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SOLUTION OF LINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BASED ON THE OPERATOR MATRIX OF FRACTIONAL BERNSTEIN POLYNOMIALS AND ERROR CORRECTION

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Abstract. *In this paper, firstly, a new method which makes a modification of the Bernstein polynomials is introduced to solve the linear fractional partial differential equations (FPDEs). The biggest advantage of the fractional Bernstein polynomials is that the order can be changed with the order of the fractional partial differential equations. For the first time, we try to use this method to solve the linear fractional partial differential equations. Secondly, convergence analysis and error correction are also given to make the calculation results more accurate. The concrete content of this method and error correction are explained briefly and numerical examples are given to demonstrate the validity and accuracy of the method.*

Keywords: Fractional Bernstein polynomials, Linear fractional partial differential equation, Operator matrix, Convergence analysis, Error correction

1. **Introduction.** Since 1974, Oldam and Spanier published the first book on the theory of fractional order calculus book [1], and the fractional order calculus is developed quickly. At present, the main definitions of fractional operator include Riemann-Liouvile, Caputo, Grunwald-Letnikov, Weyl, Erdelyi-Kober, Riesz and Marchaud-Hadamard [2, 3]. With the efforts of many scholars, fractional order calculus theory to a certain extent was established. Modeling problem in the field of engineering as well as the complex mechanics, physics problems promoted fractional order calculus theory and application

research, and in these problems, fractional calculus had certain practical significance and geometric interpretation. In soft matter physics, environmental mechanics, non-Newton fluid mechanics, viscous elastic mechanics, porous media dynamics and anomalous diffusion, fractional order differential research has caused a high degree of attention, and at the same time, it has been widely used. For fractional calculus, an important part is the study of fractional differential equation. When talking about the equation, solution is an inevitable problem. Due to the fact that overwhelming majority of the accurate solution of fractional differential equations is not easy to get, more and more scholars dedicated to studying its numerical solution. The most commonly used methods are like variational iteration method [4], finite difference method [5], generalized differential transform method [6, 7], Adomian decomposition method [8, 9], wavelet method [10] and operational matrix method [11, 12], for example, Adomian decomposition method, the homotopy perturbation method, the homotopy analysis method, the variational iteration method, the wavelet operator matrix analysis method, and block pulse function method. In [13], Wang et al. solved the fractional differential equation based on the Bernstein polynomials, proving the feasibility of the Bernstein polynomial for fractional differential equation. In [14], Chen et al. solved the FPDEs based on the fractional Legendre polynomial, introducing some theories to the fractional partial differential equations. Based on these, first, we will construct a class of fractional orthogonal polynomial (namely fractional Bernstein polynomial) with the help of Bernstein polynomial. Then, by using the properties of fractional calculus, we deduce the fractional differential operator matrix. At last, we combine the operator matrix and collocation method [15] to solve the FPDEs. Compared with [13], the maximum advantage of this method is that the order of polynomials used to approximate functions has variability with the order of the equation, which makes this method more flexible. Compared with [14], in this paper, the Bernstein polynomial is more conciseness than Legendre polynomial, which increases the calculation speed and accuracy. In this chapter, we consider the FPDEs on $\Omega = [0,1] \times [0,1]$

$$\frac{\partial^{\upsilon} u(x,t)}{\partial x^{\upsilon}} + \frac{\partial^{\gamma} u(x,t)}{\partial t^{\gamma}} = g(x,t) \tag{1}$$

$$u(0,t) = u(x,0) = 0 \tag{2}$$

where $u(x,t)$ is unknown function and $g(x,t)$ is known function, and υ and γ are the order. This paper is organized as follows. In Section 2, some definitions of fractional calculus are presented. In Section 3, we give the definition and matrix form of fractional Bernstein polynomial. And we introduce the theory of two-dimensional function approximation. In Section 4, we present the error analysis and error correction. In Section 5, we use the method to deduce the fractional differential operator matrix and find the fundamental matrix equation, and use collocation points to obtain a linear system. Next, correction solution is obtained through the error correction. In Section 6, two numerical examples are presented in order to support our work. The paper is concluded in Section 7 by a brief conclusion.

2. **Preliminaries and Notations.** In this section, we give some definitions of the fractional calculus.

Definition 2.1. [16] *A real α, $[\alpha]$ is the biggest integer which $[\alpha] \le \alpha$, if $f(t)$ is defined $[a,t]$ has $m+1$ order continuous derivative, when α>0, m is α at least, and the α order derivative of $f(t)$ is defined as follows:*

$$ {}_{a}^{G}\mathrm{D}_{t}^{\alpha} f(t) = \lim_{h\to 0,\, mh=t-a} h^{-\alpha} \sum_{k=0}^{m} g_k f(t-kh) \tag{3}$$

where ${}_a^G\mathrm{D}_t^\alpha$ *is the derivative operator, G represents Grunwald-Letnikov (G-L),* α *is the order, a and t are integral lower bound and higher bound respectively, and* α *is initial value of* t*, constant* $g_k = \frac{(-\alpha)(-\alpha+1)(-\alpha+2)\cdots(-\alpha+k-1)}{k!}$.

Because $\alpha>0$*, (3) is given as:*

$$ {}_a^G\mathrm{D}_t^\alpha f(t) = \sum_{k=0}^{m} \frac{f^k(a)(t-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(-\alpha+k+1)} \int_a^t (t-\tau)^{m-\alpha} f^{m+1}(\tau)\, d\tau \tag{4} $$

where $\Gamma(\alpha) = \int_0^\infty \mathrm{e}^{-x} x^{\alpha-1}\, dx = (\alpha-1)!$ *is gamma function.*

In order to simplify the calculation in the practical application, under the premise that satisfies the conditions and properties of fractional calculus, on the basis of G-L definition, we can find the improved Riemann-Liouville (R-L) calculus.

Definition 2.2. [17] *If* $f(t)$ *is a continuous function on the interval* $[0,+\infty)$*, namely* $f(t) \in \mathbb{C}[0,+\infty)$*, and is integrable on any finite subinterval of* $[0,+\infty)$*, so R-L integral is defined as*

$$ {}_a^R\mathrm{D}_t^{-\beta} f(t) = \begin{cases} f(t) & \beta = 0 \\ \dfrac{1}{\Gamma(\beta)} \displaystyle\int_a^t \frac{f(\tau)}{(t-\tau)^{1-\beta}}\, d\tau & \beta > 0,\ \tau > 0 \end{cases} \tag{5} $$

in order to distinguish, the ${}_a^R\mathrm{D}_t^{-\beta}$ *becomes* I_t^β*, so (5) can be written as*

$$ \mathrm{I}_t^\beta f(t) := \frac{1}{\Gamma(\beta)} \int_a^t \tau^{\beta-1} f(t-\tau)\, dt \qquad \beta > 0 \tag{6} $$

α *order R-L differential definition as*

$$ {}_a^R\mathrm{D}_t^\alpha f(t) = \begin{cases} \dfrac{d^n f(t)}{dt^n} & \alpha = n \in N \\ \dfrac{d^n}{dt^n} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau & 0 \le n-1 < \alpha < n \end{cases} \tag{7} $$

Definition 2.3. [18] *If* $f(t) \in \mathbb{C}^n(0,+\infty)$*, the Caputo fractional derivative is*

$$ {}_a^C\mathrm{D}_t^\alpha f(t) = \begin{cases} \dfrac{d^n f(t)}{dt^n} & \alpha = n \in N^+ \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau & 0 \le n-1 < \alpha < n \end{cases} \tag{8} $$

where ${}_a^C\mathrm{D}_t^\alpha$ *is Caputo fractional differential operator. In this paper, we use the Caputo definition. In order to be convenient, we use* D^α *to represent it.*

Particularly, when $f(t) = t^m$,

$$ \mathrm{D}^\alpha x^m = \begin{cases} 0 & \alpha \in N \text{ and } m < \lceil\alpha\rceil \\ \dfrac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} x^{x-\alpha} & m \in N_0 \text{ and } m \ge \lceil\alpha\rceil \text{ or } m \notin N \text{ and } m > \lfloor\alpha\rfloor \end{cases} \tag{9} $$

where $N_0 = \{0,1,2,\ldots\}$, $N = \{1,2,\ldots\}$. Upper limit function $\lceil\alpha\rceil$ is the smallest positive integral where $\lceil\alpha\rceil > \alpha$; lower limit function $\lfloor\alpha\rfloor$ is the biggest positive integral where $\lfloor\alpha\rfloor < \alpha$.

3. **Fractional Bernstein Polynomial and Function Approximate.**

3.1. **Fractional Bernstein polynomial.** We can find the definitions of Bernstein polynomial in [9, 10, 11, 12, 13] as follows.

Definition 3.1. [18, 19] *n order Bernstein polynomial in $[0,1]$ is*

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} \tag{10}$$

Using binomial theorem in (10), we can obtain

$$B_{i,n}(x) = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} x^{i+k} \tag{11}$$

Definition 3.2. [19, 20] *n order Bernstein polynomial in $[0,R]$ is*

$$B_{i,n}(x) = \binom{n}{i} \frac{x^i (R-x)^{n-i}}{R^n} = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} \frac{x^{i+k}}{R^{i+k}} \tag{12}$$

Definition 3.3. [21, 22] *n order Bernstein polynomial in $[a,b]$ is*

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-1)^i (b-x)^{n-i}}{(b-a)^n} \tag{13}$$

Substituting $x \to x^\alpha$ in (10), we can obtain fractional Bernstein polynomial is $[0,1]$ as follows:

$$B_{i,n}^{\alpha}(x) = \binom{n}{i} x^{i\alpha} (1-x^\alpha)^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} x^{(i+k)\alpha} \tag{14}$$

Then, we deduce the matrix form of fractional Bernstein polynomial:

$$\Phi(x) = \left[B_{0,n}^{\alpha}(x), B_{1,n}^{\alpha}(x), \ldots, B_{n,n}^{\alpha}(x)\right]^T = AX \tag{15}$$

where

$$A = \begin{bmatrix} (-1)^0 \binom{n}{0}\binom{n}{0} & (-1)^1 \binom{n}{0}\binom{n}{1} & \cdots & (-1)^{n-0} \binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0 \binom{n}{1}\binom{n-1}{0} & \cdots & (-1)^{n-1} \binom{n}{1}\binom{n-1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^0 \binom{n}{n} \end{bmatrix} \tag{16}$$

$$X = [1, x^\alpha, \ldots, x^{n\alpha}] \tag{17}$$

3.2. **Two-dimensional function approximation theory.** We can find one-dimensional function approximation based on fractional Bernstein polynomial in [15].

Square-integrable function $u(x) \in [0, x_u]$ can be approximated, and we consider the $n+1$ terms.

$$u(x) \approx \sum_{i=0}^{n} c_i B_{i,n}^{\alpha}(x) = C^{\mathrm{T}} \Phi(x) \tag{18}$$

where $C = [c_0, c_1, \ldots, c_n]^T$ is coefficient matrix which needs to obtain. C also can be obtained by $C = Q^{-1} \langle u, \Phi(x) \rangle$, and $\langle * \rangle$ is defined $\langle u, \Phi(x) \rangle = \int_0^{x_u} u(x)\Phi(x)dx$ where

$$Q = \int_0^{x_u} \Phi(x)\, \Phi^{\mathrm{T}}(x)\, dx = \int_0^{x_u} (AX)(AX)^{\mathrm{T}}\, dx = A \left(\int_0^{x_u} XX^{\mathrm{T}}\, dx \right) A^{\mathrm{T}} = AHA^{\mathrm{T}}$$

Particularly, when $x_u = 1$, $\alpha = 1$, H is Hilbert matrix.

Two-dimensional function approximation based on Bernstein polynomial exists in [23, 24]. Now, we give the two-dimensional function approximation based on fractional Bernstein polynomial as follows.

If $u(x,t) \in L^2([0,1) \times [0,1))$ considers $n+1$ terms,

$$u(x,t) \approx \sum_{i=0}^{n} \sum_{j=0}^{n} u_{i,j} B_{i,n}^{\alpha}(x) B_{j,n}^{\beta}(t) = \Phi^T(x) U \Phi(t) \tag{19}$$

where $u_{i,j}$ $(i = 0,1,\dots,n;\ j = 0,1,\dots,n)$ is undetermined coefficient

$$U = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0n} \\ u_{10} & u_{11} & \cdots & u_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n0} & u_{n1} & \cdots & u_{nn} \end{bmatrix} \tag{20}$$

The U can be obtained by the equation of $U = Q^{-1} \langle \Phi(x), \langle \Phi(t), u(x,t) \rangle \rangle Q^{-1}$.

4. **Convergence Analysis and Error Correction.**

4.1. **Convergence analysis.** We can find the error analysis about approximation based on Bernstein polynomial in [21, 25, 26]. Now, we deduce the error analysis about approximation based on fractional Bernstein polynomial.

Theorem 4.1. [21, 25, 26] *If H is Hilbert space, Y is random subspace, and $\dim Y < \infty$. If $\{y_1, y_2, \dots, y_n\} \in Y$ is a set of orthogonal basis, $x \in H$, y_0 is the best approximation of $x \in Y$,*

$$\|x - y_0\|_2^2 = \frac{G(x, y_1, y_2, \dots, y_n)}{G(y_1, y_2, \dots, y_n)} \tag{21}$$

where

$$G(x, y_1, y_2, \dots, y_n) = \begin{vmatrix} \langle x, x \rangle & \langle x, y_1 \rangle & \cdots & \langle x, y_n \rangle \\ \langle y_1, x \rangle & \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_n, x \rangle & \langle y_n, y_2 \rangle & \cdots & \langle y_n, y_n \rangle \end{vmatrix} \tag{22}$$

Corollary 4.1. [21, 25, 26] *If $Y = Span\left\{B_{0,n}^{\alpha}, B_{1,n}^{\alpha}, \dots, B_{n,n}^{\alpha}\right\}$, the absolute error in Theorem 4.1 can be written as*

$$\|x - y_0\| = \frac{\det\left[\int_{x_0}^{x_u} \Psi(x)\Psi^T(x)\,dx\right]}{\det\left[\int_{x_0}^{x_u} \Phi(x)\Phi^T(x)\,dx\right]} \tag{23}$$

where

$$\Psi^T = \left[x, B_{0,n}^{\alpha}, B_{1,n}^{\alpha}, \dots, B_{n,n}^{\alpha}\right] \text{ and } \Phi^T = \left[B_{0,n}^{\alpha}, B_{1,n}^{\alpha}, \dots, B_{n,n}^{\alpha}\right] \tag{24}$$

Theorem 4.2. *If $g \in C^{n+1}[x_0, x_u]$ and $Y = Span\left\{B_{0,n}^{\alpha}, B_{1,n}^{\alpha}, \dots, B_{n,n}^{\alpha}\right\}$, $C^T\Phi(x)$ is the best approximation of g in Y, so the biggest error is*

$$\left\|g - C^T\Phi(x)\right\|_2 \le \frac{M(x_u - x_0)^{\frac{2n+3}{2}}}{(n+1)\sqrt{2n+3}} \tag{25}$$

where $M = \max_{x\in[x_0,x_u]} \left|g^{(n+1)}(x)\right|$.

Proof: The Taylor expansion of $g(x)$ is

$$g_1(x) = g(x_0) + g'(x_0)(x - x_0) + g''(x_0)\frac{(x-x_0)^2}{2} + \cdots + g^{(n)}(x_0)\frac{(x-x_0)^n}{n!} \tag{26}$$

Using Lagrange mean value theorem, we can obtain

$$|g(x) - g_1(x)| = \left|g^{(n+1)}(x)\right| \frac{(x-x_0)^{n+1}}{(n+1)!} \qquad \exists \epsilon \in (x_0, x_u) \tag{27}$$

$C^T\Phi(x)$ is the best approximation of g in Y, so

$$\begin{aligned}
\left\|g - C^T\Phi(x)\right\|_2^2 \le \|g - y_1\|_2^2 &= \int_{x_0}^{x_u} |g - y_1(x)|^2 \, dx \\
&\le \int_{x_0}^{x_u} \left[\left|g^{(n+1)}(\epsilon)\right| \frac{(x-x_0)^{n+1}}{(n+1)!}\right]^2 dx \\
&\le \frac{M^2}{(n+1)!^2} \int_{x_0}^{x_u} (x-x_0)^{2n+2} \, dx \\
&= \frac{M^2 (x_u - x_0)^{2n+3}}{(n+1)!^2(2n+3)}
\end{aligned} \tag{28}$$

After square, we can obtain the biggest error, end.

Definition 4.1. *$f \in [a, b]$, the convergence coefficient form is*

$$\omega(f, \delta) = \sup_{x,y\in[a,b],|x-y|\le\delta} |f(x) - f(y)| \tag{29}$$

Theorem 4.3. [21, 25, 26] *$f \in [a, b]$ is uniform convergence if and only if*

$$\lim_{\delta\to 0} \omega(f, \delta) = 0 \tag{30}$$

Theorem 4.4. [21, 25, 26] *$f \in [0, 1]$ and bound, so $\|f - p(f, n)\|_\infty \le \frac{3}{2}\omega\left(f, \frac{1}{\sqrt{n}}\right)$ where $p(f, n) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}^{\alpha}$ and $\|f\|_\infty = \sup |f(x)|$.*

Theorem 4.5. [21, 25, 26] *If $f \in [0, 1]$ satisfies α order Lipschitz condition,*

$$\|f - p(f, n)\|_\infty \le \frac{3}{2} k m^{-\frac{\alpha}{2}} \tag{31}$$

where k is Lipschitz constant.

Theorem 4.6. *If $f \in [0, 1]$ and bound, $Y = Span\left\{B_{0,n}^{\alpha}, B_{1,n}^{\alpha}, \ldots, B_{n,n}^{\alpha}\right\}$, if $C^T\Phi(x)$ is the best approximation of f in Y,*

$$\left\|f - C^T\Phi\right\|_\infty \le \frac{3}{2}\omega\left(f, \frac{1}{\sqrt{n}}\right) \tag{32}$$

Proof: Because $C^T\Phi(x)$ is the best approximation of f in Y and $p(f, m) \in Y$, combining $\|f\|_2 \le \|f\|_\infty$, we can obtain

$$\left\|f - C^T\Phi\right\|_2 \le \|f - p(f, n)\|_2 \le \|f - p(f, n)\|_\infty \le \frac{3}{2}\omega\left(f, \frac{1}{\sqrt{n}}\right) \tag{33}$$

4.2. **Error correction.** First, we assume that $u_*(x,t)$ is the approximate solution of Equation (1) and Equation (2). Thus, $u_*(x,t)$ satisfies the problem $\frac{\partial^{\upsilon} u_*(x,t)}{\partial x^{\upsilon}} + \frac{\partial^{\gamma} u_*(x,t)}{\partial t^{\gamma}} = g(x,t)$. And $u(x,t)$ is the exact solution of Equation (1) and Equation (2). So, we define the error function as

$$e(x,t) = u(x,t) - u_*(x,t) \tag{34}$$

Next, we write Equation (1) for $L\left[u(x,t)\right] = \frac{\partial^{\upsilon} u(x,t)}{\partial x^{\upsilon}} + \frac{\partial^{\gamma} u(x,t)}{\partial t^{\gamma}} = g(x,t)$. Thus, we can define residual items as $R_*(x,t) = L\left[u_*(x,t)\right] - g(x,t)$. So, we can obtain

$$L\left[u_*(x,t)\right] = R_*(x,t) + g(x,t) \tag{35}$$

Because $u(x,t)$ is the exact solution, there are no residual items, namely residual item is 0, and we can obtain

$$L\left[u(x,t)\right] = 0 + g(x,t) \tag{36}$$

Then, through Equation (34), Equation (35) and Equation (36), we can infer error fractional partial differential equations as follows:

$$\begin{aligned} L\left[e_*(x,t)\right] &= L\left[u(x,t)\right] - L\left[u_*(x,t)\right] \\ &= 0 + g(x,t) - R_*(x,t) - g(x,t) \\ &= -R_*(x,t) \end{aligned} \tag{37}$$

Namely:

$$\frac{\partial^{\upsilon} e(x,t)}{\partial x^{\upsilon}} + \frac{\partial^{\gamma} e(x,t)}{\partial t^{\gamma}} = -R_*(x,t) \tag{38}$$

Here, $e_*(x,t)$ is exact solution of Equation (38). We can obtain approximate solution $e_{**}(x,t)$ of Equation (38) by some methods.

Last, approximate solution $u_*(x,t)$ plus error approximate solution $e_{**}(x,t)$ is defined correction solution $u_{**}(x,t)$, namely:

$$u_{**}(x,t) = u_*(x,t) + e_{**}(x,t) \tag{39}$$

5. **Numerical Method.** First, we deduce the fractional differential operator matrix of fractional Bernstein polynomial. Combining Equation (9) and Equation (15), we can obtain

$$D^{\alpha}\Phi\left(x\right) = D^{\alpha}AX = AD^{\alpha}X$$

$$= AD^{\alpha}\begin{bmatrix} 1 \\ x^{\alpha} \\ \vdots \\ x^{n\alpha} \end{bmatrix} = A\begin{bmatrix} D^{\alpha}1 \\ D^{\alpha}x^{\alpha} \\ \vdots \\ D^{\alpha}x^{n\alpha} \end{bmatrix} = A\begin{bmatrix} 0 \\ \Gamma\left(\alpha+1\right) \\ \dfrac{\Gamma\left(2\alpha+1\right)}{\Gamma\left(\alpha+1\right)}x^{\alpha} \\ \vdots \\ \dfrac{\Gamma\left(n\alpha+1\right)}{\Gamma\left(\left(n-1\right)\alpha+1\right)}x^{(n-1)\alpha} \end{bmatrix}$$

$$= A\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \Gamma\left(\alpha+1\right) & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{\Gamma\left(2\alpha+1\right)}{\Gamma\left(\alpha+1\right)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\Gamma\left(n\alpha+1\right)}{\Gamma\left(\left(n-1\right)\alpha+1\right)} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ x^{\alpha} \\ \vdots \\ x^{(n-1)\alpha} \end{bmatrix}$$

$$
\begin{aligned}
&= AP \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x^{\alpha} \\ \vdots \\ x^{n\alpha} \end{bmatrix} \\
&= A * P * E1 * X = A * P * E1 * A^{-1} * \Phi(x)
\end{aligned} \tag{40}
$$

So

$$
D^{\alpha} = A * P * E1 * A^{-1} \tag{41}
$$

where

$$
P = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \Gamma(\alpha+1) & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} \end{bmatrix} \tag{42}
$$

$$
E1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{43}
$$

Then, combining Equation (19) and Equation (41), we can obtain

$$
\frac{\partial^{v} u(x,t)}{\partial x^{v}} \cong \frac{\partial^{v}\left(\Phi^{T}(x) U \Phi(t)\right)}{\partial x^{v}} = \left(\frac{\partial^{v}\Phi(x)}{\partial x^{v}}\right)^{T} U\Phi(t) \approx \Phi^{T}(x)\left(D^{v}\right)^{T} U\Phi(t) \tag{44}
$$

$$
\frac{\partial^{\gamma} u(x,t)}{\partial t^{\gamma}} \cong \frac{\partial^{\gamma}\left(\Phi^{T}(x) U \Phi(t)\right)}{\partial x^{\gamma}} = \Phi^{T}(x) U \frac{\partial^{\gamma}\Phi(t)}{\partial t^{\gamma}} \approx \Phi^{T}(x) U D^{\gamma} \Phi(t) \tag{45}
$$

where

$$
D^{v} = A * M * E1 * A^{-1} \tag{46}
$$

$$
M = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \Gamma(v+1) & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{\Gamma(2v+1)}{\Gamma(v+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\Gamma(nv+1)}{\Gamma((n-1)v+1)} \end{bmatrix} \tag{47}
$$

$$
D^{\gamma} = A * N * E1 * A^{-1} \tag{48}
$$

$$
N = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \Gamma(\gamma+1) & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{\Gamma(2\gamma+1)}{\Gamma(\gamma+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\Gamma(n\gamma+1)}{\Gamma((n-1)\gamma+1)} \end{bmatrix} \tag{49}
$$

$$
u(0,t) \cong \Phi^{T}(0) U \Phi(t) \tag{50}
$$

$$u(x,0) \cong \Phi^T(x)\, U \Phi(0) \tag{51}$$

Using Equation (44) to Equation (51) in Equation (1) and Equation (2), we can deduce the initial fractional partial differential equation as follows:

$$\Phi^T(x)\,(D^v)^T U\Phi(t) + \Phi^T(x)\, U D^{\gamma T}\Phi(t) = g(x,t) \tag{52}$$

$$\Phi^T(0)\, U\Phi(t) = \Phi^T(x)\, U\Phi(0) = 0 \tag{53}$$

Finally, discrete variable $(x,t) \to (x_i, t_i)$ by collocation method is

$$x_i = \frac{2k_i - 1}{2n} \text{ and } t_i = \frac{2k_i - 1}{2n} \quad (i = 1, 2, \ldots, n) \tag{54}$$

so Equation (52) and Equation (53) become system of linear equations. Combining Matlab and least square fitting, we can obtain the undetermined coefficient $u_{i,j}$. Through Equation (19) we deduce the numerical solution of Equation (1) to Equation (2).

6. **Numerical Example.**

Example 6.1. *Considering the fractional partial differential equation as*

$$\frac{\partial^{\frac{1}{4}} u(x,t)}{\partial x^{\frac{1}{4}}} + \frac{\partial^{\frac{1}{2}} u(x,t)}{\partial t^{\frac{1}{2}}} = g(x,t) \tag{55}$$

$$u(0,t) = u(x,0) = 0 \tag{56}$$

the exact solution is x^2t^2, *where* $g(x,t) = \frac{8x^2t^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{32t^2x^{\frac{7}{4}}}{21\Gamma\left(\frac{3}{4}\right)}$.

The exact solution, numerical solution, exact and numerical solution error, approximate solution of error, correction solution and exact and correction solution error for different $n = 4$, $ne = 7$; $n = 5$, $ne = 8$; $n = 6$, $ne = 8$ *are reported in Figures 1-6. What can be found in these images is that the correction solution by error correction is closer to the exact solution than the previous numerical solution, which shows that the error correction can reduce the error. And with the increase of the number of nodes selected, the accuracy*

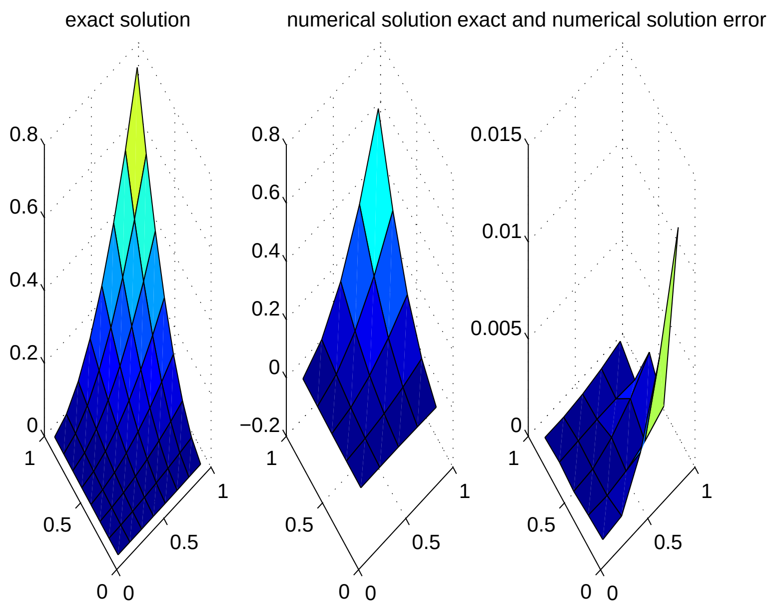

FIGURE 1. Exact solution, numerical solution and exact and numerical solution error to Example 6.1, for $n = 4$, $ne = 7$

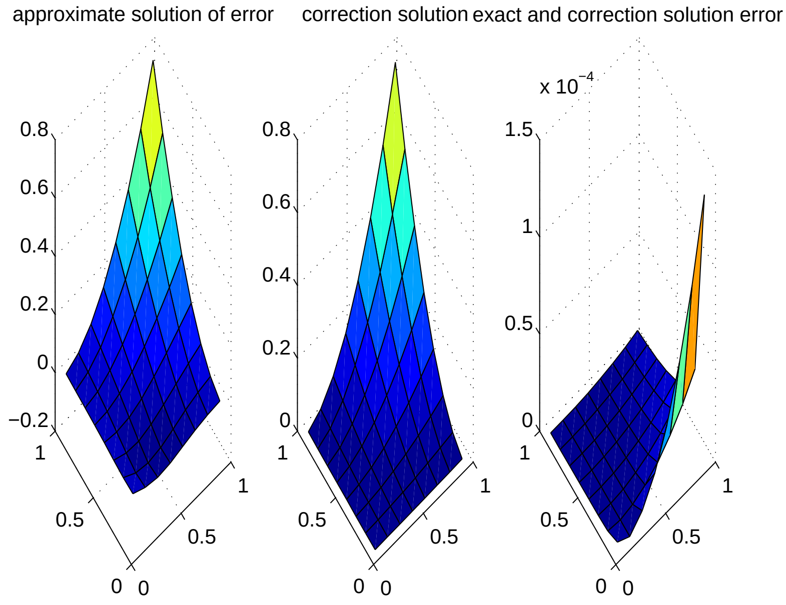

FIGURE 2. Approximate solution of error, correction solution and exact and correction solution error to Example 6.1, for $n = 4$, $ne = 7$

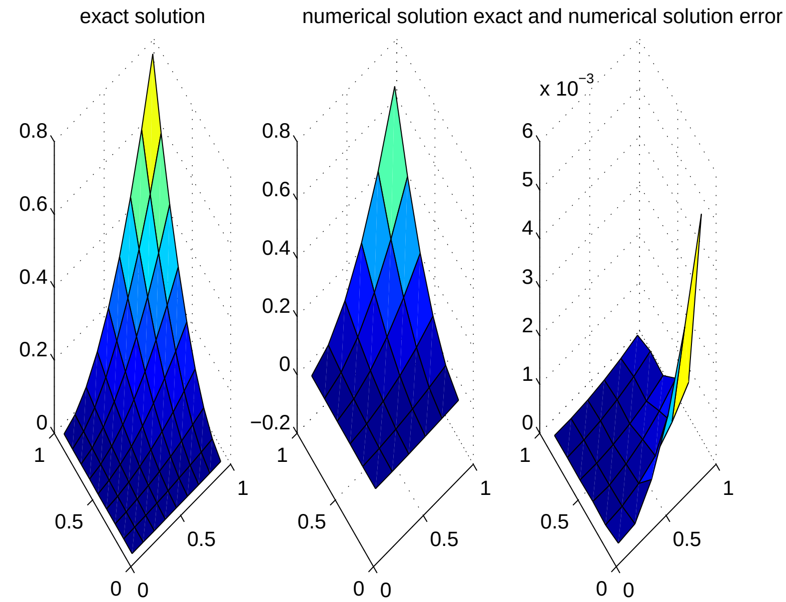

FIGURE 3. Exact solution, numerical solution and exact and numerical solution error to Example 6.1, for $n = 5$, $ne = 8$

of the solution has been improved. The results show the validity and applicability of the presented method and error correct, and we can obtain more accurate solution.

Example 6.2. *Considering the fractional partial differential equation as*

$$\frac{\partial^{\frac{5}{4}} u(x,t)}{\partial x^{\frac{5}{4}}} + \frac{\partial^{\frac{5}{4}} u(x,t)}{\partial t^{\frac{5}{4}}} = g(x,t) \tag{57}$$

$$u(0,t) = u(x,0) = 0 \tag{58}$$

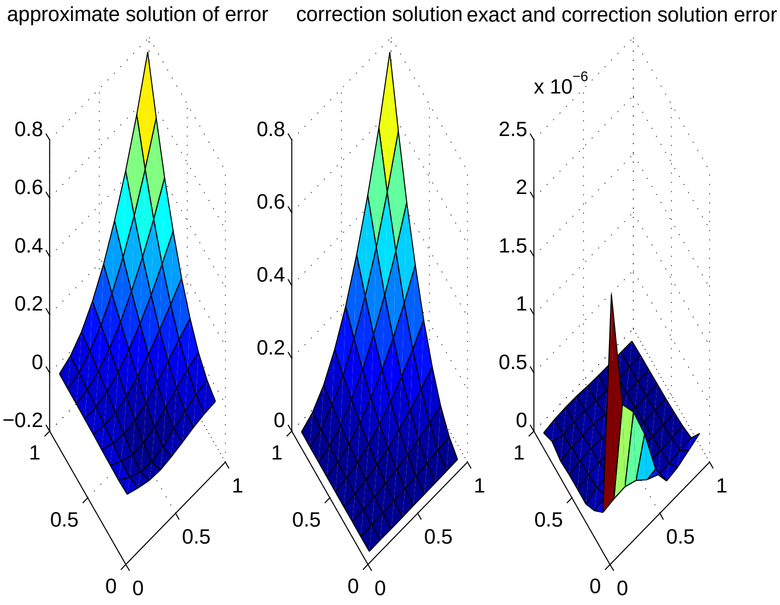

FIGURE 4. Approximate solution of error, correction solution and exact and correction solution error to Example 6.1, for $n = 5$, $ne = 8$

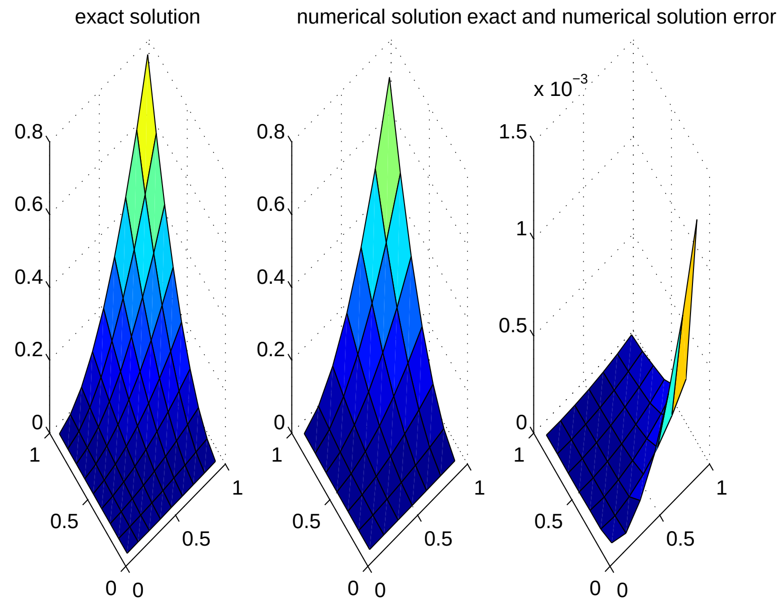

FIGURE 5. Exact solution, numerical solution and exact and numerical solution error to Example 6.1, for $n = 6$, $ne = 8$

the exact solution is $x^{\frac{5}{2}}t^{\frac{5}{4}} + x^{\frac{5}{4}}t^{\frac{15}{4}}$, *where* $g(x,t) = \frac{5\Gamma\left(\frac{1}{4}\right)}{16}x^{\frac{5}{2}} + \frac{5\Gamma\left(\frac{1}{4}\right)}{16}t^{\frac{15}{4}} + \frac{30\sqrt{\pi}x^{\frac{5}{4}}t^{\frac{5}{4}}}{16\Gamma\left(\frac{9}{4}\right)} + \frac{2x^{\frac{5}{4}}t^{\frac{5}{2}}\Gamma\left(\frac{15}{4}\right)}{\sqrt{\pi}}$.

The exact solution, numerical solution, exact and numerical solution error, approximate solution of error, correction solution and exact and correction solution error for different $n = 3$, $ne = 5$; $n = 5$, $ne = 7$; $n = 6$; $ne = 7$ *are reported in Figures 7-12. In these*

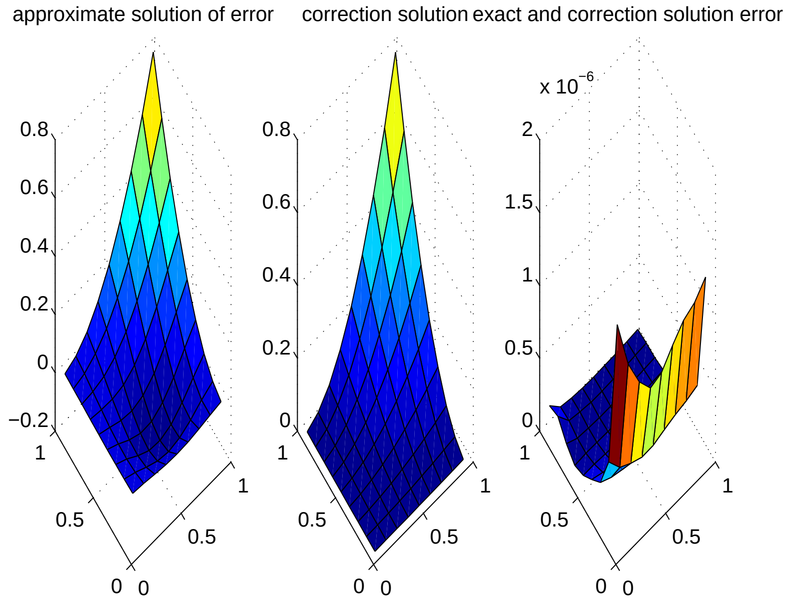

FIGURE 6. Approximate solution of error, correction solution and exact and correction solution error to Example 6.1, for $n = 6$, $ne = 8$

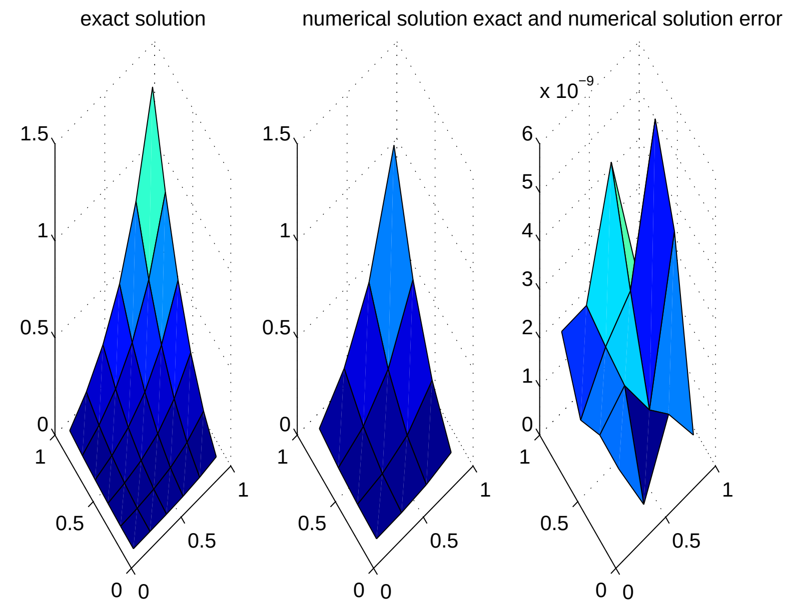

FIGURE 7. Exact solution, numerical solution and exact and numerical solution error to Example 6.2, for $n = 3$, $ne = 5$

images, we can also find the correction solution by error correction is closer to the exact solution than the previous numerical solution, which shows that the error correction can reduce the error. And we can get very high resolution by picking very few nodes n. The results show the validity and applicability of the presented method and error correct, and we can obtain more accurate solution.

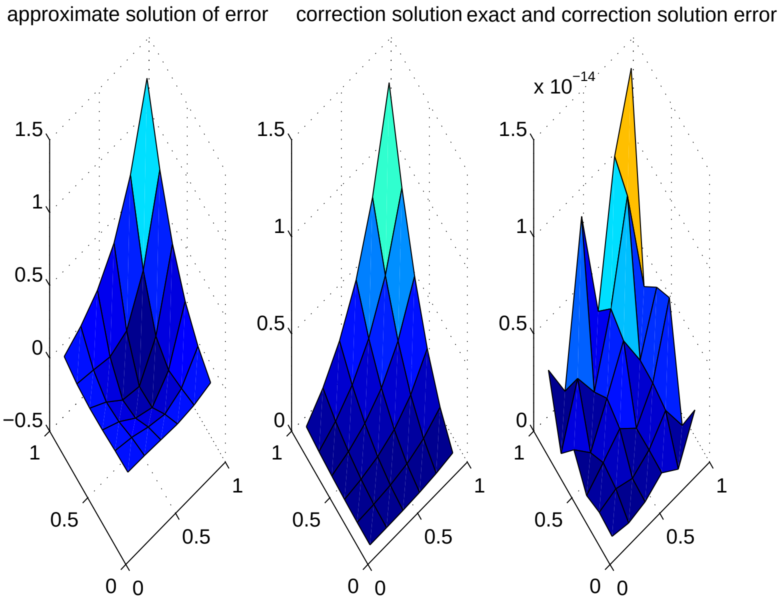

Figure 8. Approximate solution of error, correction solution and exact and correction solution error to Example 6.2, for $n = 3$, $ne = 5$

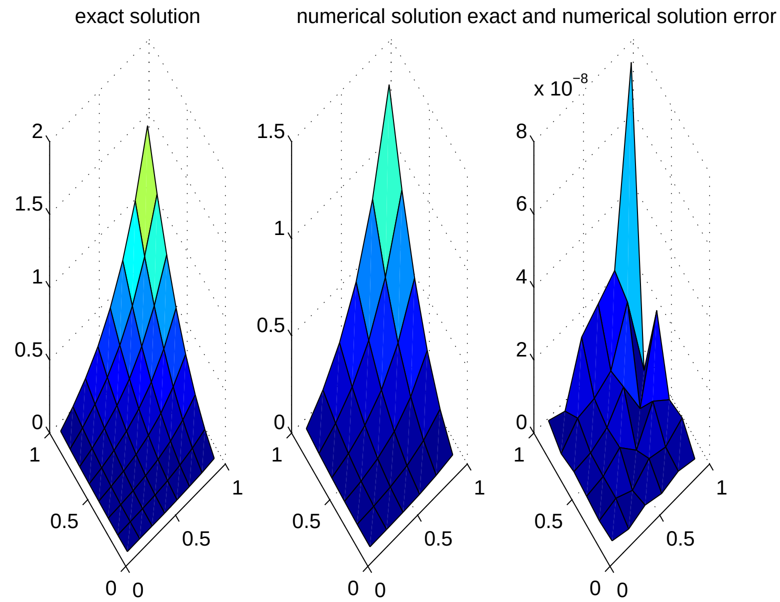

Figure 9. Exact solution, numerical solution and exact and numerical solution error to Example 6.2, for $n = 5$, $ne = 7$

7. **Conclusions.** In this paper, first according to the theory of function approximation with fractional Bernstein polynomial approximate unknown functions $u(x,t)$; then fractional differential operator matrix of fractional Bernstein polynomial is derived by using the properties of fractional calculus; next combining the ideas of the operator matrix and collocation method to discrete variable (x,t) become (x_i,t_i), converting the problem to solve the algebraic equations and obtaining the numerical $u_*(x,t)$; then by using the residual function, an error fractional partial differential equation is constructed and thus

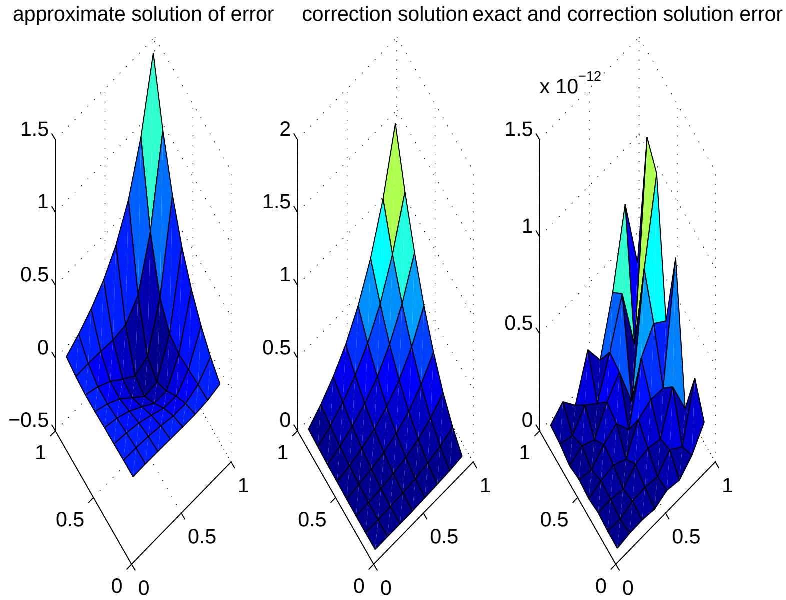

FIGURE 10. Approximate solution of error, correction solution and exact and correction solution error to Example 6.2, for $n = 5$, $ne = 7$

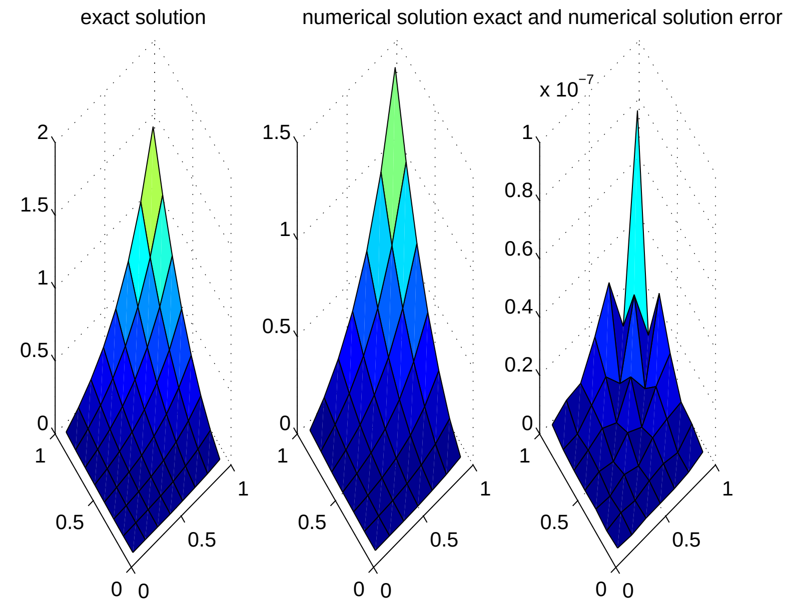

FIGURE 11. Exact solution, numerical solution and exact and numerical solution error to Example 6.2, for $n = 6$, $ne = 7$

the approximate solution of error obtained by fractional Bernstein polynomial is corrected as $e_{**}(x,t)$, and we can obtain the correction solution $u_*(x,t) + e_{**}(x,t)$; at last two numerical examples are given to prove the feasibility and effectiveness of the method and we can obtain more accurate solution by error correct, and we can find high accuracy as long as small n and ne. In the future, we can try to apply this method to studying the partial differential equation of variable coefficient, and expand the applicability of this method.

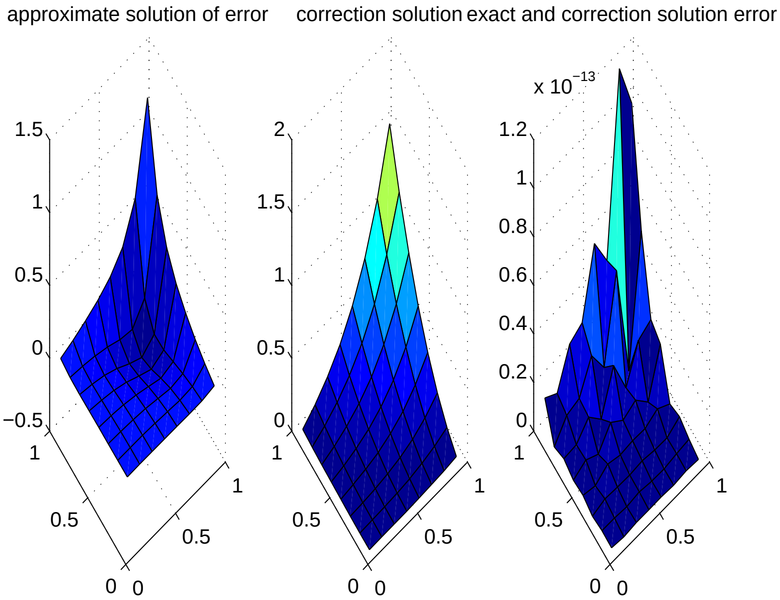

FIGURE 12. Approximate solution of error, correction solution and exact and correction solution error to Example 6.2, for $n = 6$, $ne = 7$

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