International Journal of Innovative Computing, Information and Control
Volume **11**, Number **1**, February **2015**

ICIC International ©2015 ISSN 1349-4198
pp. 153–162

ROBUST BIFURCATION ANALYSIS BASED ON OPTIMIZATION OF DEGREE OF STABILITY

Hiroyuki Kitajima[1], Tetsuya Yoshinaga[2], Jun-ichi Imura[3] and Kazuyuki Aihara[4]

[1]Faculty of Engineering
Kagawa University
2217-20 Hayashi, Takamatsu, Kagawa 761-0396, Japan
kitaji@eng.kagawa-u.ac.jp

[2]Institute of Health Biosciences
The University of Tokushima
3-18-15 Kuramoto, Tokushima 770-8509, Japan
yosinaga@medsci.tokushima-u.ac.jp

[3]Graduate School of Information Science and Engineering
Tokyo Institute of Technology
2-12-1 O-Okayama, Meguro, Tokyo 152-8552, Japan
imura@mei.titech.ac.jp

[4]Institute of Industrial Science
The University of Tokyo
4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, Japan
aihara@sat.t.u-tokyo.ac.jp

Received March 2014; revised July 2014

Abstract. *Robust bifurcation analysis of nonlinear dynamical systems with parameters is introduced. It provides a direct method of finding the values of system parameters at which the system has a steady state with a high degree of stability using a method of optimization. The approach is based on the idea of characterizing the degree of stability as a function of the parameters. As a result, we could design a system that was robust to unexpected factors, such as environmental changes and major incidents in applications. In particular, an example we present demonstrates that our method can avoid voltage collapse in an electrical power system.*
Keywords: Dynamical system, Bifurcation, Robust system, Optimization

1. **Introduction.** Consider nonlinear dynamical systems represented by parameterized differential and difference equations that are widely used for mathematical modeling of physical systems. When the values of the system parameters vary from those at which the system is currently operated, it can exhibit qualitative changes in behavior and a steady state may disappear or become unstable through a bifurcation [1, 2]. Bifurcation analysis is clearly useful and a bifurcation diagram composed of bifurcation sets projected into parameter space displays various nonlinear phenomena in dynamical systems. However, when considering a steady state that is closely approaching a bifurcation point due to unexpected factors like environmental changes, major incidents, and aging, self recovery is an important strategy for constructing a robust system. Traditional bifurcation analysis is not effective for this purpose because the global features of a bifurcation diagram in parameter space need to be computed to enable system behavior to be totally understood with variations in parameters treated as measurable and directly controlled variables.

We present a new innovative approach to analyze the system parameters based on the idea of characterizing a steady state with the degree of stability as a function of the parameters. The robust bifurcation analysis defined in this paper provides a direct method of finding the values of the parameters at which the dynamical system acquires a steady state with a high degree of stability using a method of optimization; this makes it possible to avoid bifurcations caused by the adverse effects of unexpected factors, without having to prepare the bifurcation diagrams that are required in advance for methods using traditional bifurcation analysis. The algorithm in our method is simpler than those of the others [3, 4] because the previous methods need a normal vector to a bifurcation curve on a parameter plane. Since such information is not necessary for our algorithm, it is easy to apply our method to multi-parameter space. Although our method needs model equations, the system behavior to parameter perturbation in engineering systems, such as electrical circuit systems, electrical power systems, and mechanical systems, coincides well with that in their mathematical models. Thus, our method can be applied to such systems in practice. Section 4 presents the results obtained when our method was applied to electrical circuit and power systems. A numerical example demonstrates that our method can avoid voltage collapse in an electrical power system. Stability often needs to be maintained for equilibria and periodic solutions against perturbation in such engineering systems. Bifurcation control methods have also been proposed for such purposes [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Some of them have required feedback systems to be constructed by adding control input, whereas our method uses prescribed parameters to optimize stability. As a result, it can be used to design a system that is robust to unexpected factors.

2. System Description and Robust Bifurcation Analysis.

This section provides an exact definition of robust bifurcation analysis, which we describe both for equilibria in continuous-time dynamical systems and fixed points in discrete-time dynamical systems.

2.1. Continuous-time systems.

Let us consider a parameterized autonomous differential equation for continuous-time systems described by:

$$\frac{dx}{dt} = f(x, \lambda), \quad t \in R, \tag{1}$$

where $x(t) = (x_1, x_2, \ldots, x_N)^\top \in R^N$ is a state vector, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M)^\top \in R^M$ is a measurable and directly controlled parameter vector, and f is assumed to be a C^∞ function for simplicity. Assume that there exists an equilibrium, x^*, satisfying:

$$f(x^*, \lambda) = 0, \tag{2}$$

and it can be expressed locally as a function of the parameters, λ, viz., $x^*(\lambda)$. The Jacobian or the derivative of f with respect to x at $x = x^*$ denoted by:

$$D(x^*(\lambda), \lambda) := \left.\frac{\partial f(x, \lambda)}{\partial x}\right|_{x=x^*(\lambda)} \tag{3}$$

provides information to determine the stability of the equilibrium. The eigenvalues, μ, of D at x^* are obtained by solving the characteristic equation:

$$\det(\mu I - D(x^*(\lambda), \lambda)) = 0, \tag{4}$$

where I is the identity matrix. We call x^* a hyperbolic equilibrium of the system, if D is hyperbolic, i.e., all the real parts of the eigenvalues of D are different from zero. If all eigenvalues lie on the left-half plane, then the equilibrium is asymptotically stable. A bifurcation occurs if an equilibrium loses its hyperbolicity due to continuous parameter variations. The generic bifurcations of an equilibrium are the saddle-node or tangent bifurcation and the Hopf bifurcation.

The main objective of bifurcation analysis is to find sets of bifurcation values. The bifurcation sets can be obtained by solving simultaneous equations consisting of the equilibrium equation in Equation (2) and the characteristic equation in Equation (4) for unknown variables x^* and λ_m for $m = 1, 2, \ldots, M$. Thus, the bifurcation sets for all m's need to be computed to enable system behavior to be totally understood with parameter variations.

2.2. **Discrete-time systems.** Next, let us consider a parameterized difference equation for discrete-time systems:

$$x(k+1) = g(x(k), \lambda), \quad k = 1, 2, \ldots, \tag{5}$$

or, equivalently, a map defined by:

$$g : R^N \times R^M \to R^N; \ \ (x, \lambda) \mapsto g(x, \lambda), \tag{6}$$

where $x(k)$ and x are state vectors in R^N, $\lambda \in R^M$ are the parameters, and the function, g, is assumed to be C^∞. Note that we have used the same symbols for the state variables, x, the parameters, λ, and others in continuous-time and discrete-time systems for simplicity. A discrete-time system can be a Poincaré map so that periodic solutions can be taken into consideration in an autonomous or periodic non-autonomous differential equation. The existence of a fixed point, x^*, satisfying:

$$x^* - g(x^*, \lambda) = 0 \tag{7}$$

is assumed. The Jacobian of g at fixed point x^* is indicated by:

$$D(x^*(\lambda), \lambda) := \left. \frac{\partial g(x, \lambda)}{\partial x} \right|_{x = x^*(\lambda)}. \tag{8}$$

The roots of the characteristic equation denoted by:

$$\det(\mu I - D(x^*(\lambda), \lambda)) = 0 \tag{9}$$

become the characteristic multipliers of the fixed point. The fixed point is hyperbolic, if all absolute values of the eigenvalues of D are different from unity. If every characteristic multiplier of the hyperbolic fixed point is located inside the unit circle, then it is asymptotically stable. When hyperbolicity is destroyed by varying the parameters, a bifurcation occurs. Generic bifurcations are the tangent, period-doubling, and Neimark-Sacker bifurcations, which correspond to the critical distribution of characteristic multiplier μ such that $\mu = +1$, $\mu = -1$, and $\mu = e^{i\theta}$ with $i = \sqrt{-1}$, respectively. Further, a pitchfork bifurcation can appear in a symmetric system as a degenerate case of the tangent bifurcation.

We can simultaneously solve equations consisting of the fixed point equation in Equation (7) and the characteristic equation in Equation (9) with a fixed μ depending on the bifurcation conditions to obtain unknown bifurcation sets x^* and λ_m.

2.3. **Robust bifurcation analysis.** Let $\mu_{\max}(D)$ be the maximum value of the real parts (or the absolute values) of eigenvalues with respect to the matrix, D, for a continuous-time system in Equation (3) (or a discrete-time system in Equation (8)). We denote this as a function of the parameters as:

$$\rho(\lambda) := \mu_{\max}(D(x^*(\lambda), \lambda)). \tag{10}$$

Consider that the dynamics, f or g, with parameter values λ^* defines a system after being affected by unexpected factors, and the steady state, $x^*(\lambda^*)$, has a low degree of stability, which means the value of $\rho(\lambda^*)$ is near the condition of a bifurcation. Then, the purpose of robust bifurcation analysis is to find $\lambda \in R^M$ such that:

$$\rho(\lambda) < \rho(\lambda^*) \tag{11}$$

for given $\lambda^* \in R^M$ satisfying:

$$\left.\frac{\partial \rho(\lambda)}{\partial \lambda_m}\right|_{\lambda=\lambda^*} \neq 0 \tag{12}$$

for some m's in $\{1, 2, \ldots, M\}$. We assume that the unexpected factors do not change during the process.

The corresponding eigenvalues can be used to create a contour along which the real part of an eigenvalue is equal to zero to analyze the stability of an equilibrium observed in a continuous-time system. This should be a curve on the parameter plane defining the boundary of a region in which the equilibrium stably exists. The level curves at a fixed maximum eigenvalue in the real part similarly imply the degree of stability in parameter space. Robust bifurcation analysis provides a method of finding the values of the parameters at which stable equilibrium has a higher degree of stability. Obtaining a set of parameters:

$$\Lambda := \arg \min_{\lambda \in R^M} \rho(\lambda) \tag{13}$$

subject to $\rho(\lambda) < \rho(\lambda^*)$, then enables us to design a system that has a steady state with a high degree of stability. The same argument can be applied to a fixed point observed in a discrete-time system by taking into consideration the absolute values of eigenvalues instead of the real parts.

3. **Method of Robust Bifurcation Analysis.**

We present a method of finding the set of parameters for the minimization problem in Equation (13), assuming that the maximum characteristic multiplier is real in the case of discrete-time systems.

Because the eigenvalues are not differentiable with respect to the parameters at points where they coalesce, we have considered the optimization problem below instead of directly solving Equation (13):

$$\min_{\lambda \in R^M,\, \nu \geq \rho(\lambda)} J(\lambda, \nu), \tag{14}$$

where

$$J(\lambda, \nu) := \det\left(\nu I - D(x^*(\lambda), \lambda)\right). \tag{15}$$

The characteristic polynomial, J, is non-negative for $\nu \geq \rho(\lambda)$ and the optimization problem is formulated to minimize the maximum value of the real parts of eigenvalues of $D(x^*(\lambda), \lambda)$ by varying the parameters, λ. We use a continuous gradient method to obtain the values of the parameters, λ, and the supplementary variable, ν. The descent flow is given by the initial value problem of the following differential equations:

$$\begin{aligned} \frac{d\lambda}{d\tau} &= -(\nu - \rho(\lambda)) \frac{\partial J}{\partial \lambda}^{\top}, \\ \frac{d\nu}{d\tau} &= -(\nu - \rho(\lambda)) \frac{\partial J}{\partial \nu}. \end{aligned} \tag{16}$$

Here, the solution, $(\lambda(\tau), \nu(\tau))$, is a function of the independent variable, τ, with the initial conditions, $\lambda(0) = \lambda^*$ and $\nu(0) > \rho(\lambda^*)$. When $\nu \neq \rho(\lambda)$, the gradient parts on the right hand sides in Equations (16) are given by:

$$\frac{\partial J}{\partial \lambda_m} = -\operatorname{tr}\left(\operatorname{adj}(\nu I - D)\frac{\partial D}{\partial \lambda_m}\right) \tag{17}$$
$$= -J\operatorname{tr}\left((\nu I - D)^{-1}\frac{\partial D}{\partial \lambda_m}\right),$$
$$\frac{\partial J}{\partial \nu} = \operatorname{tr}\left(\operatorname{adj}(\nu I - D)\right) \tag{18}$$
$$= J\operatorname{tr}\left((\nu I - D)^{-1}\right),$$

for $m = 1, 2, \ldots, M$, where $\operatorname{tr}(\cdot)$ and $\operatorname{adj}(\cdot)$ correspond to the trace and adjugate. We need the derivative of the (i, j) element of D in Equation (17) with respect to the parameter, λ_m. This is expressed by:

$$\frac{\partial}{\partial \lambda_m}\frac{\partial g_i}{\partial x_j} = \sum_{n=1}^{N}\frac{\partial^2 g_i}{\partial x_j \partial x_n}\frac{\partial x_n^*}{\partial \lambda_m} + \frac{\partial^2 g_i}{\partial x_j \partial \lambda_m},$$

for the difference system, g. Here, $\partial x_n^*/\partial \lambda_m$, $n = 1, 2, \ldots, N$ can be obtained by differentiating the fixed point equation in Equation (7) with respect to λ_m. Then, we have

$$\frac{\partial x^*}{\partial \lambda_m} = \left(I - \left.\frac{\partial g}{\partial x}\right|_{x=x^*}\right)^{-1}\left.\frac{\partial g}{\partial \lambda_m}\right|_{x=x^*}.$$

A similar argument can be applied to the differential dynamics, f.

Let us present a theoretical result for the behavior of the solution to the dynamical system in Equation (16). Note that the subspace of the state, $(\lambda, \nu) \in R^{M+1}$, satisfying $\nu = \rho(\lambda)$ is an equilibrium set of Equation (16). Therefore, the trajectories cannot pass through the set, according to the uniqueness of solutions to the initial value problem. This leads to the fact that, if we choose initial values $(\lambda(0), \nu(0))$ with $\nu(0) > \rho(\lambda(0))$, any solution $(\lambda(\tau), \nu(\tau))$ stays in the subspace, $\nu > \rho(\lambda)$, for all $\tau > 0$. Under this condition, the derivative of J along the solution to Equations (16) is given by:

$$\left.\frac{dJ}{d\tau}\right|_{(16)} = \begin{pmatrix} \frac{\partial J}{\partial \lambda} & \frac{\partial J}{\partial \nu}\end{pmatrix}\begin{pmatrix}\frac{d\lambda}{d\tau} \\ \frac{d\nu}{d\tau}\end{pmatrix}$$
$$= -(\nu - \rho(\lambda))\left(\left\|\frac{\partial J}{\partial \lambda}\right\|_2^2 + \left(\frac{\partial J}{\partial \nu}\right)^2\right)$$
$$< 0,$$

where the last inequality is obtained because $\nabla J \neq 0$ if $\nu > \rho(\lambda)$. Then, we can see that the value of $J(\lambda(\tau), \nu(\tau))$ with $J > 0$, for all τ, monotonically decreases as time passes.

An interior point method [16] is effective when implementing the algorithm to solve the optimization problem.

4. **Numerical Examples.**

Here, we present three representative examples of our robust bifurcation analysis for continuous-time and discrete-time systems.

4.1. **Continuous-time system 1.**

It is known that voltage collapse and blackouts can emerge in electrical power systems when load powers vary so that the systems lose stability in a tangent bifurcation [3]. The next example is a model for a two bus power system [3]. The load flow equations are given by:

$$-YE_Sx_1 \sin x_2 - \lambda_2 = 0, \qquad (19)$$
$$-Yx_1^2 + YE_Sx_1 \cos x_2 - \lambda_1 = 0,$$

where λ_1 and λ_2 correspond to reactive and real powers consumed by load, E_S is the voltage of a generator slack bus, Y is the admittance of the lossless line, and $x_1\angle x_2$ is the load voltage phasor. Here, we fix the parameter values at $Y = 4$ and $E_S = 1$ and consider the following differential equations:

$$\frac{dx_1}{dt} = -4x_1 \sin x_2 - \lambda_2, \qquad (20)$$
$$\frac{dx_2}{dt} = -4x_1^2 + 4x_1 \cos x_2 - \lambda_1.$$

We can easily see that the equilibria in Equation (20) are the same as the solutions in Equation (19). Moreover, the tangent bifurcation point on (λ_1, λ_2) in Equation (20) is the same as the disappearance point of the solutions in Equation (19). This disappearance point of the solutions is referred to as voltage collapse. Thus, our method can be applied to Equation (20) to avoid the voltage collapse in Equation (19), which means that we need to control the load power to obtain a highly stable voltage. Note that there are two eigenvalues in the Jacobian matrix of Equation (20) and one of them corresponds to the zero eigenvalue causing the voltage collapse in Equation (19). We called this the corresponding eigenvalue.

The results with our method are presented in Figure 1. A contour plot of the corresponding eigenvalue is also shown. The curve T indicates the tangent bifurcation of the stable equilibrium, which corresponds to voltage collapse. If the real and reactive powers consumed by the load approach this unfavorable state, then our method begins. The eigenvalue is -0.73 at the initial parameter values, λ^*, labeled a. The eigenvalue became -2.91 at the parameter value, b, after our method was applied. Thus, we could avoid the emergence of voltage collapse and find parameter values with sufficient margin against voltage collapse.

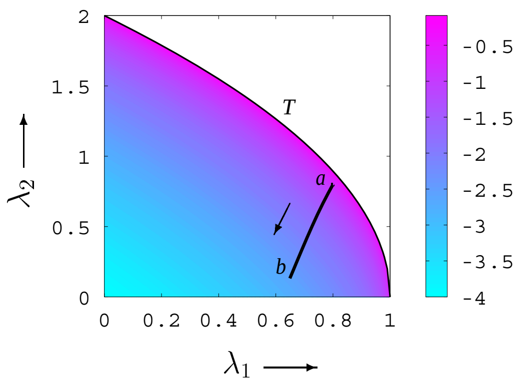

FIGURE 1. Bold solid line indicates trace of corresponding eigenvalue of equilibrium in Equation (20). Colored contour plot represents the corresponding eigenvalue indicated by the color bar. The corresponding eigenvalue is equal to 0 on the tangent bifurcation curve denoted by T.

4.2. **Continuous-time system 2.** The second example involves the following two-dimensional differential equations, known as BvP (Bonhöffer-van der Pol) equations:

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{3}{2}\left(x_1 - \frac{1}{3}x_1^3 + x_2\right), \\ \frac{dx_2}{dt} &= -\frac{2}{3}(-\lambda_1 + x_1 + \lambda_2 x_2). \end{aligned} \tag{21}$$

These equations describe the behavior of an electric circuit containing an inductor, a capacitor, and linear and nonlinear resistors [17]. The results with our method are in the graph in Figure 2. A contour plot of an eigenvalue with the maximum real part is also presented. The curve T indicates the tangent bifurcation of an equilibrium. Equation (21) has three equilibria in the colored region. A pair of stable and unstable equilibria disappears at the tangent bifurcation. Here, we will try to find the parameter values at which the equilibrium has a high degree of stability. The eigenvalue is -0.04 at the initial parameter values, λ^*, labeled a. The eigenvalue became -0.77 at the parameter value, b, after our method was applied. Thus, we could avoid the occurrence of the tangent bifurcation and obtain a high degree of stability by automatically changing the parameters when considering that the situation of the equilibrium with a low degree of stability at the parameter values, λ^*, near the bifurcation was caused by the effect of unexpected factors.

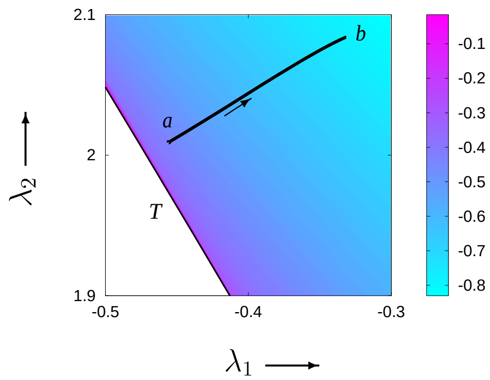

FIGURE 2. Bold solid line indicates trace of maximum eigenvalue of equilibrium in BvP equations. Colored contour plot represents the maximum eigenvalue, as indicated by the color bar. The maximum eigenvalue is equal to 0 on the tangent bifurcation curve denoted by T.

Figure 3 outlines the basins of attraction for stable equilibrium before and after our method was applied. The phase diagrams in Figures 3(a) and 3(b) correspond to the parameter values a and b in Figure 2. If we place the initial states in the gray region, we can obtain the targeted equilibrium labeled e in Figure 3 as a steady state. We can see that our method expanded the basins of attraction by comparing these figures.

4.3. **Discrete-time system.** Here, we consider a ring of 64-coupled quadratic maps ($N = 64$) as the second example where the nth element of the function vector, g, in Equation (6) is defined by:

$$\begin{aligned} g_n(x, \lambda) = x_n^2 - \lambda_1 - \lambda_2(x_{n-1} - 2x_n + x_{n+1}), \\ n = 1, 2, \ldots, N, \quad x_0 \equiv x_N, \quad x_{N+1} \equiv x_1. \end{aligned} \tag{22}$$

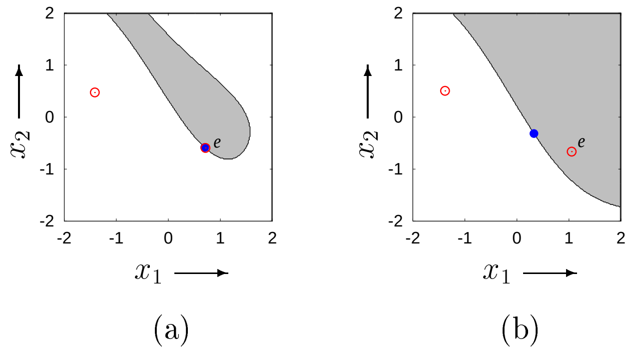

FIGURE 3. Basins of attraction of equilibria for BvP equations. Red open and blue closed circles correspond to stable and unstable equilibria. Blue and red circles are very close in (a) because the parameter value is near the tangent bifurcation. The stable manifolds of the saddle-type equilibrium (blue circle) form the basin boundary between the gray and white regions.

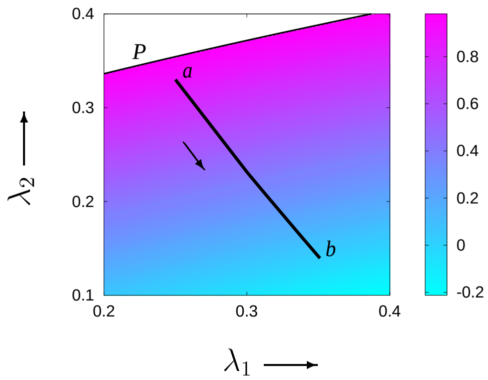

FIGURE 4. Bold solid line plots trace of maximum characteristic multiplier of fixed point in coupled quadratic map. Colored contour plot represents the maximum characteristic multiplier, as indicated by the color bar. The maximum characteristic multiplier is equal to 1 on the pitchfork bifurcation curve denoted by P.

One can demonstrate that any characteristic multiplier is real in this system with $\lambda_1 > -1/4$.

Figure 4 shows a trace of the optimization process using our method of robust bifurcation analysis and a contour plot of the maximum characteristic multiplier on the (λ_1, λ_2)-plane. The curve P indicates the pitchfork bifurcation of a fixed point. The white region denotes the existence of an unstable fixed point. The value of the characteristic multiplier at the initial parameter value, λ^*, labeled a is 0.98, which is close to that of the pitchfork bifurcation. The value is automatically decreased by applying our algorithm. Its value becomes zero after calculation with the parameter value, b. Thus, the trace denoted by the bold solid line means one of the sets Λ in Equation (13), and the parameters changed to move away from the bifurcation set.

The histograms of characteristic multipliers at the parameter values, a and b, in Figure 4 are given in Figures 5(a) and 5(b). We can see that the minimization of the maximum characteristic multiplier results in a narrow histogram distribution, which represents a high degree of stability.

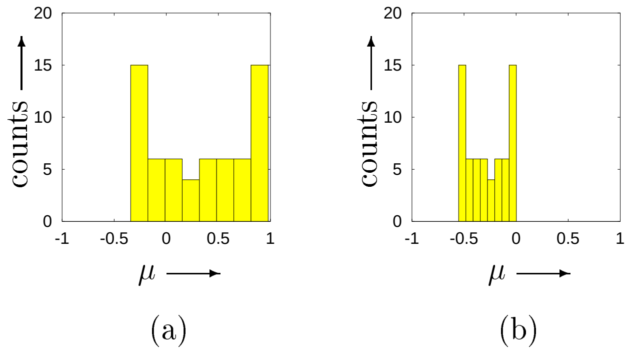

FIGURE 5. Histograms of 64 real characteristic multipliers in coupled quadratic map

5. **Conclusion.**

Conventional bifurcation analysis in parameter space deals with the contour or level sets of the eigenvalue for a bifurcation, whereas our robust bifurcation analysis is used for finding parameter sets that cause a gradient decrease in the maximum eigenvalue. An automatic trace of the gradient based on our method can effectively construct a robust system that has a steady state with a high degree of stability. Numerical examples demonstrated that our method could be applied to engineering systems to maintain high levels of stability against parameter perturbation.

Acknowledgements. This research was partially supported by the Aihara Innovative Mathematical Modelling Project, the Japan Society for the Promotion of Science (JSPS) through the "Funding Program for World-Leading Innovative R&D on Science and Technology (FIRST Program)", initiated by the Council for Science and Technology Policy (CSTP), and JSPS KAKENHI Grant Numbers 23500367 and 24560522.

REFERENCES

[1] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, 2nd Edition, Springer, 2003.
[2] Y. Kuznetsov, *Elements of Applied Bifurcation Theory*, 3rd Edition, Springer, 2004.
[3] I. Dobson and L. Lu, New methods for computing a closest saddle node bifurcation and worst case load power margin for voltage collapse, *IEEE Transactions on Power Systems*, vol.8, no.3, pp.905-913, 1993.
[4] H. Kitajima and T. Yoshinaga, A method for finding optimal parameter values using bifurcation-based procedure, *Int. J. Modern Nonlinear Theory and Application*, vol.3, no.2, pp.37-43, 2014.
[5] G. Chen, J. L. Moiola and H. O. Wang, Bifurcation control: Theories, methods, and applications, *Int. J. Bifurcation Chaos*, vol.10, no.3, pp.511-548, 2000.
[6] M. Basso, R. Genesio and A. Tesi, A frequency method for predicting limit cycle bifurcations, *Nonlinear Dynamics*, vol.13, no.4, pp.339-360, 1997.
[7] D. W. Berns, J. L. Moiola and G. Chen, Predicting period-doubling bifurcations and multiple oscillations in nonlinear time-delayed feedback systems, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol.45, no.7, pp.759-763, 1998.
[8] Z. Wu and P. Yu, A method for stability and bifurcation control, *IEEE Transactions on Automatic Control*, vol.51, no.6, pp.1019-1023, 2006.
[9] Y. Xie, L. Chen, Y. M. Kang and K. Aihara, Controlling the onset of Hopf bifurcation in the Hodgkin-Huxley model, *Phys. Rev. E*, vol.77, 2008.
[10] G. Wen and D. Xu, Control algorithm for creation of Hopf bifurcations in continuous-time systems of arbitrary dimension, *Phys. Lett. A*, vol.337, pp.93-100, 2005.
[11] F. Verduzco, Control of codimension one stationary bifurcations, *Int. J. Bifurcation Chaos*, vol.17, no.2, pp.575-585, 2007.

[12] M. K. Liu and C. S. Suh, Simultaneous time-frequency control of bifurcation and chaos, *Commun. Nonlinear Sci. Numer. Simul.*, vol.17, pp.2539-2550, 2012.
[13] W. Xiang, Equilibrium points and bifurcation control for Lorenz-Stenflo system, *ICIC Express Letters*, vol.3, no.1, pp.61-66, 2009.
[14] M. Ishikawa, On the simulation analysis for the stochastic infectious model with vaccination, *International Journal Innovative Computing, Information and Control*, vol.7, no.1, pp.3-12, 2011.
[15] N. I. Kalashnykova, V. V. Kalashnikov and L. R. Chavez Delgadillo, Consistent conjectures in a human migration model: Definition, existence and computation, *International Journal Innovative Computing, Information and Control*, vol.7, no.4, pp.1949-1957, 2011.
[16] J. Imae, T. Furudate and S. Sugawara, A simple numerical method for minimizing the maximum eigenvalues of symmetric matrices via nonlinear differential equation solvers, *Trans. Japan Soc. Mech. Eng.*, vol.63, no.608, pp.87-92, 1997.
[17] R. Fitzhugh, Impulses and physiological states in theoretical models of nerve membrane, *Biophysical Journal*, vol.1, no.6, pp.445-466, 1961.