SIMPLE OPTIMAL PID TUNING METHOD BASED ON ASSIGNED ROBUST STABILITY –TRADE-OFF DESIGN BASED ON SERVO/REGULATION PERFORMANCE–

RYO KUROKAWA¹, NAOYA INOUE¹, TAKAO SATO¹, ORLAND ARRIETA²
RAMON VILANOVA³ AND YASUO KONISHI¹

¹Graduate School of Engineering
University of Hyogo
2167 Shosha, Himeji, Hyogo 671-2280, Japan
{tsato; konishi}@eng.u-hyogo.ac.jp

²Department of Automatic Control
School of Electrical Engineering
University of Costa Rica
11501-2060 San Jose, Costa Rica
Orlando.Arrieta@ucr.ac.cr

³Department of Telecommunications and Systems Engineering
Universitat Autònoma de Barcelona
Edifici Q-Campus de la UAB, 08193 Bellaterra, Barcelona, Spain
ramon.vilanova@uab.cat

Received April 2017; revised August 2017

ABSTRACT. The present paper investigates a design method for a one-degree-of-freedom control system with proportional-integral-derivative (PID) compensation. In the control system, the tracking performance and stability are in a trade-off relationship, and the servo performance and the regulation performance are also in a trade-off relationship depending on the design of the tracking performance. The present paper proposes a simple design method of the PID parameters. In the proposed system, the PID parameters are decided such that the tracking performance is optimized subject to the assigned robust stability. Furthermore, the tracking performance is seamlessly adjusted between the servo performance and the regulation performance. The effectiveness of the proposed method is demonstrated through numerical examples.

Keywords: PID control, Sensitivity function, Robust stability, Servo performance, Regulation performance, Trade-off design

1. Introduction. The proportional-integral-derivative (PID) control [1, 2, 3] has been widely used in industry because its performance is adjusted intuitively based on user experiments, even if the control structure is simple. Furthermore, the meanings of the control parameters are clear, i.e., proportional, integral, and derivative compensation and denoted by P, I, and D, respectively. Since the control performance of the PID control is decided by the PID parameters, numerous studies have examined PID parameter tuning.

The present study discusses a design method for controlling a one-degree-of-freedom (1DOF) system. In a 1DOF system, trade-off design is necessary because multiple performance characteristics can change simultaneously based on the design of the controller. The trade-off relationship between the tracking performance and stability is well known. If the stability of a control system is insufficient, the control system will become unstable due to even a slight plant perturbation. On the other hand, poor tracking performance is obtained when stability is ensured too much.
Since the stability margin is adjusted using the sensitivity function, the stability margin is assigned by designing the maximum value of the sensitivity function [4]. Using the sensitivity function, the PID parameters are decided such that the tracking performance is optimized, in which case the assigned stability margin is achieved [5, 6, 7, 8]. In order to obtain intermediate performance for a servo with regulation optimization, Arrieta and Vilanova proposed an intermediate design method [9, 10]. Using this design method, the PID parameters are decided by selecting servo-optimized, regulation-optimized, or intermediate performance.

In the present study, we propose a new trade-off design method. In the proposed method, the tracking performance is optimized subject to the assigned stability margin. Moreover, the tracking performance is designed between the servo performance and regulation performance. As a result, the optimal PID parameters are decided seamlessly between the reference response and the disturbance response.

This paper is organized as follows. Section 2 presents a control system and the control objective, and Section 3 presents the proposed control system. In Section 4, we present and analyze numerical simulations using our design strategy. Concluding remarks are presented in Section 5.

2. Problem Formulation. Consider the PID control system illustrated in Figure 1. In this system, the controlled plant is a first-order plus dead-time model, which is given as follows:

\[ P(s) = \frac{K}{Ts + 1} e^{-Ls} \]  

where \( K, T, \) and \( L \) are the gain, time-constant, and dead-time, respectively. The controller is represented by a PID control law as follows:

\[ U(s) = K_p \left\{ \left(1 + \frac{1}{T_i s}\right) E(s) - \left(\frac{T_d s}{T_d s / N + 1}\right) Y(s) \right\} \]

\[ E(s) = R(s) - Y(s) \]

where \( U(s), Y(s), \) and \( R(s) \) are the control input, plant output, and reference input, respectively, and \( K_p, T_i, \) and \( T_d \) are the proportional gain, integral time, and derivative time, respectively. Moreover, \( N \) denotes the derivative filter constant and is set to 10, as is the usual practice in industry. This control law is rearranged as follows:

\[ U(s) = C_r(s) R(s) - C_y(s) Y(s) \]  

\[ C_r(s) = K_p \left(1 + \frac{1}{T_i s}\right) \]  

\[ C_y(s) = K_p \left(1 + \frac{1}{T_i s} + \frac{T_d s}{T_d s / N + 1}\right) \]

Figure 1. Block diagram of a PID control system
The tracking performance is evaluated using the integral absolute error (IAE), which is defined as follows:

\[ J = \int_0^\infty |e(t)| \, dt \]  \hspace{1cm} (6)

\[ e(t) = r(t) - y(t) \]

Robust stability is guaranteed using the following sensitivity function:

\[ S(s) = \frac{1}{1 + P(s)C_y(s)} \]  \hspace{1cm} (7)

The relationships between the maximum value of the sensitivity function, \( M_s \), and the gain \( g_m \), and the phase margin, \( \phi_m \), respectively, are as follows [11]:

\[ g_m \geq \frac{M_s}{M_s - 1} \]  \hspace{1cm} (8)

\[ \phi_m \geq 2 \arcsin \left( \frac{1}{2M_s} \right) \]  \hspace{1cm} (9)

where \( M_s \) is defined as:

\[ M_s \triangleq \max_\omega |S(j\omega)| = \max_\omega \frac{1}{|1 + P(j\omega)G_y(j\omega)|} \]  \hspace{1cm} (10)

Although the stability margin is broadened with small \( M_s \), since the relationship between the tracking performance and the stability margin is a trade-off relationship, the recommended range of \( M_s \) is from 1.4 to 2.0 [3]. The gain and the phase margins for \( M_s = 1.4, 2.0 \) are given as follows:

- \( M_s = 1.4 : g_m \geq 3.5, \phi_m \geq 41^\circ \)
- \( M_s = 2.0 : g_m \geq 2.0, \phi_m \geq 28^\circ \)

In order to achieve the assigned robust stability, the PID parameters are decided such that the desired value of \( M_s \) is obtained, i.e., the following constraint condition is satisfied [7]:

\[ |M_s - M_s^d| = 0 \]  \hspace{1cm} (11)

where \( M_s^d \) denotes the desired value of \( M_s \). Therefore, the performance index Equation (6) is optimized such that the constraint is satisfied.

The objective of the present study is to obtain a simple design method of the PID parameters such that the performance function is optimized subject to the desired robust stability.

3. Controller Design Based on a Trade-off Relationship.

3.1. Servo/regulation tuning. The tuning points for the reference/disturbance optimization are shown in Figure 2. In this figure, \( J_r \) on the vertical axis indicates the evaluation of the reference response, and \( J_d \) on the horizontal axis is that of the disturbance response. Moreover, \( J_r^* \) denotes the evaluation on the reference response optimization design, and \( J_d^* \) denotes the evaluation on the disturbance response optimization design. Hence, the meanings of \( J_r^r, J_r^d, J_r^*, J_d^d \) are as follows:

- \( J_r^r \): Reference response evaluation on the reference response optimization design
- \( J_r^d \): Reference response evaluation on the disturbance response optimization design
- \( J_r^* \): Disturbance response evaluation on the reference response optimization design
- \( J_d^d \): Disturbance response evaluation on the disturbance response optimization design
On the other hand, \( J_{rd} \) denotes the optimization design of both the reference and disturbance responses, and hence, \( J_{rd}^r \) and \( J_{rd}^d \) are the reference and disturbance response evaluations on the optimization design of the reference and disturbance responses, respectively.

In a 1DOF system, since \( J_r \) and \( J_d \) are not independently optimized, the ideal point described in Figure 2 is not achieved. Therefore, the servo performance \( J_r \) and the regulation performance \( J_d \) must be mediated based on the trade-off between the reference optimization \( J^r \) and the disturbance optimization \( J^d \) [5, 6]. An intermediate tuning between \( J^r \) with \( J^d \) has been proposed [9, 10]. In this method, the performance index for the intermediate design is defined as follows:

\[
J_{rd} = \sqrt{(J_{rd}^r - J_r^r)^2 + (J_{rd}^d - J_d^d)^2}
\]  

\( (12) \)
The intermediate tuning $J_r$ provides a normalized form of the controlled plant and the PID compensators as follows:

$$J_r^\alpha = \sqrt{\alpha (J_r^d - J_r^o)^2 + (1 - \alpha) (J_d^d - J_d^o)^2} \quad (0 \leq \alpha \leq 1)$$  \hspace{1cm} (13)

In Equation (13), $\alpha = 0$ corresponds to the disturbance response optimization, and $\alpha = 1$ corresponds to the reference response optimization. Moreover, Equation (13) with $\alpha = 0.5$ is comparable to Equation (12), and hence, the proposed performance index includes the conventional trade-off design method \cite{9, 10}. The new trade-off design image using $\alpha$ is shown in Figure 3.

3.2. Simple decision of PID parameters.

3.2.1. Optimal decision method. In the present study, we propose a simple decision method of the PID parameters for a normalized system. The use of the transformation $\hat{s} = Ts$ provides a normalized form of the controlled plant and the PID compensators as follows:

$$\hat{P}(\hat{s}) = \frac{e^{-\tau \hat{s}}}{\hat{s} + 1}$$  \hspace{1cm} (14)

$$\hat{C}_r(\hat{s}) = \kappa_p \left(1 + \frac{1}{\tau_i \hat{s}}\right)$$  \hspace{1cm} (15)

$$\hat{C}_p(\hat{s}) = \kappa_p \left(1 + \frac{1}{\tau_i \hat{s}} + \frac{\tau_d \hat{s}}{\tau_d \hat{s}/N + 1}\right)$$  \hspace{1cm} (16)

where the normalized PID parameters are defined as follows:

$$\kappa_p = K_p K, \quad \tau_i = \frac{T_i}{T}, \quad \tau_d = \frac{T_d}{T}, \quad \tau = \frac{L}{T}$$  \hspace{1cm} (17)

where $\kappa_p$, $\tau_i$, $\tau_d$, and $\tau$ are the normalized gain, normalized integral time, and normalized dead-time, respectively.

For the normalized control system, the normalized PID parameters optimized subject to the stability margin constraint, in which $M_s^d = 1.4, 1.6, 1.8, \text{and} 2.0$. In the present study, the optimal parameters are obtained numerically with the MATLAB function $fmincon$.\footnote{Mathworks, Inc.} In Figure 4, the calculated $\kappa_p$ for each $\tau$, $\alpha$, is plotted by a $\circ$ symbol, where $\tau$ is set to be from 0.2 to 1.2 in 0.1 increments, and $\alpha$ is also set to be from 0 to 1.0 in 0.1 increments. Moreover, $\tau_i$ and $\tau_d$ are plotted in Figure 5 and Figure 6, respectively. Using the obtained data, the decision rule of the normalized PID parameters is proposed. In the present study, the calculated normalized parameters are approximated by the following equations:

$$\kappa_p = a_0(\alpha) + a_1(\alpha) \tau^{a_2(\alpha)}$$  \hspace{1cm} (18)

$$a_0(\alpha) = x_{00} + x_{01} \alpha, \quad a_1(\alpha) = x_{10} + x_{11} \alpha, \quad a_2(\alpha) = x_{20} + x_{21} \alpha$$

$$\tau_i = b_0(\alpha) + b_1(\alpha) \tau + b_2(\alpha) \tau^2 + b_3(\alpha) \tau^3$$  \hspace{1cm} (19)

$$b_0(\alpha) = y_{00} + y_{01} \alpha, \quad b_1(\alpha) = y_{10} + y_{11} \alpha, \quad b_2(\alpha) = y_{20} + y_{21} \alpha, \quad b_3(\alpha) = y_{30} + y_{31} \alpha$$

$$\tau_d = c_0(\alpha) + c_1(\alpha) \tau$$  \hspace{1cm} (20)

$$c_0(\alpha) = z_{00} + z_{01} \alpha, \quad c_1(\alpha) = z_{10} + z_{11} \alpha$$

where the coefficients for each $M_s^d$ are given in Table 1.
Figure 4. Optimal $\kappa_p$ for $\tau$ and $\alpha$ and the approximated surface

Figure 5. Optimal $\tau_i$ for $\tau$ and $\alpha$ and the approximated surface
**Figure 6.** Optimal $\tau_d$ for $\tau$ and $\alpha$ and the approximated surface

**Table 1.** $x_{ij}$, $y_{ki}$ and $z_{mn}$ in Equation (18)-Equation (20) for $M^d_s$

<table>
<thead>
<tr>
<th>$M^d_s$</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{00}$</td>
<td>0.1521</td>
<td>0.2484</td>
<td>0.3235</td>
<td>0.3635</td>
</tr>
<tr>
<td>$x_{01}$</td>
<td>0.04199</td>
<td>0.01711</td>
<td>-0.02680</td>
<td>-0.06098</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>0.4659</td>
<td>0.5794</td>
<td>0.6641</td>
<td>0.7586</td>
</tr>
<tr>
<td>$x_{11}$</td>
<td>-0.02869</td>
<td>-0.008122</td>
<td>0.03400</td>
<td>0.04427</td>
</tr>
<tr>
<td>$x_{20}$</td>
<td>-0.9807</td>
<td>-1.011</td>
<td>-1.0395</td>
<td>-1.035</td>
</tr>
<tr>
<td>$x_{21}$</td>
<td>-0.03788</td>
<td>-0.02278</td>
<td>0.02127</td>
<td>0.03574</td>
</tr>
<tr>
<td>$y_{00}$</td>
<td>0.2027</td>
<td>0.08962</td>
<td>0.1040</td>
<td>0.07031</td>
</tr>
<tr>
<td>$y_{01}$</td>
<td>0.7444</td>
<td>1.025</td>
<td>1.115</td>
<td>1.203</td>
</tr>
<tr>
<td>$y_{10}$</td>
<td>1.718</td>
<td>2.180</td>
<td>2.019</td>
<td>2.106</td>
</tr>
<tr>
<td>$y_{11}$</td>
<td>-1.399</td>
<td>-1.904</td>
<td>-1.764</td>
<td>-1.880</td>
</tr>
<tr>
<td>$y_{20}$</td>
<td>-1.435</td>
<td>-1.742</td>
<td>1.410</td>
<td>-1.440</td>
</tr>
<tr>
<td>$y_{21}$</td>
<td>1.304</td>
<td>1.759</td>
<td>1.410</td>
<td>1.507</td>
</tr>
<tr>
<td>$y_{30}$</td>
<td>0.4870</td>
<td>0.5874</td>
<td>0.4268</td>
<td>0.4591</td>
</tr>
<tr>
<td>$y_{31}$</td>
<td>-0.4421</td>
<td>-0.5908</td>
<td>-0.4268</td>
<td>-0.4738</td>
</tr>
<tr>
<td>$z_{00}$</td>
<td>0.001796</td>
<td>0.02239</td>
<td>0.02739</td>
<td>0.02729</td>
</tr>
<tr>
<td>$z_{01}$</td>
<td>-0.013732</td>
<td>-0.02483</td>
<td>-0.02678</td>
<td>-0.02991</td>
</tr>
<tr>
<td>$z_{10}$</td>
<td>0.3695</td>
<td>0.2929</td>
<td>0.2624</td>
<td>0.2490</td>
</tr>
<tr>
<td>$z_{11}$</td>
<td>-0.05715</td>
<td>-0.02371</td>
<td>0.02220</td>
<td>0.07068</td>
</tr>
</tbody>
</table>
3.2.2. **Confirmation of the assigned stability and the trade-off design.** The reliability of the proposed method is confirmed by two comparisons: a comparison of the optimized normalized parameters and the calculated normalized parameters using Equation (18) through Equation (20), and a comparison of the assigned $M_s^d$ and the actual calculated $M_s$. The achieved trade-off performance is also shown, and the feature of the proposed method is shown.

The normalized parameters $\kappa_p$, $\tau_i$, and $\tau_d$ calculated using Equation (18) through Equation (20) are shown as surfaces on Figure 4, Figure 5, and Figure 6, respectively. These surfaces are calculated by not only the preliminarily used $\tau$ and $\alpha$ for obtaining the ◦ symbols in Figure 4, Figure 5, and Figure 6, but also their interpolated values among the preliminarily used $\tau$ and $\alpha$. Here, Figure 4, Figure 5, and Figure 6 show that the calculated surfaces are well approximated by the optimized points.

The obtained $M_s$ corresponding to the assigned $M_s^d$ is shown in Figure 7, where $0.2 \leq \tau \leq 1.2$ and $0 \leq \alpha \leq 1$. The maximum errors of the assigned $M_s^d$ with respect to the actual obtained $M_s$ are shown in Table 2. Since the errors are quite small, the assigned stability margin is achieved using the proposed design method.

The trade-off performance of the proposed method is confirmed. The trade-off relationship between $J_r$ and $J_d$ is shown in Figure 8, where $\tau$ is 1.2, and $\alpha$ is changed from 0 to 1.0 in 0.05 increments. This figure indicates that the trade-off design between $J_r$ and $J_d$ is accomplished by designing $\alpha$.

![Figure 7](image-url)

**Figure 7.** Obtained $M_s$ for the assigned $M_s^d$

<table>
<thead>
<tr>
<th>$M_s^d$</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max</td>
<td>M_s - M_s^d</td>
<td>\max$</td>
<td>0.0094</td>
<td>0.013</td>
</tr>
</tbody>
</table>
4. **Numerical Examples.** Consider the following controlled plant:

\[ P(s) = \frac{2.5}{19.6s + 1} e^{-4.9s} \]  

(21)

where the reference input is set as a unit step function, and the control input is disturbed by the signal generated by a unit step function after 150s.

4.1. **Performance comparison for \( \alpha \).** The simulation results with respect to \( M_s^d = 1.4, 1.6, 1.8, \) and 2.0 are shown in Figure 9 through Figure 12, where \( \alpha \) is set to 0, 0.25, 0.5, 0.75, and 1.0. These figures show that the control performance improves as \( M_s^d \) increases. Furthermore, the trade-off design between the tracking performance and the regulation performance is adjusted by tuning \( \alpha \).

4.2. **Performance comparison for \( M_s^d \).** The robust stability for \( M_s^d \) is confirmed. The controlled plant is changed to \( \tilde{P}(s) \) after 200s.

\[ \tilde{P}(s) = \frac{5.1}{21.5s + 1} e^{-6.2s} \]  

(22)

Using \( \alpha = 0.25 \), the PID parameters are designed based on \( P(s) \) for \( M_s^d = 1.4, 1.6, 1.8, \) and 2.0, respectively. The simulation results are shown in Figure 13. It can be seen that the transient response is superior with large \( M_s^d \), whereas the robust margin is large with small \( M_s^d \).

5. **Conclusions.** In the present study, we proposed a simple tuning method of a PID control system. In the proposed method, the trade-off between the tracking performance and the regulation performance is adjusted seamlessly. Since the robustness for plant
Figure 9. Transient responses on $M_s^d = 1.4$

Figure 10. Transient responses on $M_s^d = 1.6$

Figure 11. Transient responses on $M_s^d = 1.8$

Figure 12. Transient responses on $M_s^d = 2.0$

Figure 13. Transient responses for each $M_s^d$ ($\alpha = 0.25$)
perturbation is assigned using the sensitivity function, the robust stability is designed based on the accuracy of the plant model. Our future work is an extension for a second-order system.

Acknowledgment. The present study was supported by JSPS Grant-in-Aid for Scientific Research(C) Number 16K06425. Ramon Vilanova acknowledges the support by the Ministerio de Economía e Innovación of Spain under the project DPI2016-77271-R (including FEDER).

REFERENCES