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MARKET SHARE MODELLING AND FORECASTING USING MARKOV CHAINS AND ALTERNATIVE MODELS

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Abstract. *Markov chain has been a popular approach for market share modelling and forecasting in many industries. This paper presents four mathematical models for the same market share problem based on different underlying assumptions. The four models include a homogeneous Markov model, a time-varying Markov model, a new extended time-varying Markov model, and a novel non-Markov model. A numerical example in the telecommunications industry is included to illustrate that all four models can be used for market share forecasting. Although Markov models are popular, forecasters should be cautious in choosing Markov or the alternative models for their problems in hand. In order to achieve the best forecasting results, forecasters should have in-depth understanding of the industries, market conditions, and trends, then make appropriate assumptions, and apply or even develop the most suitable models.*
Keywords: Markov chain, Market share, Forecasting, Brand switching, Competition

1. **Introduction.** Markov chains have been widely studied and applied in brand switching problems and market share forecasting [1,2]. The Markov brand switching model studies customer loyalty and forecasts the brands, products, or service that a customer is likely to purchase next. As aggregation of individual customer choices, market shares of companies and their competitors can also be studied using similar approach. In the literature, most applications used a memory-less process, or first order Markov chain to model the probability that a customer switches from one provider to another over a period of time, and to forecast future market shares.

An early example in the literature presented a Markovian analysis of newspaper subscriptions [3]. Similarly, it was proposed in the work of [4], that mobile subscriptions offer a more contemporary example to introduce students to Markov chains. More recently, Markov chains were applied to model market shares of mobile operators [5,6]. In the work of [7], a diffusion growth model was included in the Markov formulation to forecast changes in market share according to phases in the product life cycles. In another study related to the telecommunications industry, a non-homogeneous Markov model was applied to forecast market shares of all competitors, using non-stationary transition probability matrices, which could be modified to reflect consequences due to actions such as marketing activities taken by providers [8]. In all these previous studies, the Markov chain model was used to investigate the movements of a group of buyers among a number of sellers, and market shares as proportions are used in the state vectors. Markov chains have also been applied to other problems that are similar in nature but in different areas, such as manpower supply [9,10], home heating units forecasting [11], credit risk analysis [12], tourist destinations [13], and student progressions in a regional university [14]. In a very recent paper [15], an integrated mathematical model linking the Markov chain model with a product life cycle model was proposed in forecasting market shares of all competitors, and their market share components including customers retained, customers gained from overall market growth, and customers gained from competitors over all stages of the product life cycle. This was an interesting and unique approach in which the impact of varying overall market size on the actual market share of individual sellers was studied.

This paper aims to review the popular homogeneous Markov model and the time-varying Markov model; and to present two new models. The first new model, an extension based on the time-varying Markov model, is capable of modelling the effect of changes in market sizes; while the second model is completely new and non-Markov. The focus of Markov models is on the transition probabilities. The non-Markov model is novel with a new focus on the trends of categories of customers. Further details will be provided later. As these models have their own assumptions and focuses in the formulation, each model would be particularly suitable for certain types of scenarios. Although Markov chain models are elegant, popular in academia, and have found wide applications in many research disciplines, they might not be the only possible or appropriate model for a particular problem in hand. The main contribution of this paper is to propose a number of alternatives and demonstrate how they have been developed based on different assumptions. In practice, it is important to understand in depth the nature and characteristics of the problem, and then choose the most suitable forecasting model in order to achieve the most accurate results. This paper will present the formulations of the four models and compare the different approaches. A numerical example will be provided to illustrate how these models can be applied to the same market share forecasting problem.

2. Four Mathematical Formulations.

This section will present the mathematical formulations of the four brand switching models for market share modelling and forecasting. They are:

1. Homogeneous Markov model (*m1*)
2. Time-varying Markov model (*m2*)
3. Extended Time-varying Markov model (*m3*)
4. Non-Markov model (*m4*)

For simplicity, we will refer to these models as *m1* to *m4* in the following.

2.1. Homogeneous Markov model (*m1*) formulation.

In a Markov model, a population of individuals moves among a finite set of states in a sequence of trials at discrete points of time. For the brand switching problem, the individuals are subscribers or customers. The finite set of states, denoting the set of providers or competitors, is defined as the state space, S:

$$s \in S,\ S = \{s_1, s_2, \cdots, s_{M-1}, s_M\},\ M < \infty.$$

There are M providers in total. The discrete points of time, referred to as epochs or time steps, are defined by a set of time steps, T:

$$t \in T,\ T = \{1, 2, \cdots, t_{\max}\},\ t_{\max} \leq \infty.$$

A customer is with a provider at each time step, and will stay or move to another provider at the next time step. Let X_t be the provider of a particular customer at time step t. And X_t is a random variable taking values in the state space S. The sequence $X = X_1, X_2, X_3, \cdots$ is a Markov chain if the following equality holds:

$$P\{X_{t+1} = j \mid X_t = i, X_{t-1} = i_{t-1}, \cdots, X_1 = i_1\} = P\{X_{t+1} = j \mid X_t = i\}, \forall t.$$

This equality is the so-called Markov condition that states that, if the system is in state i at time step t, then the probability that it will be in state j at time step $t+1$ does not depend on the states of the system in earlier times. In other words, this is a model of a memory-less process, or first order Markov chain. For a homogeneous Markov chain, the transition probability

$$P\{X_{t+1} = j \mid X_t = i\} = p_{i,j}$$

is independent of time step t. The probability $p_{i,j}$ represents the chance that a customer of provider i at time step t will change to provider j at the next time step $t+1$. The transition probabilities can be placed together in a matrix P, called the transition probability matrix, where $P(i,j) = p_{i,j}$. The transition probability matrix has the following properties:

$$0 \le p_{i,j} \le 1 \text{ and } \sum_{j=1}^{M} p_{i,j} = 1, \text{ for } i = 1, 2, \cdots, M.$$

Transition probability matrices are usually estimated based on historical data. Let us also define the market share vector, Y_t, at time $t \ge 1$, as

$$Y_t = \begin{bmatrix} y_t(s_1) & y_t(s_2) & \cdots & y_t(s_{M-1}) & y_t(s_M) \end{bmatrix}.$$

The components of $Y_t, y_t(s_1), \cdots, y_t(s_M)$, represent the market shares of providers s_1 to s_M at time t. The market share of a provider is a proportion. Therefore, the combined market share of all providers is equal to one, i.e., $\sum_{k=1}^{M} y_t(s_k) = 1$. To predict the next market share vector Y_{t+1}, we simply multiply Y_t by the transition probability matrix P:

$$Y_{t+1} = Y_t P.$$

To forecast market shares at time step $t+1$ based on the initial market share vector at time step $t = 1$, we have

$$\begin{aligned} &Y_2 = Y_1 P, \text{ and} \\ &Y_3 = Y_2 P = Y_1 P^2, \\ &Y_{t+1} = Y_1 P^t. \end{aligned} \tag{1}$$

The numbers of customers of the providers at time step t can be represented by a population vector, Q_t,

$$Q_t = M_t Y_t,$$

where M_t is the overall market size, or total number of customers of all providers, at time step t. We have now completed the mathematical formulation of model *m1*.

Figure 1 illustrates a simple scenario of model *m1*. There are two providers: *Us* and *Competitor*. In this diagram, each circle represents the market share of a provider at a certain time step. Each row of circles shows the market shares of all providers at a certain time step. Each column shows the market shares captured by a specific provider changing over time. The arrows represent the market share transitions from one time step to the next. The vertical downward arrows represent the market share that a provider retains; while the diagonal ones represent the market share that a provider gains from, or loses to, its competitor. For example, at time step t, *Competitor* or provider s_k has a market share $y_t(s_k)$. From time step t to $t+1$, *Competitor* gains $p_{1,k}$ of the market share of *Us* or provider s_1, and retains $p_{k,k}$ of its own market share. At time step $t+1$, provider s_k will have a market share $y_{t+1}(s_k) = y_t(s_1)p_{1,k} + y_t(s_k)p_{k,k}$. The transition probabilities

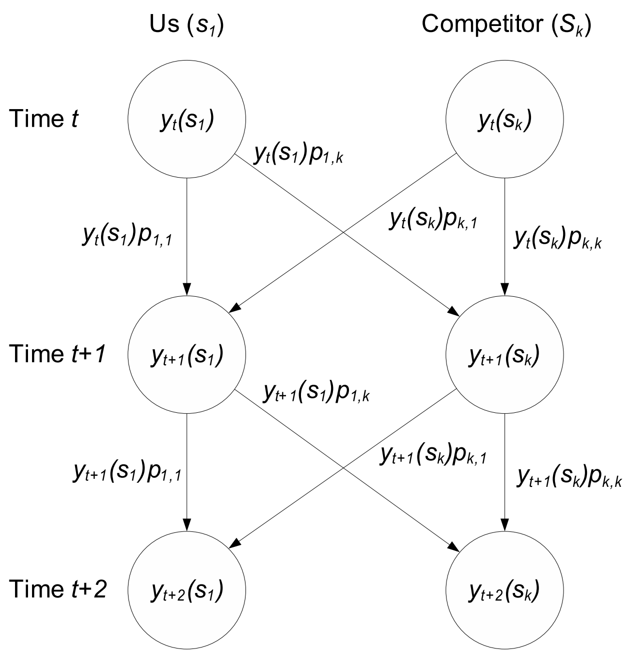

Figure 1. Homogeneous Markov model (*m1*)

$p_{1,k}$ and $p_{k,k}$ are elements of the transition probability matrix P. The term $y_{t+1}(s_k)$ is the k^{th} component of the market share vector Y_{t+1}. The market share vectors at future time steps can be easily calculated using Equation (1).

To decide if the Markov chain model *m1* is a suitable model for a forecasting problem, a major consideration should be given to whether the Markov condition is an appropriate assumption for the problem [16].

2.2. **Time-varying Markov model (*m2*) formulation.** If the transition probability matrix P is constant, the Markov chain is time homogeneous (*m1*). When P varies over time, the sequence becomes a time-varying Markov chain (*m2*), and P_t will be used instead of P. And the following equations will be used instead of Equation (1):

$$\begin{aligned} &Y_2 = Y_1 P_1, \text{ and} \\ &Y_3 = Y_2 P_2 = Y_1 P_1 P_2, \\ &Y_{t+1} = Y_1 P_1 P_2 \cdots P_t = Y_1 \prod_{n=1}^{t} P_n. \end{aligned} \tag{2}$$

Figure 2 illustrates a simple case of model *m2*. There are two providers: *Us* and *Competitor*. This diagram is similar to Figure 1 with the only difference that the transition probability $p_{i,j}$ used in model *m1* becomes time-dependent; and therefore, the time-varying transition probability $p_{t|i,j}$ is used instead. For example, at time step $t+1$, provider s_k will have a total market share $y_{t+1}(s_k) = y_t(s_1)p_{t|1,k} + y_t(s_k)p_{t|k,k}$. The transition probabilities $p_{t|1,k}$ and $p_{t|k,k}$ are elements of the transition probability matrix P_t.

Whether a homogeneous (*m1*) or time-varying (*m2*) Markov chain model should be used depends on the nature of problem, data availability, and many other considerations. The transition probability, $p_{i,j}$ for model *m1*, or $p_{t|i,j}$ for model *m2*, defines the proportion of market share that provider s_i loses to provider s_j (or s_j gains from s_i) at time step t. In general, model *m1* that has a constant P would be suitable for modelling competition in a more stable market condition, while *m2* that has a time-varying P_t for a more dynamic

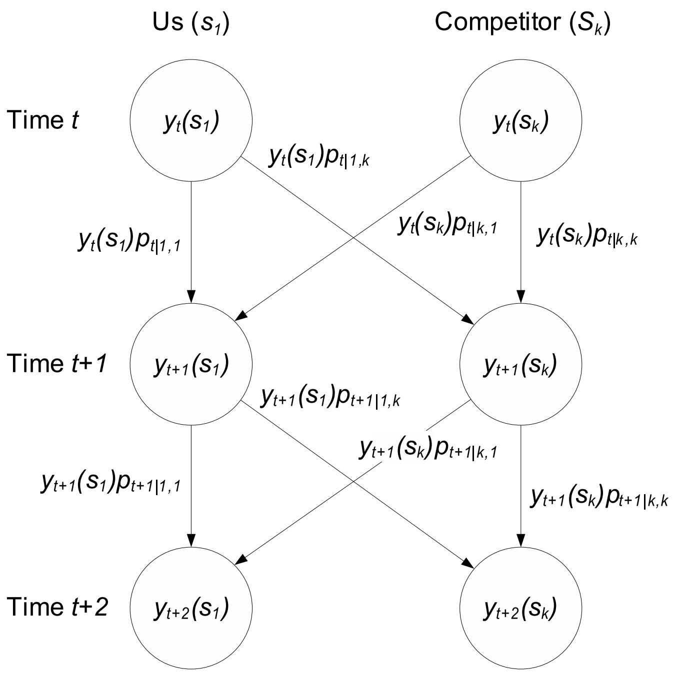

FIGURE 2. Time-varying Markov model (*m2*)

or changing market condition. For example, during the growing phase of a product, or in a highly competitive environment where players are usually more aggressive, and creative in offering special deals in order to gain more market shares, one would expect a time-varying Markov model (*m2*) to perform better than a homogeneous Markov model (*m1*), as it is more likely that the transition probabilities would be changing rather than staying constant. Similar to model *m1*, model *m2* is suitable for forecasting problems that meet the Markov condition.

2.3. **Extended time-varying Markov model (*m3*) formulation.** In both models *m1* and *m2*, the components of the state vector Y_t are market shares, i.e., proportions, at all time steps. These models can be applied to forecast market shares, but not actual numbers of customers. In other words, these models can inform the forecasters that the total market share of all competitors is equal to one at any time, but are not able to tell what the total number of customers is, or whether the overall market is growing or shrinking. The model *m3* proposed in this section aims to address this limitation, and is developed to forecast the overall market size in terms of customer numbers, as well as market share and number of customers of each competitor at each time step.

The extended Markov model (*m3*) makes use of the same time-varying transition probability matrix, P_t, as in model *m2*. However, the market share vector Y_t used in the previous models, *m1* and *m2*, is replaced with the population vector, Q_t. This model assumes that the Markov condition applies, and the transition probabilities are non-stationary.

The mathematical formulation of model *m3* is given as follows. Let the growth multipliers of market sizes of providers be expressed as a $1 \times M$ vector G:

$$G = \begin{bmatrix} g_1 & g_2 & \cdots & g_M \end{bmatrix}.$$

The growth multiplier, g_i, is used to model the growth and shrinkage in market size captured by a provider, s_i, from one time step to the next. The population vector at the next time step, $t = 2$, is given as:

$$Q_2 = G \odot Q_1 P_1 = \begin{bmatrix} g_1 q_1(s_1) & g_2 q_1(s_2) & \cdots & g_M q_1(s_M) \end{bmatrix} P_1,$$

where $\odot$ denotes the element-wise multiplication operator. To forecast the future population vectors based on the initial population vector Q_1,

$$Q_3 = G \odot Q_2 P_2 = G \odot G \odot Q_1 P_1 P_2 = \begin{bmatrix} g_1^2 q_1(s_1) & g_2^2 q_1(s_2) & \cdots & g_M^2 q_1(s_M) \end{bmatrix} P_1 P_2,$$

$$Q_{t+1} = G^{\odot,t} \odot Q_1 \prod_{n=1}^{t} P_n = \begin{bmatrix} g_1^t q_1(s_1) & g_2^t q_1(s_2) & \cdots & g_M^t q_1(s_M) \end{bmatrix} P_1 P_2 \cdots P_t,$$

where $G^{\odot,t}$ denotes the t^{th} power of vector G using element-wise multiplications. To forecast the next population vector Q_{t+1} from Q_t,

$$Q_{t+1} = G \odot Q_t P_t = \begin{bmatrix} g_1 q_t(s_1) & \cdots & g_i q_t(s_i) & \cdots & g_M q_t(s_M) \end{bmatrix} P_t. \quad (3)$$

This single equation serves two purposes at each time step. The first purpose, which is the element-wise product $G \odot Q_t$, is to adjust the market size of each provider s_i by multiplying $q_t(s_i)$ with a growth multiplier g_i. The term $g_i q_t(s_i)$ takes into account the customers who are completely new to the service, and those who leave permanently. To clarify, the new customers indicate those who have never used the service provided by any provider before, and the ones who leave the provider indicate those who will cease to use the same service provided by any provider. The purpose is to model the effects to the customer numbers of individual providers due to natural growth or shrinkage of the overall market. The second purpose, which is similar to that of the previous model *m2*, is to take care of customer transitions among providers by multiplying with the time-varying transition probability matrix P_t.

Figure 3 illustrates a simple case of model *m3*. There are two providers: *Us* and *Competitor*. Unlike models *m1* and *m2*, the circles in model *m3* represent customer numbers, $q_t(s_i)$, rather than market shares, $y_t(s_i)$. Compared to model *m2* shown in Figure 2, there is an extra row of circles (drawn in dash lines) between two consecutive time steps. This extra row shows the effect due to market size changing with customers joining and leaving. The combined effect of customers newly joined (the big arrow Join) and left permanently (the big arrow Leave) is shown in the dashed circles, which contains the term $g_i q_t(s_i)$ in Equation (3). Assuming this diagram represents a product which is in the growing phase of its product life cycle, the net increase in the overall market size would be equal to $[g_1 q_t(s_1) + g_k q_t(s_k)] - [q_t(s_1) + q_t(s_k)]$ from time step t to $t+1$. If the overall market is shrinking, this number will be negative. Following this intermediate step of market size adjustment, multiplying the transition probabilities will give us the final numbers of customers at the next time step $t+1$. For example, our competitor s_k will have a total number of customers, $y_{t+1}(s_k) = g_1 q_t(s_1) p_{t|1,k} + g_k q_t(s_k) p_{t|k,k}$, at time step $t+1$.

The main advantage of *m3* is that it can model the effect of changes in market sizes, and forecast in terms of, not just market shares, but numbers of customers. The growth vector, G, which can be estimated based on historical data or the experiences of the forecasters, contribute mostly to this advantage. As the growth vector G is a constant in the formulation, one would expect this model to predict more accurately for problems in which the growth rates are steady, or to be more appropriate for short term prediction. However, this growth vector can be modified to become a time-varying vector, G_t. In that case, the model would be more suitable for longer term prediction that will take into account the variations of growth rates, or different stages of the product life cycle.

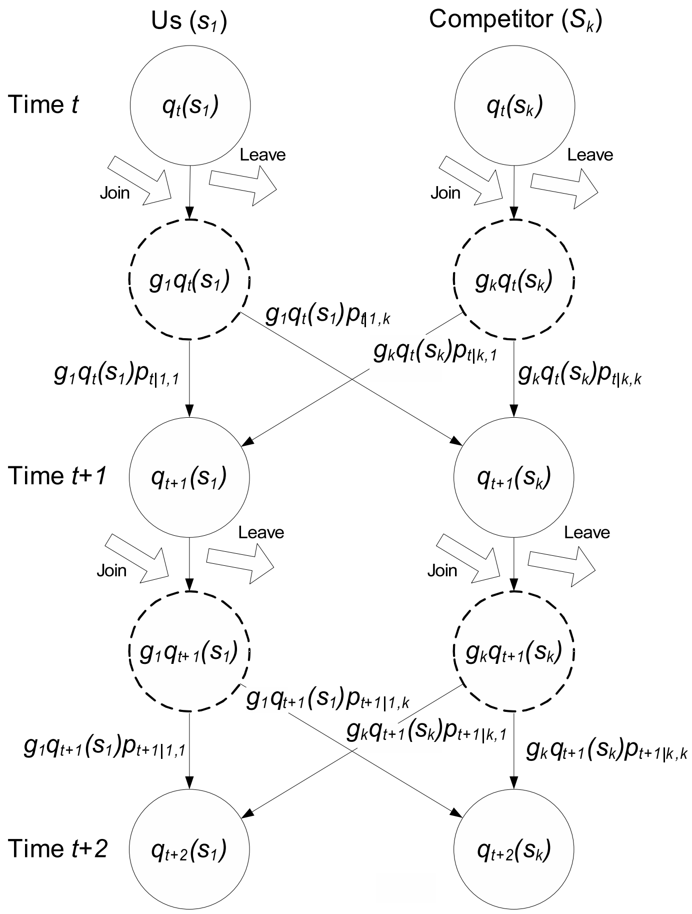

FIGURE 3. Extended time-varying Markov model (*m3*)

2.4. **Non-Markov model (*m4*) formulation.** In the previous three models, the transition probability matrix, which models the transitions of market shares among providers over time steps, plays a key role in the formulation. In this section, we propose a new, non-Markov forecast model (*m4*) in which transition probability is not a notion in the formulation. The focus of the model is on the individual trends of each category of customers. The group of customers who moves from a specific provider to another, or to itself, over time steps is called a category. This model assumes that the trends are independent of each other, and aims to forecast the future trend of each category of customers. In other words, each category follows its own trend. Similar to model *m3*, this model uses customer numbers rather than market shares in the formulation.

Let us first define the customer number matrix N_t as

$$N_t = \begin{bmatrix} n_{t|1,1} & \cdots & n_{t|1,M} \\ \vdots & \ddots & \vdots \\ n_{t|M,1} & \cdots & n_{t|M,M} \end{bmatrix},$$

where $n_{t|i,j}$ is the number of customers switching from provider, s_i, to s_j, at time t. Each element, $n_{t|i,j}$, of the matrix N_t is the number of a particular category of customers at time step t. The diagonal elements in the matrix, $n_{t|i,i}$, are the net numbers of those who stay with, newly join, and have left provider s_i, during time step t to $t+1$. The newly joined customers are totally new to the service, and not a previous customer of any other providers. The ones who left, are leaving permanently and will cease using the service again. Similar to model *m3*, this model also allows us to model the effect of market growth or shrinkage.

The total customer number of each provider at time step t can be determined, and placed together in a population vector, Q_t, where

$$\begin{aligned} Q_t &= \begin{bmatrix} q_t(s_1) & \cdots & q_t(s_k) & \cdots & q_t(s_M) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^{m} n_{t|i,1} & \cdots & \sum_{i=1}^{m} n_{t|i,k} & \cdots & \sum_{i=1}^{m} n_{t|i,M} \end{bmatrix}. \end{aligned} \tag{4}$$

The summation $\sum_{i=1}^{m} n_{t|i,k}$ is the total number of customers that provider s_k has at time step t. In addition, we define a constant $M \times M$ matrix N' to represent the relative changes of customer numbers from one time step to the next as:

$$N' = \begin{bmatrix} n'_{1,1} & \cdots & n'_{1,M} \\ \vdots & \ddots & \vdots \\ n'_{M,1} & \cdots & n'_{M,M} \end{bmatrix}.$$

We assume that the relative change $n'_{i,j}$ of customer numbers remains constant over the whole forecasting period. We use the following equation to find the customer numbers at the next time step:

$$N_{t+1} = \begin{bmatrix} \max(0, n_{t|1,1} + q_t(s_1)n'_{1,1}) & \cdots & \max(0, n_{t|1,M} + q_t(s_1)n'_{1,M}) \\ \vdots & \ddots & \vdots \\ \max(0, n_{t|M,1} + q_t(s_M)n'_{M,1}) & \cdots & \max(0, n_{t|M,M} + q_t(s_M)n'_{M,M}) \end{bmatrix}. \tag{5}$$

The maximum function ensures that the number of customers in each category is non-negative at all time. Once N_{t+1} is determined, we can apply Equation (4) to calculate the number of customers for all the providers, and their corresponding market shares.

Figure 4 illustrates model *m4* and how customer numbers change from one time step to the next. In this diagram, the circles represent the total numbers of customers owned by a provider, $q_t(s_k)$, while the ellipses represent the numbers of customers in particular customer categories at certain time step t. As shown in Equation (4), the total number of customers of provider, $q_t(s_k)$, is equal to the sum of the categories of customers switched to provider s_k, i.e., $\sum_{i=1}^{m} n_{t|i,k}$. This can be seen in a row that the customer number in a circle is equal to the sum of that in the ellipses. From time step t to $t+1$, each category of customers changes according to Equation (5). For example, $n_{t+1|1,k} = n_{t|1,k} + q_t(s_k)n'_{1,k}$, and $n_{t+1|1,k} \geq 0$. This shows the number of customers, in the category of customers switching from provider s_1, to s_k, changing from $n_{t|1,k}$ to $n_{t+1|1,k}$, from time step t to $t+1$. In summary, model *m4* takes a completely different approach compared with the previous Markov models, and the focus is placed on the trends of categories of customers rather than transition probabilities among providers. In the next section, we will apply these four models to a numerical example.

3. **Numerical Example.** We will present a numerical example of subscribers switching providers in the telecommunications industry, a typical Markov chain application. The four models will be applied to the same problem. In this example, the historical data from year 2010 to 2011 is available. There are four providers: $s_1 = Incumbent$; $s_2 = NextGen$; $s_3 = CowBoy$; $s_4 = Others$. The numbers of subscribers switching providers in year 2010 and 2011 are given in Tables 1 and 2.

The data is hypothetical. Over the period of 2010, provider *Incumbent* had 1600000 subscribers that include existing customers retained from the previous year, new customers who were not previously with any other providers, subtracting those who left completely

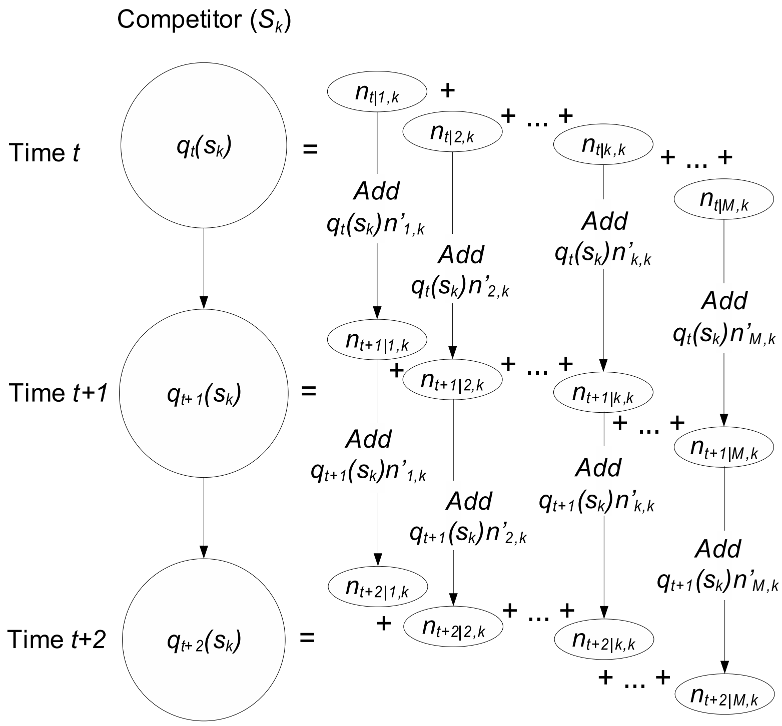

FIGURE 4. Non-Markov model (*m4*)

TABLE 1. Subscriber numbers in 2010

2010	*Incumbent*	*NextGen*	*CowBoy*	*Others*	Total
Incumbent	1600000	12000	7000	8300	1627300
NextGen	8900	750000	20000	1400	780300
CowBoy	1800	1200	125000	2776	130000
Others	3200	1750	2250	166000	173200
Total	1613900	764950	154250	178476	2711576
Market Share	0.5952	0.2821	0.0569	0.0658	1

TABLE 2. Subscriber numbers in 2011

2011	*Incumbent*	*NextGen*	*CowBoy*	*Others*	Total
Incumbent	1633800	12800	8300	9900	1664800
NextGen	9500	777200	19500	1650	807850
CowBoy	2100	1800	133500	1868	139268
Others	4100	2400	2778	185200	194478
Total	1649500	794200	164078	198618	2806396
Market Share	0.5878	0.2830	0.0585	0.0708	1

and ceased to be customers of any providers. In the same period, *Incumbent* lost 12000 customers to *NextGen*, 7000 to *CowBoy*, and 8300 to *others*. The numbers in the rightmost column show the total of these numbers for all providers. On the other hand, the numbers in the second last row show the total of existing and new customers, plus customers gained from competitors. For example, provider *Incumbent* has a total of 1613900 subscribers, including 1600000 existing and new subscribers joined in 2010; 8900 customers gained from competitor *NextGen*; 1800 from *CowBoy*; and 3200 from *Others*.

In practice, the amount and format of data available may be different. The forecasters may need to use different approaches in estimating the transition probability matrices, and apply statistical tests to ensure best estimates of historical data are used.

3.1. **Numerical example – forecasting using model *m1*.** In a homogeneous Markov model (*m1*), the transition probability matrix P is assumed constant over time. Therefore, we will take averages to estimate $p_{i,j}$ based on the customer numbers in year 2010 and 2011. For example, $p_{2,1}$ can be estimated:

$$p_{2,1} = \frac{1}{2}\left(\frac{8900}{780300} + \frac{9500}{807850}\right) = 0.01158.$$

Other $p_{i,j}$ values in the matrix P can be determined in a similar manner,

$$P = \begin{bmatrix} 0.98230 & 0.00753 & 0.00464 & 0.00552 \\ 0.01158 & 0.96161 & 0.02489 & 0.00192 \\ 0.01442 & 0.01105 & 0.95721 & 0.01732 \\ 0.01978 & 0.01122 & 0.01364 & 0.95536 \end{bmatrix}.$$

The numbers in the last row of Table 2 can be used as the components in the initial market share vector Y_1:

$$Y_1 = \begin{bmatrix} 0.5878 & 0.2830 & 0.0585 & 0.0708 \end{bmatrix}, \text{ and}$$

with the above values and Equation (1), we can forecast market shares in the next few years. The results are summarized in Table 3.

TABLE 3. Forecast by *m1*

t	*Incumbent*	*NextGen*	*CowBoy*	*Others*
1	0.5878	0.2830	0.0585	0.0708
2	0.5829	0.2780	0.0667	0.0724
3	0.5782	0.2733	0.0745	0.0741
4	0.5737	0.2688	0.0818	0.0758
5	0.5693	0.2645	0.0887	0.0775
6	0.5651	0.2605	0.0951	0.0792

This *m1* model has been widely applied to many problems in the literature. It is able to forecast the market shares of all the competitors, but not the actual numbers of customers. In other words, this approach cannot determine the overall market size and the customer number of each provider.

3.2. **Numerical example – forecasting using model *m2*.** If the probabilities that subscribers switching providers are changing or following certain trends rather than remaining constants, it would be more appropriate to forecast using a time-varying Markov chain (*m2*). Let us define a difference matrix ΔP, where

$$P_{t+1} = P_t + \Delta P; \text{ and } \Delta P(i,j) = \delta p_{i,j}.$$

We can estimate ΔP based on the subscriber numbers in Tables 1 and 2. For example, $\delta p_{2,1}$ can be calculated:

$$\delta p_{2,1} = \frac{9500}{807850} - \frac{8900}{780300} = 0.000354.$$

Other $\delta p_{i,j}$ values can be calculated in a similar manner:

$$\Delta P = \begin{bmatrix} -1.8446 & 0.3144 & 0.6840 & 0.8462 \\ 0.3538 & 0.8910 & -1.4930 & 0.2483 \\ 1.3149 & 3.7487 & 2.7506 & -7.8142 \\ 2.6063 & 2.2368 & 1.2936 & -6.1368 \end{bmatrix} \times 10^{-3}.$$

Once ΔP is determined, we can calculate P_t at each time step to forecast the next Y_{t+1} using Equation (3). All elements in P_t must not be negative as they are probabilities. If $(p_{t|i,j} + \delta p_{i,j}) < 0$, the next transition probability $p_{t+1|i,j}$ should be set to 0; and other probabilities in the same row need to be adjusted so that the sum of row i remains one. By using Equation (2), the market share vector can be determined at each time step. The forecast results are shown in Table 4. If we were to ask which model, *m1* or *m2*, is better, the answer would be it depends. It depends on whether constant or varying transition probabilities are better estimates of the reality. The forecaster will need to make the assumption based on past experiences, or gather more data and resort to some statistical means to decide which model to use.

TABLE 4. Forecast by *m2*

t	*Incumbent*	*NextGen*	*CowBoy*	*Others*
1	0.5878	0.2830	0.0585	0.0708
2	0.5818	0.2792	0.0674	0.0716
3	0.5754	0.2765	0.0756	0.0725
4	0.5685	0.2749	0.0832	0.0733
5	0.5614	0.2744	0.0901	0.0742
6	0.5539	0.2750	0.0962	0.0750

3.3. **Numerical example – forecasting using model *m3*.** A limitation of both the homogeneous and time-varying Markov models presented is that they can only forecast market shares, but not actual customer numbers. Being able to predict the actual numbers of customers can be very useful for any service provider. In the following we will show how models *m3* and *m4* can be used to obtain this information.

From Tables 1 and 2, we can estimate the growth vector G as follows:

$$G = \begin{bmatrix} \frac{1664800}{1613900} & \cdots & \frac{194478}{178476} \end{bmatrix} = \begin{bmatrix} 1.0315 & 1.0561 & 0.9029 & 1.0897 \end{bmatrix}.$$

From Table 2, we can start with Q_1 as:

$$Q_1 = \begin{bmatrix} 1649500 & 794200 & 164078 & 198618 \end{bmatrix}.$$

By using the same P_t as in Section 3.2, and Equation (3), we can forecast market shares in the next few years. The results are summarized in Table 5. There is an additional column which shows the overall market size in terms of numbers of customers at each time step. The previous models (*m1* and *m2*) are not able to provide this information due to their limitations.

3.4. **Numerical example – forecasting using model *m4*.** Unlike the previous three models, model *m4* is a non-Markov model. Based on the data given in Tables 1 and 2, the N' matrix can be estimated easily. For example,

$$n'_{1,1} = \frac{1633800 - 1600000}{1613900} = 0.02094.$$

TABLE 5. Forecast by *m3*

t	*Incumbent*	*NextGen*	*CowBoy*	*Others*	Q_t
1	0.5878	0.2830	0.0585	0.0708	2806396
2	0.5799	0.2847	0.0603	0.0751	2904830
3	0.5716	0.2871	0.0618	0.0795	3006878
4	0.5630	0.2904	0.0629	0.0838	3112896
5	0.5540	0.2944	0.0636	0.0879	3223242
6	0.5448	0.2991	0.0642	0.0919	3338269

To begin forecasting, we need the following:

$$Q_1 = \begin{bmatrix} 1649500 & 794200 & 164078 & 198618 \end{bmatrix}, \text{ and}$$

$$N' = \begin{bmatrix} 2.09431 & 0.04957 & 0.08055 & 0.09914 \\ 0.07844 & 3.55579 & -0.06536 & 0.03268 \\ 0.19449 & 0.38898 & 5.51053 & -0.58865 \\ 0.50647 & 0.36579 & 0.29713 & 10.80473 \end{bmatrix} \times 10^{-2}.$$

Using the customer numbers in Table 2, for the matrix N_1, and Equation (5), we can determine the customer number matrix at the next time step. Then Equation (4) can be applied to determine the customer number and market share of each provider. Repeating the same calculations at each time step, the forecasting results are summarized in Table 6.

TABLE 6. Forecast by *m4*

t	*Incumbent*	*NextGen*	*CowBoy*	*Others*	Q_t
1	0.5878	0.2830	0.0585	0.0708	2806396
2	0.5801	0.2838	0.0601	0.0760	2906144
3	0.5723	0.2844	0.0616	0.0817	3011215
4	0.5642	0.2847	0.0632	0.0879	3122858
5	0.5558	0.2850	0.0648	0.0945	3240716
6	0.5472	0.2850	0.0663	0.1014	3365282

4. **Discussion and Conclusion.** Two existing and two new mathematical models have been presented for the brand switching problem. All of these models have also been applied to the same numerical example for market share forecasting. This example was provided to show that the same problem can be modelled differently, with different assumptions and model focuses. The prediction results are only sensible and accurate if the model is sound and appropriate, and the assumptions made are correct, and the historical data is reliable.

For a steady or stable market, it would be appropriate to assume that the market size is fixed, and the probabilities that customers switching providers are constant. For such market conditions, the homogeneous Markov model *m1* would be appropriate. However, if certain providers have been constantly advertising and gaining momentum in gaining customers from their competitors, it would be reasonable to assume that the transition probabilities would change and continue their trends, and therefore, the time-varying Markov model *m2* would be more appropriate. However, for a growing market in which new customers are joining regularly, models *m3* and *m4* would be appropriate alternatives as they are capable of modelling market sizes.

Table 7. Model comparison

m1	*m2*	*m3*	*m4*
Homogeneous Markov model	Time-varying Markov model	Extension of time-varying Markov model	Non-Markov model
Market share based	Market share based	Population based	Population based
Predicting market shares only	Predicting market shares only	Predicting market shares and actual customer numbers	Predicting market shares and actual customer numbers
Satisfying Markov condition	Satisfying Markov condition	Satisfying Markov condition	Markov condition not applicable
Modelling stationary transition probabilities	Modelling time-varying transition probabilities	Modelling time-varying transition probabilities	Modelling customer numbers in categories, and their relative changes over time
Incapable of modelling total market growth, assuming a fixed market size	Incapable of modelling total market growth, assuming a fixed market size	Explicitly modelling growth of each provider over time	Implicitly modelling population growth of providers over time
Incapable of modelling growth of individual provider	Incapable of modelling growth of individual provider	Growth rate of each competitor is assumed constant.	Growth rate of each category of customers is assumed constant.

In summary, a comparison table of the four proposed models (*m1* to *m4*) is provided in Table 7.

This paper reviewed two commonly used Markov chain models and presented two new models for market share forecasting. The purpose of this paper is to illustrate that the same problem can be modelled in different ways, based on different assumptions and modelling focuses. Certain models would be more suitable for certain types of applications or business environment. Although Markov chains have been popular for the brand switching problem, forecasters should not simply fit Markov chains to their problems. Rather, forecasters should have good understanding of the problems, and market conditions and trends, then make the most appropriate assumptions; and choose, or develop, the most suitable models in order to achieve the best prediction accuracy.

REFERENCES

[1] J. S. Armstrong and J. U. Farley, A note on the use of Markov chains in forecasting store choice, *Management Science*, vol.16, no.4, pp.B281-B285, 1969.

[2] A. S. C. Ehrenberg, An appraisal of Markov brand-switching models, *Journal of Marketing Research*, vol.2, no.4, pp.347-362, 1965.

[3] W. E. Deming and G. J. Glasser, A Markovian analysis of the life of newspaper subscriptions, *Management Science*, vol.14, no.6, pp.B283-B293, 1968.

[4] K. C. Chan, C. T. Lenard and T. M. Mills, An introduction to Markov chains, *Proc. of the 49th Annual Conference of Mathematical Association of Victoria*, Melbourne, pp.40-47, 2012.

[5] G. Datong, A Markov chain model analysis of GSM network service providers marketing mix, *International Journal of Engineering and Technology*, vol.11, no.4, pp.49-57, 2011.

[6] J. Yang and Q. Sha, Research and application by Markov chain operators in the mobile phone market, *Proc. of the 2nd International Conference on Artificial Intelligence, Management Science and Electronic Commerce*, pp.7156-7159, 2011.

[7] M. Sokele, L. Moutinho and V. Hudek, Advanced market share modelling based on Markov chains, *Proc. of the 10th International Conference on Telecommunications*, Zagreb, Croatia, pp.339-343, 2009.

[8] K. C. Chan, C. T. Lenard and T. M. Mills, Modelling competition in telecommunications, *Proc. of the 8th International Conference on Information Technology and Applications*, Sydney, Australia, pp.15-20, 2013.

[9] S. H. Zanakis and M. W. Maret, A Markov chain application to manpower supply planning, *Journal of Operational Research Society*, vol.31, no.12, pp.1095-1102, 1980.

[10] K. M. Rowland and M. G. Sovereigh, Markov chain analysis of internal manpower supply, *Industrial Relations*, vol.9, no.1, pp.88-99, 1969.

[11] A. Ezzati, Forecasting market shares of alternative home-heating units by Markov process using transition probabilities estimated from aggregate time series data, *Management Science*, vol.21, no.4, pp.462-472, 1974.

[12] M. T. Jones, *Estimating Markov Transition Matrices Using Proportions Data: An Application to Credit Risk*, International Monetary Fund, http://www.imf.org/external/pubs/ft/wp/2005/wp05219.pdf, 2005.

[13] J. G. Choi, J. W. Mok and J. S. Han, The use of Markov chains to estimate destination switching and market share, *Tourism Economics*, vol.17, no.6, pp.1181-1196, 2011.

[14] A. Waugh, K. Chan and M. Martin, A framework for analysis of the local dependencies of a regional undergraduate program, *Proc. of Informing Science and IT Education Conference*, Wollongong, Australia, pp.337-348, 2014.

[15] K. C. Chan and T. Mills, Modelling competition over product life cycles, *Asia-Pacific Journal of Operational Research*, vol.32, no.3, 2014.

[16] K. C. Chan, C. T. Lenard and T. M. Mills, On Markov chains, *The Mathematical Gazette*, vol.97, no.540, pp.515-520, 2013.