] introduced a new extension of fuzzy set, called the Pythagorean fuzzy set (PFS). The reason of putting forward the PFS is that in some practical decision-making situations, the sum of the support (membership) degree and the against (nonmembership) degree may be bigger than 1 but their square sum is equal to or less than. As a new extension of fuzzy set, several studies of the Pythagorean fuzzy set have been carried out from researchers to manage the complex uncertainty in decision making problems. For example, Zhang and Xu [11] defined some novel operational laws of PFS, and extended the traditional TOPSIS (technique for order preference by similarity to ideal solution) approach to deal effectively with the decision-making problems under Pythagorean fuzzy environment. Peng and Yang [12] developed several Pythagorean fuzzy functions and studied their fundamental properties. Zhang [13] put forward some similarity measure for Pythagorean fuzzy set and studied their application MAGDM problems. Zhang [14] developed a closeness index-based Pythagorean fuzzy QUALIFLEX method to address hierarchical multicriteria decision-making problems under Pythagorean fuzzy environment. Zeng et al. [15] developed a hybrid method for MAGDM problems.

Aggregation operators play a vital role during the information fusion process in decision making. Up to now, a large number of aggregation operators are found in the literature. One common aggregation method is the ordered weighted averaging (OWA) operator [16]. It provides a parameterized family of aggregation operators that include as special cases the maximum, the minimum and the average. An interesting extension of OWA operator is the induced OWA (IOWA) operator [17], in which the reordering step is no longer determined only by the values of the arguments, but the values of their associated order-inducing variables. A further generalization has been suggested by using generalized means. This operator is known as the induced generalized OWA (IGOWA) operator [18]. The main advantage of this operator is that it generalizes the OWA operator, including the main characteristics of both the generalized OWA (GOWA) [19] and the IOWA operator. Therefore, it provides a wide range of mean operators including all the particular cases of the IOWA, the GOWA operator and the induced ordered weighted geometric (IOWG) operator, etc. In the last few years, the IGOWA operator has received more and more attention from researcher [20-22].

Although the IGOWA operator is proved to be an effective operator, there is no investigation on the application of IGOWA in the Pythagorean fuzzy set, which is able to model the uncertainty in the practical decision making problems more accurately than IFS. Inspired by this idea, in this paper, we should introduce the Pythagorean fuzzy IGOWA (PFIGOWA) operator. It is a new aggregation operator that uses generalized means in the IOWA operator and in Pythagorean fuzzy situations where the available information cannot be represented with exact numbers but it is possible to use Pythagorean fuzzy information. Going a step further, we investigate some of its main properties and different families. Finally, we develop an MAGDM approach for evaluating photovoltaic cell selection based on the PFIGOWA operator.

The remainder of this article is organized as follows. First, we briefly review Pythagorean fuzzy set theory, the IOWA and the IGOWA operator. Second, we present the PFIGOWA operator and next we study a wide range of particular cases. Third, we discuss the applicability of the PFIGOWA operator with an MAGDM example and we end the paper summarizing the main conclusions.

## 2. **Preliminaries.** 

In this section we briefly review the Pythagorean fuzzy set theory, the IOWA operator and IGOWA approach.

### 2.1. **Pythagorean fuzzy set.** 

Intuitionistic fuzzy set introduced by Atanassov [2] is defined as follows.

**Definition 2.1.** *Let a set $X = \{x_1, x_2, \ldots, x_n\}$ be fixed, and an IFS $I$ is given as follows:*

$$I = \{\langle x, I(\mu_I(x), v_I(x))\rangle \,| x \in X\} \tag{1}$$

*The numbers $\mu_I(x)$ and $v_I(x)$ represent, respectively, the membership degree and non-membership degree of the element $x$ to the set $I$, $0 \leq \mu_I(x) + v_I(x) \leq 1$, for all $x \in X$. For any IFS $I$ and $x \in X$, $\pi_I(x) = 1-\mu_I(x)-v_I(x)$ is called the degree of indeterminacy of $x$ to the set $I$. The pair $(\mu_I(x), v_I(x))$ is called intuitionistic fuzzy number (IFN) and each IFN can be simply denoted as $\alpha = I(\mu_\alpha, v_\alpha)$, where $\mu_\alpha \in [0, 1]$, $v_\alpha \in [0, 1]$, $\mu_\alpha + v_\alpha \leq 1$.*

As an extension of the classical fuzzy set, the IFS is a suitable way to deal with vagueness. However, in various real-world decision-making problems the decision makers (or experts) may express their preferences of an alternative with a criterion satisfying the condition that the sum of the degree to which the alternative satisfies the criterion and dissatisfies the criterion is bigger than 1. Obviously, the experts' preferences cannot be described by using the IFS in this situation. For such cases, Yager [10] developed Pythagorean fuzzy set (PFS) characterized by a membership degree and nonmembership degree, which satisfies the condition that the square sum is less than or equal to 1.

**Definition 2.2.** *Let a set $X = \{x_1, x_2, \ldots, x_n\}$ be fixed, and a PFS $P$ is given as follows:*

$$P = \{\langle x, P(\mu_P(x), v_P(x))\rangle \,|x \in X\} \tag{2}$$

*The numbers $\mu_P(x)$ and $v_P(x)$ represent, respectively, the membership degree and non-membership degree of the element $x$ to the set $P$, $0 \leq (\mu_P(x))^2 + (v_P(x))^2 \leq 1$, for all $x \in X$. For any PFS $P$ and $x \in X$, $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (v_P(x))^2}$ is called the degree of indeterminacy of $x$ to the set $P$. For simplicity, the pair $P(\mu_P(x), v_P(x))$ is called Pythagorean fuzzy number (PFN) and each PFN can be simply denoted as $\beta = P(\mu_\beta, v_\beta)$, where $\mu_\beta \in [0,1]$, $v_\beta \in [0,1]$ and $(\mu_\beta)^2 + (v_\beta)^2 \leq 1$.*

Given three PFNs $\beta = P(\mu_\beta, v_\beta)$, $\beta_1 = P(\mu_{\beta_1}, v_{\beta_1})$ and $\beta_2 = P(\mu_{\beta_2}, v_{\beta_2})$, Zhang and Xu [11] gave some operations on them, shown as:

(1) $\beta_1 \oplus \beta_2 = P\left(\sqrt{\mu^2_{\beta_1} + \mu^2_{\beta_2} - \mu^2_{\beta_1} \cdot \mu^2_{\beta_2}}, v_{\beta_1} \cdot v_{\beta_2}\right)$;

(2) $\beta_1 \otimes \beta_2 = P\left(\mu_{\beta_1} \cdot \mu_{\beta_2}, \sqrt{v^2_{\beta_1} + v^2_{\beta_2} - v^2_{\beta_1} \cdot v^2_{\beta_2}}\right)$;

(3) $\lambda\beta = P\left(1 - \left(1 - \mu^2_\beta\right)^\lambda, (v_\beta)^\lambda\right)$, $\lambda > 0$;

(4) $\beta^\lambda = P\left((\mu_\beta)^\lambda, 1 - \left(1 - v^2_\beta\right)^\lambda\right)$, $\lambda > 0$.

**Definition 2.3.** [11] *Let $\beta_j = P\left(\mu_{\beta_j}, v_{\beta_j}\right)$ $(j = 1, 2)$ be two PFNs, a nature quasi-ordering on the PFNs is defined as follows: $\beta_1 \geq \beta_2$ if and only if $\mu_{\beta_1} \geq \mu_{\beta_2}$ and $v_{\beta_1} \leq v_{\beta_2}$.*

To compare the PFSs, Zhang and Xu [11] defined the following comparison laws.

**Definition 2.4.** *For a PFN $\beta = P(\mu_\beta, v_\beta)$, $s(\beta) = (\mu_\beta)^2 - (\nu_\beta)^2$ is called the score function of $\beta$. For two PFNs $\beta_1 = P(\mu_{\beta_1}, v_{\beta_1})$ and $\beta_2 = P(\mu_{\beta_2}, v_{\beta_2})$, if $s(\beta_1) > s(\beta_2)$, then $\beta_1 > \beta_2$; if $s(\beta_1) = s(\beta_2)$, then $\beta_1 = \beta_2$.*

To aggregate PFNs, Yager [10] introduced a Pythagorean fuzzy weighted averaging aggregation (PFWA) operator. While Zhang [13] pointed out that Yager's PFWA operator was different from the basic operational laws of PFNs introduced by Zhang and Xu [11]. To overcome this drawback, Zhang [13] defined the Pythagorean fuzzy ordered weighted averaging (PFOWA) operator as follows.

**Definition 2.5.** *Let $\beta_j = P\left(\mu_{\beta_j}, v_{\beta_j}\right)$ $(j = 1, 2)$ be two PFNs, and a PFOWA operator of dimension $n$ is a mapping* PFOWA*: $\Omega^n \to \Omega$ that has an associated weighting $W$ with $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$ such that:*

$$PFOWA(\beta_1, \beta_2, \ldots, \beta_n) = \sum_{j=1}^n w_j\gamma_j = P\left(\sqrt{1 - \prod_{j=1}^n \left(1 - \mu^2_{r_j}\right)^{w_j}}, \prod_{j=1}^n v^2_{r_j}\right) \tag{3}$$

*where $\gamma_j$ is the $j$th largest of the $\beta_i$.*

2.2. **IOWA and IGOWA operator.** The IOWA operator is characterized by its reordering step that is not carried out with the values of the arguments $a_i$. It can be defined as follows.

**Definition 2.6.** *An IOWA operator of dimension $n$ is a mapping IOWA: $R^n \times R^n \to R$ that has an associated weighting $W$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:*

$$IOWA\left(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle\right) = \sum_{j=1}^{n} w_j b_j \tag{4}$$

*where $b_j$ is $a_i$ value of the IOWA pair $\langle u_i, a_i\rangle$ having the $j$th largest $u_i$, $u_i$ is the order inducing variable and $a_i$ is the argument variable.*

The generalized OWA (GOWA) operator introduced by Yager [19] can be defined as follows.

**Definition 2.7.** *A GOWA operator of dimension $n$ is a mapping GOWA: $R^n \to R$ that has an associated weighting $W$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:*

$$GOWA(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \tag{5}$$

*where $b_j$ is the $j$th largest of the $a_i$, and $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.*

As demonstrated in previous studies, the GOWA operator is commutative, monotonic, bounded, and idempotent. A further interesting extension of the GOWA operator is the induced GOWA (IGOWA) operator [18] that uses order-inducing variables in the GOWA operator. The IGOWA operator can be defined as follows.

**Definition 2.8.** *An IGOWA operator of dimension $n$ is a mapping IGOWA: $R^n \times R^n \to R$ that has an associated weighting $W$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:*

$$IGOWA\left(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle\right) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \tag{6}$$

*where $b_j$ is $a_i$ value of the IGOWA pair $\langle u_i, a_i\rangle$ having the $j$th largest $u_i$, $u_i$ is the order inducing variable and $a_i$ is the argument variable. $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$. As we can see, if $\lambda = 1$, we obtain the IOWA operator. If $\lambda = 0$, the IOWG operator, if $\lambda = -1$, the IOWHA operator, and if $\lambda = 2$, the IOWQA operator.*

When using the IGOWA operator, it is assumed that the available information can be represented in the form of exact numbers. Next we should extend the IGOWA operator to Pythagorean fuzzy environment and introduce the PFIGOWA operator.

3. **The PFIGOWA Operator.** The PFIGOWA operator is an extension of the IGOWA operator that uses uncertain information in the aggregation represented in the form of PFNs. Note that the PFIGOWA can also be seen as an aggregation operator that uses the IOWA operator, the GOWA and PFNs in the same formulation. The reason for using this operator is that sometimes, the uncertain factors that affect our decisions are not clearly known and in order to assess the problem we need to use PFNs. Another reason is we should deal with complex attitudinal characters (or complex degrees of orness) in the decision process by using order-inducing variables in some group decision-making problems. This operator can be defined as follows.

**Definition 3.1.** *A PFIGOWA operator of dimension $n$ is a mapping PFIGOWA: $\Omega^n \times R^n \to \Omega$ that has an associated weighting $W$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:*

$$PFIGOWA\left(\langle u_1, \beta_1\rangle, \langle u_2, \beta_2\rangle, \ldots, \langle u_n, \beta_n\rangle\right) = \left(\sum_{j=1}^{n} w_j \gamma_j^{\lambda}\right)^{1/\lambda} \tag{7}$$

*where $\gamma_j$ is $\beta_i$ value of the PFIGOWA pair $\langle u_i, \beta_i\rangle$ having the $j$th largest $u_i$, $u_i$ is the order inducing variable and $\beta_i$ is the argument variable. $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.*

An important issue is to consider the measures for characterizing the weighting vector $W$ of the PFIGOWA operator based on the attitudinal character:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right) \tag{8}$$

It can be shown that $\alpha \in [0,1]$. The more weight is located near the top of $W$, the closer $\alpha$ is to 1, while the more weight is located toward the bottom of $W$, the closer $\alpha$ is to 0.

Next we present a simple numerical example to illustrate the PFIGOWA's application in aggregation process

**Example 3.1.** *Assume the following arguments in an aggregation process: $\beta = (P(0.8, 0.3), P(0.4, 0.5), P(0.7, 0.4), P(0.9, 0.2))$ with the following order-inducing variables $U = (4, 7, 1, 9)$. Assume the following weighting vector $W = (0.2, 0.1, 0.4, 0.3)$ and without loss of generality, let $\lambda = 2$, then we can aggregate this information by using the PFIGOWA operator, we will get the following result:*

$$\begin{aligned} & PFIGOWA\left(\langle u_1, \beta_1\rangle, \langle u_2, \beta_2\rangle, \langle u_3, \beta_3\rangle, \langle u_4, \beta_4\rangle\right) \\ &= \left(0.2 \times (0.9, 0.2)^2 + 0.1 \times (0.4, 0.5)^2 + 0.4 \times (0.8, 0.3)^2 + 0.3 \times (0.7, 0.4)^2\right)^{1/2} \\ &= P(0.634, 0.387). \end{aligned}$$

*Note that if the weighting vector is not normalized, i.e., $\hat{V} = \sum_{j=1}^{n} \hat{\upsilon}_j \neq 1$, then, the PFIGOWA operator can be expressed as:*

$$PFIGOWA\left(\langle u_1, \beta_1\rangle, \langle u_2, \beta_2\rangle, \ldots, \langle u_n, \beta_n\rangle\right) = \frac{1}{\hat{V}} \sum_{j=1}^{n} \hat{\upsilon}_j \gamma_j \tag{9}$$

*The PFIGOWA operator is monotonic, bounded and idempotent. These properties can be proved with the following theorems. It is monotonic because $\beta_i \geq \beta_i'$ for all $i$, then $PFIGOWA\left(\langle u_1, \beta_1\rangle, \ldots, \langle u_n, \beta_n\rangle\right) \geq PFIGOWA\left(\langle u_1, \beta_1'\rangle, \ldots, \langle u_n, \beta_n'\rangle\right)$. It is commutative because any permutation of the arguments has the same evaluation. That is, $PFIGOWA\left(\langle u_1, \beta_1\rangle, \ldots, \langle u_n, \beta_n\rangle\right) = PFIGOWA\left(\langle u_1', \beta_1'\rangle, \ldots, \langle u_n', \beta_n'\rangle\right)$, where $(\langle u_1', \beta_1'\rangle, \ldots, \langle u_n', \beta_n'\rangle)$ is any permutation of the arguments $(\langle u_1, \beta_1\rangle, \ldots, \langle u_n, \beta_n\rangle)$. It is bounded because the PFIGOWA aggregation is delimitated by the minimum and the maximum. That is, $Min(\beta_i) \leq PFIGOWA\left(\langle u_1, \beta_1\rangle, \ldots, \langle u_n, \beta_n\rangle\right) \leq Max(\beta_i)$. It is idempotent because if $\beta_i = \beta$, for all $i$, $PFIGOWA\left(\langle u_1, \beta_1\rangle, \ldots, \langle u_n, \beta_n\rangle\right) = \beta$.*

4. **Families of PFIGOWA Operators.** In the following we study different families of PFIGOWA operators. Basically, we distinguish between the families found in the parameter $\lambda$ and the weighting vector $W$.

Firstly, if we analyze the parameter $\lambda$ in the PFIGOWA operator, we will obtain a wide range of Pythagorean fuzzy aggregation operators such as the Pythagorean fuzzy induced

OWA (PFIOWA) operator, the Pythagorean fuzzy induced Euclidean ordered weighted averaging (PFIEOWA) operator, the Pythagorean fuzzy induced ordered weighted geometric (PFIOWG) operator, the Pythagorean fuzzy induced ordered weighted harmonic averaging (PFIOWHA) operator and a lot of other cases.

**Remark 4.1.** *If $\lambda = 1$, then, we get the PFIOWA operator.*

$$PFIGOWA\left(\langle u_1, \beta_1\rangle, \langle u_2, \beta_2\rangle, \ldots, \langle u_n, \beta_n\rangle\right) = \sum_{j=1}^{n} w_j \gamma_j \tag{10}$$

*where $\gamma_j$ is $\beta_i$ value of the PFIGOWA pair $\langle u_i, \beta_i\rangle$ having the $j$th largest $u_i$, $u_i$ is the order inducing variable and $\beta_i$ is the argument variable.*

Note that we will obtain the Pythagorean fuzzy weighted averaging (PFWA) is obtained if $u_i > u_{i+1}$ for all $i$ in the PFIOWA operator, and the Pythagorean fuzzy ordered weighted averaging distance (PFOWA) is obtained if the ordered position of $u_i$ is the same as the ordered position of $\gamma_j$.

**Remark 4.2.** *If $\lambda = 2$, we get the PFIEOWA operator.*

$$PFIGOWA\left(\langle u_1, \beta_1\rangle, \langle u_2, \beta_2\rangle, \ldots, \langle u_n, \beta_n\rangle\right) = \left(\sum_{j=1}^{n} w_j \gamma_j^2\right)^{1/2} \tag{11}$$

**Remark 4.3.** *When $\lambda = 0$, we get the PFIOWG operator.*

$$PFIGOWA\left(\langle u_1, \beta_1\rangle, \langle u_2, \beta_2\rangle, \ldots, \langle u_n, \beta_n\rangle\right) = \prod_{j=1}^{n} \gamma_j^{w_j} \tag{12}$$

**Remark 4.4.** *When $\lambda = -1$, we get the PFIOWHA operator.*

$$PFIGOWA\left(\langle u_1, \beta_1\rangle, \langle u_2, \beta_2\rangle, \ldots, \langle u_n, \beta_n\rangle\right) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{\gamma_j}} \tag{13}$$

Moreover, by choosing a different parameter $W$ in the PFIGOWA operator, we are able to obtain different types of Pythagorean fuzzy aggregation operators, such as the Pythagorean fuzzy maximum (PFMax) operator, the Pythagorean fuzzy minimum (PFMin) operator, the Pythagorean fuzzy generalized weighted operator (PFWG), the Step-PFIGOWA and the Olympic-PFIGOWA operator

- The PFMax operator is obtained if $w_p = 1$, $w_j = 0$, for all $j \neq p$, and $u_p = \max(\beta_i)$.
- The PFMin operator is obtained if $w_p = 1$, $w_j = 0$, for all $j \neq p$, and $u_p = \min(\beta_i)$.
- The PFWG is obtained if $u_i > u_{i+1}$ for all $i$.
- Step-PFIGOWA: If $w_k = 1$ and $w_j = 0$ for all $j \neq k$.
- Olympic-PFIGOWA: If $w_1 = w_n = 0$ and for all others $w_j = 1/(n-2)$.

5. **An Approach to MAGDM Based on the PFIGOWA Operator.** Similar to the some common aggregation operators such as the OWA and the IOWA operators, the proposed PFIGOWA operator can be applied in many areas, including the statistics, engineering, economics, decision theory and soft computing, etc. In this paper, we will focus on studying the application of the PFIGOWA operator in an MAGDM problem. The process can be summarized as follows.

**Step 1.** Let $A = \{A_1, A_2, \ldots, A_m\}$ be a discrete set of alternatives, and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of attributes. Let $E = \{e_1, e_2, \ldots, e_t\}$ be the set of decision makers (whose weight vector is $V = (v_1, v_2, \ldots, v_t)$, $v_k \geq 0$, $\sum_{k=1}^{t} v_k = 1$). Each decision maker provides his own Pythagorean fuzzy matrix $\left(\beta_{ij}^{(k)}\right)_{m\times n}$.

**Step 2.** Use the Pythagorean fuzzy weighted averaging (PFWA) operator [13] to aggregate the information of the decision makers $E$ by using the weighting vector $V$. The result is the fuzzy collective Pythagorean fuzzy matrix $(\beta_{ij})_{m\times n}$, where

$$\beta_{ij} = v_1\beta_{ij}^{(1)} \oplus v_2\beta_{ij}^{(2)} \oplus \cdots \oplus v_k\beta_{ij}^{(k)}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \tag{14}$$

**Step 3.** Calculate the order-inducing variables $U$ and the weighting vector $W$ of the PFIGOWA operator by Equation (8).

**Step 4.** Calculate the aggregated results using the PFIGOWA operator explained in Equation (7). Note that it is possible to consider a wide range of PFIGOWA operators, such as those described in Section 4.

**Step 5.** Adopt decisions according to the results found in the previous steps. Select the alternative/s that provides the best result/s.

6. **Illustrative Example.** In the following section, we will develop a numerical example of the new approach in MAGDM of selecting best photovoltaic cell. According to the study of García-Cascales et al. [23], there are five potential photovoltaic cells (i.e., alternatives) currently available:

- $A_1$: photovoltaic cells with crystalline silicon (monocrystalline and polycrystalline)
- $A_2$: photovoltaic cells with inorganic thin layer (amorphous silicon)
- $A_3$: photovoltaic cells with inorganic thin layer (cadmium telluride/cadmium sulfide and copper indium gallium diselenide/cadmium sulfide)
- $A_4$: photovoltaic cells with advanced III-V thin layer with tracking systems for solar concentration
- $A_5$: photovoltaic cells with advanced, low cost, thin layers (organic and hybrid cells)

After analyzing the potential photovoltaic cells, the attributes considered for the assessment of the decision problem are the following [13]: manufacturing cost ($C_1$), efficiency in energy conversion ($C_2$), market share ($C_3$), emissions of greenhouse gases generated during the manufacturing process ($C_4$), and energy payback time ($C_5$). There are three experts ($e_1$, $e_2$, $e_3$) who are specializing in photovoltaic systems and technologies are invited to evaluate these five potential photovoltaic cells according to these five decision attributes. The weights of experts of are given as $V = (0.3, 0.4, 0.3)^T$. The assessment values of the alternatives with respect to each criterion provided by the experts are assumed to be represented by PFNs as shown in the Pythagorean fuzzy group decision matrix given in Tables 1-3.

First, we aggregate the information of the three groups into one collective matrix. The results are shown in Table 4.

TABLE 1. Pythagorean fuzzy decision matrix – expert 1

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $A_1$ | $P(0.8, 0.4)$ | $P(0.8, 0.6)$ | $P(0.6, 0.7)$ | $P(0.8, 0.3)$ | $P(0.6, 0.5)$ |
| $A_2$ | $P(0.5, 0.7)$ | $P(0.9, 0.2)$ | $P(0.8, 0.5)$ | $P(0.6, 0.3)$ | $P(0.5, 0.6)$ |
| $A_3$ | $P(0.4, 0.3)$ | $P(0.3, 0.7)$ | $P(0.7, 0.4)$ | $P(0.4, 0.6)$ | $P(0.5, 0.4)$ |
| $A_4$ | $P(0.6, 0.6)$ | $P(0.7, 0.5)$ | $P(0.7, 0.2)$ | $P(0.6, 0.4)$ | $P(0.7, 0.3)$ |
| $A_5$ | $P(0.7, 0.5)$ | $P(0.6, 0.4)$ | $P(0.9, 0.3)$ | $P(0.7, 0.6)$ | $P(0.7, 0.1)$ |

TABLE 2. Pythagorean fuzzy decision matrix – expert 2

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $A_1$ | $P(0.9, 0.3)$ | $P(0.7, 0.6)$ | $P(0.5, 0.8)$ | $P(0.6, 0.3)$ | $P(0.6, 0.3)$ |
| $A_2$ | $P(0.4, 0.7)$ | $P(0.9, 0.2)$ | $P(0.8, 0.1)$ | $P(0.5, 0.3)$ | $P(0.5, 0.3)$ |
| $A_3$ | $P(0.6, 0.3)$ | $P(0.7, 0.7)$ | $P(0.7, 0.6)$ | $P(0.4, 0.4)$ | $P(0.3, 0.4)$ |
| $A_4$ | $P(0.8, 0.4)$ | $P(0.7, 0.5)$ | $P(0.6, 0.2)$ | $P(0.7, 0.4)$ | $P(0.7, 0.4)$ |
| $A_5$ | $P(0.7, 0.2)$ | $P(0.8, 0.2)$ | $P(0.8, 0.4)$ | $P(0.6, 0.6)$ | $P(0.6, 0.6)$ |

TABLE 3. Pythagorean fuzzy decision matrix – expert 3

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $A_1$ | $P(0.8, 0.6)$ | $P(0.7, 0.6)$ | $P(0.5, 0.8)$ | $P(0.5, 0.5)$ | $P(0.6, 0.1)$ |
| $A_2$ | $P(0.5, 0.6)$ | $P(0.9, 0.2)$ | $P(0.8, 0.1)$ | $P(0.5, 0.3)$ | $P(0.4, 0.3)$ |
| $A_3$ | $P(0.7, 0.4)$ | $P(0.7, 0.5)$ | $P(0.6, 0.1)$ | $P(0.9, 0.2)$ | $P(0.5, 0.6)$ |
| $A_4$ | $P(0.9, 0.2)$ | $P(0.5, 0.6)$ | $P(0.6, 0.2)$ | $P(0.6, 0.1)$ | $P(0.7, 0.4)$ |
| $A_5$ | $P(0.6, 0.1)$ | $P(0.8, 0.2)$ | $P(0.9, 0.2)$ | $P(0.5, 0.6)$ | $P(0.6, 0.4)$ |

TABLE 4. Collective results

| | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $A_1$ | $P(0.85, 0.40)$ | $P(0.74, 0.60)$ | $P(0.53, 0.77)$ | $P(0.66, 0.35)$ | $P(0.60, 0.25)$ |
| $A_2$ | $P(0.46, 0.67)$ | $P(0.90, 0.20)$ | $P(0.80, 0.16)$ | $P(0.53, 0.30)$ | $P(0.47, 0.37)$ |
| $A_3$ | $P(0.59, 0.33)$ | $P(0.63, 0.63)$ | $P(0.67, 0.31)$ | $P(0.68, 0.37)$ | $P(0.44, 0.45)$ |
| $A_4$ | $P(0.80, 0.37)$ | $P(0.65, 0.53)$ | $P(0.63, 0.20)$ | $P(0.64, 0.26)$ | $P(0.70, 0.37)$ |
| $A_5$ | $P(0.67, 0.21)$ | $P(0.76, 0.25)$ | $P(0.87, 0.30)$ | $P(0.61, 0.60)$ | $P(0.63, 0.31)$ |

TABLE 5. Order inducing variables

| | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $U$ | 12 | 13 | 16 | 15 | 19 |

Due to the fact that the attitudinal character of the selection is very complex because it involves a lot of complexities, the experts use order inducing variables to express it. Table 5 shows the results.

In this problem, the group of experts considers that the general attitudinal character of the company is given by the following weighting vector: $W = (0.1, 0.25, 0.3, 0.15, 0.2)$ (i.e., $\alpha(W) = 0.475$). With this information, it is possible to aggregate the available information based on the PFIGOWA operator and its different families for the selection of photovoltaic cell. In this example, we consider the FPMax, the FPMin, the PFWA, the PFOWA, and the PFIOWA operator. The results are shown in Table 6.

As we can see, depending on the particular type of PFIGOWA operators used, the optimal choice is different. The ranking of the alternatives based on different types of PFIGOWA is listed in Table 7. Note that the first alternative in each ordering is the optimal choice.

As we can see, depending on the aggregation operators used, the ordering of the strategies is different. Note that in this problem, the PFMax result is the most optimistic aggregation as it considers only the highest evaluation, that is, the best characteristic of an alternative. On the other hand, the PFMin aggregation is the most pessimistic

TABLE 6. Aggregated results

| | PFMax | PFMin | PFWA | PFOWA | PFIOWA |
|---|---|---|---|---|---|
| $A_1$ | $P(0.85, 0.40)$ | $P(0.53, 0.77)$ | $P(0.67, 0.48)$ | $P(0.67, 0.41)$ | $P(0.69, 0.47)$ |
| $A_2$ | $P(0.90, 0.20)$ | $P(0.46, 0.67)$ | $P(0.76, 0.25)$ | $P(0.67, 0.30)$ | $P(0.70, 0.29)$ |
| $A_3$ | $P(0.67, 0.31)$ | $P(0.44, 0.45)$ | $P(0.62, 0.40)$ | $P(0.61, 0.36)$ | $P(0.64, 0.36)$ |
| $A_4$ | $P(0.80, 0.37)$ | $P(0.65, 0.53)$ | $P(0.67, 0.32)$ | $P(0.66, 0.36)$ | $P(0.71, 0.34)$ |
| $A_5$ | $P(0.87, 0.30)$ | $P(0.61, 0.60)$ | $P(0.76, 0.31)$ | $P(0.71, 0.30)$ | $P(0.70, 0.31)$ |

TABLE 7. Ordering of the strategies

| | Ordering |
|---|---|
| PFMax | $A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$ |
| PFMin | $A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$ |
| PFWA | $A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$ |
| PFOWA | $A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$ |
| PFIOWA | $A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$ |

one. The PFWA considers the weights of the attributes, while the PFOWA operator aggregates the Pythagorean fuzzy information considering the attitudinal character of the decision maker. The PFIOWA operator takes account of complex attitudinal characters of the decision-maker by using order-inducing variables. From the above analysis, we can see that the main advantage of using the PFIGOWA operator is that it can consider a wide range of particular aggregation operators such as the PFMax, the PFMin and the PFOWA operator. Due to the fact that each particular family of PFIGOWA operator may give different results, the decision maker will select for his decision the one that is the closest to his interests.

7. **Conclusions.** To enrich application of the IGOWA operator and theory of PFS, in this paper we have presented the PFIGOWA operator. It is an aggregation operator that unifies the IOWA operator and the GOWA operator in the same formulation and in an uncertain environment that can be assessed with PFNs. We have studied some of its main properties and particular cases such as the PFMax, the PFMin, the PFWA, the PFOWA and the PFIOWA operator. We have developed an application of the new approach in MAGDM problem where a company is looking for its optimal photovoltaic cell. We have analyzed that the main advantage of the PFIGOWA operator in this type of problems is that it is possible to consider a wide range of future scenarios according to our interests and select the one that it is the closest to our real interests.

In future research, we expect to develop further extensions to this approach by using more general formulations and considering other characteristics in the problem such as the use of probabilistic information and distance measures.

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