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NOVEL DOUBLE INTEGRAL INEQUALITIES AND THEIR APPLICATION TO STABILITY OF DELAYED SYSTEMS

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Abstract. *Integral inequalities play an important role in the stability analysis for systems with time-varying delay. In this paper, the orthogonal polynomials of one variable are extended to the orthogonal system of bivariate polynomials. An orthogonal system of bivariate functions which need not be continuous is introduced by triangulating a bounded domain in the plane. The bivariate functions in this orthogonal system need not be polynomials. Based on the orthogonal decomposition of vector and orthogonal approximation of vector, some new double integral inequalities are obtained. These double integral inequalities can provide tighter bounds than most of existing inequalities. Based on these double integral inequalities, an improved sufficient condition on asymptotical stability for systems with time-varying delay is obtained. Several numerical examples are given to show the effectiveness of the stability condition proposed in this paper.*
Keywords: Double integral inequality, Time-delay, Stability, Orthogonal approximation, Discontinuous function

1. **Introduction.** The emission of an electron from an atom upon the absorption of an energetic photon is one of the most elementary quantum-mechanical phenomena. There exists a delay in photoemission. This delay implies a change in the timing of ejection of the electron pulse with respect to the arrival of the photon pulse. In realistic systems, an inclusion of time delay is natural [1-3]. From the point of view of physics, the transport of matter, energy and information through the system requires finite time which is treated as time delay. The presence of time delay may change the dynamics properties of the system, for example, time delay may bring coherence resonance, excitability, synchronously periodical oscillation and transitions [4]. Time delay may induce travelling wave solutions. Therefore, academic and industrial communities have paid extensive attention to the dynamics of time-delay systems in the past few decades. Based on a new type of augmented Lyapunov functional, some new stability criteria for linear systems with time-varying delays are obtained in terms of linear matrix inequalities (LMIs) [5]. Some new delay-dependent stability criteria have been derived for neural networks by nonuniformly dividing the whole delay interval into multiple segments [6]. Based on the reciprocally convex approach, stability of systems with time-varying delays is addressed in [7]. The event-triggered dynamic output feedback control for networked control systems has been addressed in [8]. The Lyapunov-Krasovskii functional (LKF) approach has been employed to formulate a novel stability criterion for the resultant closed-loop system with an interval time-varying delay. By fully utilizing the information on the neuron activation function,

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some new passivity conditions for the neural networks with discrete and distributed delay have been derived [9]. In [10], a fuzzy time-delay feedback controller is designed to ensure the required H_∞ performance of the uncertain Takagi-Sugeno (T-S) fuzzy systems with time-delay to be achieved. By use of delay-partitioning technique, delay-dependent robust stability criteria for the uncertain neutral-type Lur'e system are proposed in terms of LMIs [11]. Two delay-dependent mean-square asymptotical stability criteria for a class of genetic regulatory networks (GRNs) with Markovian jumping parameters and time-varying delays are derived in [12]. In 2013, Cheng et al. [13] study the problem of finite-time H_∞ control for a class of Markovian jump systems with mode-dependent time-varying delays. Several sufficient conditions are derived to guarantee the finite-time stability of the resulting closed-loop system. Furthermore, a finite-time H_∞ fuzzy control problem for a class of nonlinear Markovian jump delayed systems with partly uncertain transition descriptions is addressed in [14]. Robust finite-time sampled-data control of linear systems subject to random occurring delays is investigated in [15] by employing Wirtinger-based double integral inequality approach proposed [16] and free-weighting matrix method.

Besides free-weighting matrix method [17-19], there are many methods to reduce the conservativeness of stability criteria for delayed systems, such as convex combination technique [20-22], Lyapunov functional approach [23,24] and inequality approach [25-28]. Gu et al. [29] succeeded in applying the Jensen inequality to the stability analysis of time-delay systems. The Wirtinger-based inequality developed in [30] can give a tighter bound than what the Jensen inequality does. A new class of integral inequalities for quadratic functions via intermediate terms called auxiliary functions are established [31], which produce tighter bounds than what the Jensen inequality produces. Recently, a theoretical framework for integral inequalities is established, in which the existing integral inequalities can be almost combined into two general inequalities defined in the upper and lower forms, respectively [32]. Although a lot of single integral inequalities including some extended Jensen inequalities and Wirtinger inequalities are proved and widely used in the stability analysis of delayed systems, there are only a few results for the double integral inequalities and their applications [29,31,32]. The integral inequalities obtained in [31,32] are both based on the orthogonal polynomials of one variable. Each polynomial is a continuous function. In order to avoid the complexity of polynomials of one variable with high degree and solve the problem of large computation burden resulting from the orthogonal polynomials with high degree, it motivates us not only to consider the orthogonal bivariate polynomials but also to consider the orthogonal system of bivariate functions which need not be continuous. In order to obtain the orthogonal system of bivariate functions $\{q_i(\alpha,\beta)\}$ defined on the bounded domain $D \in R^2$, we partition D into triangles by triangulation. Function $q_i(\alpha,\beta)$ is defined as constant on each triangle to construct an orthogonal system of bivariate functions. An orthogonal group of bivariate polynomials from an orthogonal system of bivariate polynomials and an orthogonal group from an orthogonal system of bivariate functions which are non-polynomials are constructed. Based on the orthogonal decomposition of vector, orthogonal approximation of vector and the minimization of energy function, some new double integral inequalities with tighter bounds are obtained. The Jensen double integral inequality [29], the Wirtinger based double integral inequality [16] and the auxiliary function-based double integral inequalities [31] are the special cases of our double integral inequalities. Therefore, a new approach to develop the double integral inequalities is proposed.

As mentioned above, Lyapunov-Krasovskii functional (LKF) is one of the most important and effective tools for the stability analysis of time-delay systems. The achievement of the desirable stability region depends on the construction of the Lyapunov-Krasovskii functional and estimation of its derivative. The triple integral forms of Lyapunov-Krasovskii functional are beneficial for reducing the conservatism, since more system information such as the size of delay can be exploited when calculating their derivatives. Some double integral terms such as $\int_a^b\int_\beta^b y^T(\alpha)Ry(\alpha)d\alpha d\beta$, $\int_a^b\int_\beta^b \dot{x}^T(\alpha)R\dot{x}(\alpha)d\alpha d\beta$, $\int_a^b\int_a^\beta y^T(\alpha)Ry(\alpha) d\alpha d\beta$, and $\int_a^b\int_a^\beta \dot{x}^T(\alpha)R\dot{x}(\alpha)d\alpha d\beta$ may arise from the derivative of the triple integral forms of Lyapunov-Krasovskii functional. The Jensen double integral inequality is usually used to bound these double integral terms. Since the double integral inequalities obtained in this paper are sharper than the Jensen double integral inequality and the Wirtinger-based double integral inequality, they can provide more tighter upper bounds for these double integrals. Combining these double integral inequalities with a new Lyapunov-Krasovskii functional with triple integral terms, an improved asymptotical stability criterion for the systems with time-varying delay is given in this paper. This stability criterion is delay-dependent. By virtue of these new double integral inequalities, it is less conservative than some existing asymptotical stability criteria. The asymptotical stability criterion obtained in this paper only needs $29n^2+5n$ decision variables. However, the asymptotical stability criterion in [32] requires $32.5n^2+6.5n$ decision variables. Therefore, stability criterion in our paper is with less computational burden. The main result in this paper can be applied to investigating the synchronization of delayed systems through analyzing the stability of the corresponding error systems.

The main contributions of this paper are summarized as follows.

(i) The orthogonal polynomials of one variable are extended to the orthogonal system of bivariate polynomials. An orthogonal system of bivariate functions which need not be continuous is introduced by triangulating a bounded domain D in the plane. These bivariate functions in this orthogonal system need not be polynomials.

(ii) Based on the idea of orthogonal decomposition of vector, the twice orthogonal approximation of vector is obtained.

(iii) By minimizing the value of the new energy function, two double integral inequalities are established, which include almost all of the existing integral inequalities as special cases. Our double integral inequalities are different from the double integral inequalities obtained in [32].

(iv) A new Lyapunov-Krasovskii functional with four triple integral terms is constructed. Lyapunov-Krasovskii functional terms $\int_{t-\frac{h_1}{2}}^t x^T(s)Q_3x(s)ds$ and $\int_{-\frac{h_1}{2}}^0\int_{t+s}^t x^T(u)R_2x(u) duds$ with the information of delay-partition are introduced.

(v) Combining the improved integral inequalities with the reciprocally convex approach, a less conservative asymptotical stability condition with less decision variables is derived for systems with time-varying delay.

The rest of this paper is organized as follows. In Section 2, based on the orthogonal systems of bivariate functions, some new double integral inequalities are proved. The stability of system with time-varying delay is investigated in Section 3. Based on the new Lyapunov-Krasovskii functional, an improved asymptotical stability condition is presented. In Section 4, three numerical examples are given to demonstrate the effectiveness of the obtained results. Finally, the conclusions are presented in Section 5.

2. **New Inequalities.** Let $D=\{(\alpha,\beta)|\beta\le\alpha\le b, a\le\beta\le b\}$, $D^*=\{(\alpha,\beta)|a\le\alpha\le\beta, a\le\beta\le b\}$. Suppose that $\{p_i(\alpha,\beta)\}$ is an orthogonal system of bivariate polynomials defined on D, it means $\iint\limits_D p_i(\alpha,\beta)p_j(\alpha,\beta)d\alpha d\beta=0$, $i\ne j$; $\iint\limits_D p_i^2(\alpha,\beta)d\alpha d\beta>0$. Let $\{q_i(\alpha,\beta)\}$ be another orthogonal system of bivariate functions defined on D. For simplicity, let $\{1,p_1(\alpha,\beta),p_2(\alpha,\beta)\}$ be an orthogonal group chosen from orthogonal system of bivariate polynomials defined on D. Let $\{1,q_1(\alpha,\beta)\}$ be an orthogonal group chosen from

another orthogonal system of bivariate functions defined on D. In this section, some new double integral inequalities are proved.

Theorem 2.1. *For a positive definite matrix $R > 0$, an integrable vector-valued function $y(\alpha)$ defined on $[a, b]$ and functions $p_1(\alpha, \beta)$, $p_2(\alpha, \beta)$ and $q_1(\alpha, \beta)$ defined on D satisfying $\int_a^b \int_\beta^b p_i(\alpha, \beta) d\alpha d\beta = 0$ $(i = 1, 2)$, $\int_a^b \int_\beta^b q_1(\alpha, \beta) d\alpha d\beta = 0$, $\int_a^b \int_\beta^b p_1(\alpha, \beta) p_2(\alpha, \beta) d\alpha d\beta = 0$ and $\int_a^b \int_\beta^b p_2(\alpha, \beta) q_1(\alpha, \beta) d\alpha d\beta = 0$, the following inequality holds*

$$\begin{aligned} &\int_a^b \int_\beta^b y^T(\alpha) R y(\alpha) d\alpha d\beta \\ \geq\ & \frac{2}{(b-a)^2} \left(\int_a^b \int_\beta^b y(\alpha) d\alpha d\beta \right)^T R \left(\int_a^b \int_\beta^b y(\alpha) d\alpha d\beta \right) + \frac{1}{P_1} F_1^T R F_1 + \frac{1}{P_2} F_2^T R F_2 \\ &+ \frac{1}{Q_1} \left[G_1 - \frac{\int_a^b \int_\beta^b p_1(\alpha, \beta) q_1(\alpha, \beta) d\alpha d\beta}{P_1} F_1 \right]^T R \left[G_1 - \frac{\int_a^b \int_\beta^b p_1(\alpha, \beta) q_1(\alpha, \beta) d\alpha d\beta}{P_1} F_1 \right], \end{aligned} \tag{1}$$

where

$$\begin{aligned} F_i &= \int_a^b \int_\beta^b p_i(u, \beta) y(u) du d\beta, \quad i = 1, 2, \\ P_i &= \int_a^b \int_\beta^b p_i^2(u, \beta) du d\beta, \quad i = 1, 2, \\ G_1 &= \int_a^b \int_\beta^b q_1(u, \beta) y(u) du d\beta, \\ Q_1 &= \int_a^b \int_\beta^b q_1^2(u, \beta) du d\beta. \end{aligned} \tag{2}$$

Proof: Let

$$z(u, \beta) = y(u) - \frac{2}{(b-a)^2} \int_a^b \int_\beta^b y(\alpha) d\alpha d\beta - \frac{p_1(u, \beta)}{P_1} F_1 - \frac{p_2(u, \beta)}{P_2} F_2 - q_1(u, \beta) v, \tag{3}$$

where

$$v = \frac{G_1}{Q_1} - \frac{\int_a^b \int_\beta^b p_1(\alpha, \beta) q_1(\alpha, \beta) d\alpha d\beta}{P_1 Q_1} F_1. \tag{4}$$

Define the energy function $J(v)$ as

$$J(v) = \int_a^b \int_\beta^b z^T(u, \beta) R z(u, \beta) du d\beta. \tag{5}$$

Substituting $z(u, \beta)$ into $J(v)$ gives

$$\begin{aligned} J(v) =& \int_a^b \int_\beta^b z^T(u, \beta) R z(u, \beta) du d\beta \\ =& \int_a^b \int_\beta^b \Bigg[y(u) - \frac{2}{(b-a)^2} \int_a^b \int_\beta^b y(\alpha) d\alpha d\beta - \frac{p_1(u, \beta)}{P_1} F_1 \\ & - \frac{p_2(u, \beta)}{P_2} F_2 \Bigg]^T R \Bigg[y(u) - \frac{2}{(b-a)^2} \int_a^b \int_\beta^b y(\alpha) d\alpha d\beta - \frac{p_1(u, \beta)}{P_1} F_1 \\ & - \frac{p_2(u, \beta)}{P_2} F_2 \Bigg] du d\beta + \int_a^b \int_\beta^b q_1^2(u, \beta) du d\beta v^T R v - 2 \int_a^b \int_\beta^b \Big[y(u) q_1(u, \beta) \end{aligned}$$

$$
\begin{aligned}
&-\frac{2q_1(u,\beta)}{(b-a)^2}\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta-\frac{p_1(u,\beta)q_1(u,\beta)}{P_1}F_1\\
&-\frac{p_2(u,\beta)q_1(u,\beta)}{P_2}F_2\Big]^T dud\beta Rv\\
={}&\int_a^b\int_\beta^b\Big[y(u)-\frac{2}{(b-a)^2}\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta-\frac{p_1(u,\beta)}{P_1}F_1-\frac{p_2(u,\beta)}{P_2}F_2\Big]^T R\Big[y(u)\\
&-\frac{2}{(b-a)^2}\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta-\frac{p_1(u,\beta)}{P_1}F_1-\frac{p_2(u,\beta)}{P_2}F_2\Big]dud\beta-Q_1v^TRv \qquad (6)\\
={}&\int_a^b\int_\beta^b y^T(u)Ry(u)dud\beta\\
&+\frac{2}{(b-a)^2}\Big(\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta\Big)^T R\Big(\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta\Big)+\frac{1}{P_1}F_1^TRF_1\\
&+\frac{1}{P_2}F_2^TRF_2-\frac{4}{(b-a)^2}\Big(\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta\Big)^T R\Big(\int_a^b\int_\beta^b y(\alpha)d\beta d\beta\Big)\\
&-\frac{2}{P_1}F_1^TRF_1-\frac{2}{P_2}F_2^TRF_2-Q_1v^TRv\\
={}&\int_a^b\int_\beta^b y^T(u)Ry(u)dud\beta-\frac{1}{P_1}F_1^TRF_1-\frac{1}{P_2}F_2^TRF_2-Q_1v^TRv\\
&-\frac{2}{(b-a)^2}\Big(\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta\Big)^T R\Big(\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta\Big)\geq 0.
\end{aligned}
$$

Obviously, inequality (6) is equivalent to

$$
\begin{aligned}
&\int_a^b\int_\beta^b y^T(\alpha)Ry(\alpha)d\alpha d\beta\\
\geq{}&\frac{2}{(b-a)^2}\Big(\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta\Big)^T R\Big(\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta\Big)+\frac{1}{P_1}F_1^TRF_1+\frac{1}{P_2}F_2^TRF_2 \qquad (7)\\
&+\frac{1}{Q_1}\Big[G_1-\frac{\int_a^b\int_\beta^b p_1(\alpha,\beta)q_1(\alpha,\beta)d\alpha d\beta}{P_1}F_1\Big]^T R\Big[G_1-\frac{\int_a^b\int_\beta^b p_1(\alpha,\beta)q_1(\alpha,\beta)d\alpha d\beta}{P_1}F_1\Big].
\end{aligned}
$$

The proof is completed.

In Theorem 2.1, if we take

$$
\begin{aligned}
p_1(\alpha,\beta)&=3(\alpha-b)+(b-a),\\
p_2(\alpha,\beta)&=5(b-a)^2(\alpha-b)^2+4(b-a)^3(\alpha-b)+\frac{(b-a)^4}{2},\\
q_1(\alpha,\beta)&=\begin{cases}1, & \frac{a+b}{2}\leq\alpha\leq b,\quad a\leq\beta\leq\frac{a+b}{2},\\ 3, & \beta\leq\alpha\leq\frac{a+b}{2},\quad a\leq\beta\leq\frac{a+b}{2},\\ -5, & \beta\leq\alpha\leq b,\quad \frac{a+b}{2}\leq\beta\leq b,\end{cases}
\end{aligned}
\qquad (8)
$$

then

$$
\int_a^b\int_\beta^b p_i(\alpha,\beta)d\alpha d\beta=0,\quad i=1,2,
$$

$$\int_a^b \int_\beta^b p_1(\alpha,\beta)p_2(\alpha,\beta)d\alpha d\beta = 0,$$

$$\begin{aligned}
&\int_a^b \int_\beta^b q_1(\alpha,\beta)d\alpha d\beta \\
&= \int_a^{\frac{a+b}{2}} \int_{\frac{a+b}{2}}^b 1 d\alpha d\beta + \int_a^{\frac{a+b}{2}} \int_\beta^{\frac{a+b}{2}} 3 d\alpha d\beta + \int_{\frac{a+b}{2}}^b \int_\beta^b (-5) d\alpha d\beta \\
&= \frac{(b-a)^2}{4} + \frac{3(b-a)^2}{8} - \frac{5(b-a)^2}{8} = 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\int_a^b \int_\beta^b p_2(\alpha,\beta)q_1(\alpha,\beta)d\alpha d\beta \\
&= \int_a^{\frac{a+b}{2}} \int_{\frac{a+b}{2}}^b p_2(\alpha,\beta)d\alpha d\beta + 3\int_a^{\frac{a+b}{2}} \int_\beta^{\frac{a+b}{2}} p_2(\alpha,\beta)d\alpha d\beta \\
&\quad - 5\int_{\frac{a+b}{2}}^b \int_\beta^b p_2(\alpha,\beta)d\alpha d\beta \\
&= -\frac{(b-a)^6}{48} + \frac{3(b-a)^6}{64} - \frac{5(b-a)^6}{192} = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\int_a^b \int_\beta^b p_1(\alpha,\beta)q_1(\alpha,\beta)d\alpha d\beta \\
&= \int_a^{\frac{a+b}{2}} \int_{\frac{a+b}{2}}^b p_1(\alpha,\beta)d\alpha d\beta + 3\int_a^{\frac{a+b}{2}} \int_\beta^{\frac{a+b}{2}} p_1(\alpha,\beta)d\alpha d\beta \\
&\quad - 5\int_{\frac{a+b}{2}}^b \int_\beta^b p_1(\alpha,\beta)d\alpha d\beta \\
&= \frac{(b-a)^3}{16} - \frac{3(b-a)^3}{8} - \frac{5(b-a)^3}{16} = -\frac{5(b-a)^3}{8},
\end{aligned} \tag{11}$$

$$\begin{aligned}
P_1 &= \int_a^b \int_\beta^b p_1^2(\alpha,\beta)d\alpha d\beta = \frac{(b-a)^4}{4}, \\
P_2 &= \int_a^b \int_\beta^b p_2^2(\alpha,\beta)d\alpha d\beta = \frac{(b-a)^{10}}{24}, \\
Q_1 &= \int_a^b \int_\beta^b q_1^2(\alpha,\beta)d\alpha d\beta \\
&= \int_a^{\frac{a+b}{2}} \int_{\frac{a+b}{2}}^b 1 d\alpha d\beta + \int_a^{\frac{a+b}{2}} \int_\beta^{\frac{a+b}{2}} 9 d\alpha d\beta + \int_{\frac{a+b}{2}}^b \int_\beta^b 25 d\alpha d\beta \\
&= \frac{(b-a)^2}{4} + \frac{9}{8}(b-a)^2 + \frac{25}{8}(b-a)^2 = \frac{9(b-a)^2}{2},
\end{aligned} \tag{12}$$

$$\begin{aligned}
F_1 &= \int_a^b \int_\beta^b p_1(\alpha,\beta)y(\alpha)d\alpha d\beta \\
&= 3\int_a^b \int_\beta^b (\alpha-b)y(\alpha)d\alpha d\beta + (b-a)\int_a^b \int_\beta^b y(\alpha)d\alpha d\beta
\end{aligned}$$

$$= 3\int_a^b\int_\beta^b(\alpha-\beta+\beta-a+a-b)y(\alpha)d\alpha d\beta+(b-a)\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta$$
$$= 6\int_a^b\int_\beta^b\int_\alpha^b y(r)drd\alpha d\beta-2(b-a)\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta=-2(b-a)\Omega_2,$$

$$\begin{aligned}
F_2 &= \int_a^b\int_\beta^b p_2(\alpha,\beta)y(\alpha)d\alpha d\beta\\
&= (b-a)^2\Big[5\int_a^b\int_\beta^b \alpha^2 y(\alpha)d\alpha d\beta-(6b+4a)\int_a^b\int_\beta^b(\alpha-b)y(\alpha)d\alpha d\beta\\
&\quad+\left(\frac{(b-a)^2}{2}-5b^2\right)\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta\Big]\\
&= (b-a)^2\Big\{5\Big[6\int_a^b\int_s^b\int_\beta^b\int_\alpha^b y(r)drd\alpha d\beta ds\\
&\quad+4a\int_a^b\int_s^b\int_\beta^b y(r)drd\beta ds+a^2\int_a^b\int_\beta^b y(r)drd\beta\Big]\\
&\quad-(6b+4a)\Big[2\int_a^b\int_s^b\int_\beta^b y(r)drd\beta ds-(b-a)\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta\Big]\\
&\quad+\left(\frac{(b-a)^2}{2}-5b^2\right)\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta\Big\}\\
&= 30(b-a)^2\int_a^b\int_s^b\int_\beta^b\int_\alpha^b y(r)drd\alpha d\beta ds\\
&\quad-12(b-a)^3\int_a^b\int_s^b\int_\beta^b y(r)drd\beta ds+\frac{3}{2}(b-a)^4\int_a^b\int_\beta^b y(r)drd\beta\\
&= \frac{3(b-a)^4}{2}\Omega_3,
\end{aligned}\tag{13}$$

$$\begin{aligned}
G_1 &= \int_a^b\int_\beta^b q_1(\alpha,\beta)y(\alpha)d\alpha d\beta\\
&= \int_a^{\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b y(\alpha)d\alpha d\beta+3\int_a^{\frac{a+b}{2}}\int_\beta^{\frac{a+b}{2}} y(\alpha)d\alpha d\beta-5\int_{\frac{a+b}{2}}^b\int_\beta^b y(\alpha)d\alpha d\beta\\
&= \frac{b-a}{2}\int_{\frac{a+b}{2}}^b y(\alpha)d\alpha+3\int_a^{\frac{a+b}{2}}\int_\beta^{\frac{a+b}{2}} y(\alpha)d\alpha d\beta-5\int_{\frac{a+b}{2}}^b\int_\beta^b y(\alpha)d\alpha d\beta=\Omega_4,
\end{aligned}$$

$$\begin{aligned}
v &= \frac{1}{Q_1}\left[G_1-\frac{\int_a^b\int_\beta^b p_1(\alpha,\beta)q_1(\alpha,\beta)d\alpha d\beta}{P_1}F_1\right]\\
&= \frac{2}{9(b-a)^2}\left[\Omega_4-\frac{-5(b-a)^3}{8}\left(\frac{4}{(b-a)^4}\right)(-2(b-a)\Omega_2)\right]\\
&= \frac{2}{9(b-a)^2}(\Omega_4-5\Omega_2),
\end{aligned}\tag{14}$$

where Ω_i, $i=1,2,3,4$, are defined in Corollary 2.1.

From Theorem 2.1, we have

Corollary 2.1. *For a positive definite matrix $R > 0$ and an integrable vector-valued function $y(\alpha)$ defined on $[a, b]$, the following inequality holds*

$$\begin{aligned}\int_a^b \int_\beta^b y^T(\alpha)Ry(\alpha)d\alpha d\beta \geq \frac{2}{(b-a)^2}\Omega_1^T R\Omega_1 + \frac{16}{(b-a)^2}\Omega_2^T R\Omega_2 + \frac{54}{(b-a)^2}\Omega_3^T R\Omega_3 \\ +\frac{2}{9(b-a)^2}(\Omega_4 - 5\Omega_2)^T R(\Omega_4 - 5\Omega_2),\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\Omega_1 &= \int_a^b \int_\beta^b y(\alpha)d\alpha d\beta, \\
\Omega_2 &= \int_a^b \int_\beta^b y(\alpha)d\alpha d\beta - \frac{3}{b-a}\int_a^b \int_\beta^b \int_\alpha^b y(r)drd\alpha d\beta, \\
\Omega_3 &= \int_a^b \int_\beta^b y(\alpha)d\alpha d\beta - \frac{8}{b-a}\int_a^b \int_s^b \int_\beta^b y(\alpha)d\alpha d\beta ds \\
&\quad + \frac{20}{(b-a)^2}\int_a^b \int_s^b \int_\beta^b \int_r^b y(\alpha)d\alpha dr d\beta ds, \\
\Omega_4 &= \frac{b-a}{2}\int_{\frac{a+b}{2}}^b y(\alpha)d\alpha + 3\int_a^{\frac{a+b}{2}} \int_\beta^{\frac{a+b}{2}} y(\alpha)d\alpha d\beta - 5\int_{\frac{a+b}{2}}^b \int_\beta^b y(\alpha)d\alpha d\beta.
\end{aligned}$$

Let $y(\alpha) = \dot{x}(\alpha)$ in Corollary 2.1, we get

Corollary 2.2. *For a positive definite matrix $R > 0$ and a differentiable vector-valued function $x(\alpha)$ defined on $[a, b]$, the following inequality holds:*

$$\begin{aligned}\int_a^b \int_\beta^b \dot{x}^T(\alpha)R\dot{x}(\alpha)d\alpha d\beta \geq 2\Psi_1^T R\Psi_1 + 16\Psi_2^T R\Psi_2 + 54\Psi_3^T R\Psi_3 \\ +\frac{2}{9}(\Psi_4 - 5\Psi_2)^T R(\Psi_4 - 5\Psi_2),\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\Psi_1 &= x(b) - \frac{1}{b-a}\int_a^b x(\alpha)d\alpha, \\
\Psi_2 &= -\frac{1}{2}x(b) - \frac{1}{b-a}\int_a^b x(\alpha)d\alpha + \frac{3}{(b-a)^2}\int_a^b \int_\beta^b x(\alpha)d\alpha d\beta, \\
\Psi_3 &= \frac{1}{3}x(b) - \frac{1}{b-a}\int_a^b x(\alpha)d\alpha + \frac{8}{(b-a)^2}\int_a^b \int_\beta^b x(\alpha)d\alpha d\beta \\
&\quad - \frac{20}{(b-a)^3}\int_a^b \int_s^b \int_\beta^b x(\alpha)d\alpha d\beta ds, \\
\Psi_4 &= -2x(b) + x\left(\frac{b+a}{2}\right) - \frac{3}{b-a}\int_a^b x(\alpha)d\alpha + \frac{8}{b-a}\int_{\frac{b+a}{2}}^b x(\alpha)d\alpha.
\end{aligned}$$

Proof: Substituting $y(\alpha) = \dot{x}(\alpha)$ into Corollary 2.1, we can easily get that

$$\begin{aligned}
\Omega_1 &= \int_a^b \int_\beta^b \dot{x}(\alpha)d\alpha d\beta = (b-a)x(b) - \int_a^b x(\alpha)d\alpha = (b-a)\Psi_1, \\
\Omega_2 &= \int_a^b \int_\beta^b \dot{x}(\alpha)d\alpha d\beta - \frac{3}{b-a}\int_a^b \int_\beta^b \int_\alpha^b \dot{x}(r)drd\alpha d\beta \\
&= -\frac{1}{2}(b-a)x(b) - \int_a^b x(\alpha)d\alpha + \frac{3}{b-a}\int_a^b \int_\beta^b x(\alpha)d\alpha d\beta = (b-a)\Psi_2,
\end{aligned}$$

$$
\begin{aligned}
\Omega_3 &= \int_a^b \int_\beta^b \dot{x}(\alpha) d\alpha d\beta - \frac{8}{b-a} \int_a^b \int_\beta^b \int_\alpha^b \dot{x}(r) dr d\alpha d\beta \\
&\quad + \frac{20}{(b-a)^2} \int_a^b \int_s^b \int_\beta^b \int_\alpha^b \dot{x}(r) dr d\alpha d\beta ds \\
&= (b-a)x(b) - \int_a^b x(\alpha) d\alpha - \frac{8}{b-a} \left[\frac{(b-a)^2}{2} x(b) - \int_a^b \int_\beta^b x(\alpha) d\alpha d\beta \right] \\
&\quad + \frac{20}{(b-a)^2} \left[\frac{(b-a)^3}{6} x(b) - \int_a^b \int_s^b \int_\beta^b x(\alpha) d\alpha d\beta ds \right] = (b-a)\Psi_3,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\Omega_4 - 5\Omega_2 &= \frac{b-a}{2} \int_{\frac{a+b}{2}}^b \dot{x}(\alpha) d\alpha + 3 \int_a^{\frac{a+b}{2}} \int_\beta^{\frac{a+b}{2}} \dot{x}(\alpha) d\alpha d\beta \\
&\quad - 5 \int_{\frac{a+b}{2}}^b \int_\beta^b \dot{x}(\alpha) d\alpha d\beta - 5(b-a)\Psi_2 \\
&= -2(b-a)x(b) + (b-a)x\left(\frac{b+a}{2}\right) - 3 \int_a^b x(\alpha) d\alpha \\
&\quad + 8 \int_{\frac{b+a}{2}}^b x(\alpha) d\alpha - 5(b-a)\Psi_2 = (b-a)[\Psi_4 - 5\Psi_2].
\end{aligned} \tag{18}
$$

This completes the proof of Corollary 2.2.

Remark 2.1. *Compared with the auxiliary function-based double integral inequality (25) in [31], two terms $54\Psi_3^T R \Psi_3$ and $\frac{2}{9}(\Psi_4 - 5\Psi_2)^T R(\Psi_4 - 5\Psi_2)$ are added in righthand side of inequality (16) in Corollary 2.2. Since matrix R is positive definite, then $54\Psi_3^T R \Psi_3 \geq 0$ and $\frac{2}{9}(\Psi_4 - 5\Psi_2)^T R(\Psi_4 - 5\Psi_2) \geq 0$. So the Jensen double integral inequality [29], the Wirtinger based double integral inequality in [16] and the auxiliary function-based double integral inequalities [31] are the special cases of double integral inequality (16). Since $\frac{2}{9}(\Psi_4 - 5\Psi_2)^T R(\Psi_4 - 5\Psi_2) \geq 0$ in Corollary 2.2 is different from $\frac{128}{(b-a)^2}\dot{\Omega}_{0,3}^T R \dot{\Omega}_{0,3}$ in [32], the double integral inequality (16) is different from the double integral inequality (12) in [32].*

Similar to the proof of Theorem 2.1, we have

Theorem 2.2. *For a positive definite matrix $R > 0$, an integrable vector-valued function $y(\alpha)$ defined on $[a, b]$ and functions $p_1(\alpha, \beta)$, $p_2(\alpha, \beta)$ and $q_1(\alpha, \beta)$ defined on D^* satisfying $\int_a^b \int_a^\beta p_i(\alpha, \beta) d\alpha d\beta = 0$ $(i = 1, 2)$, $\int_a^b \int_a^\beta q_1(\alpha, \beta) d\alpha d\beta = 0$, $\int_a^b \int_a^\beta p_1(\alpha, \beta) p_2(\alpha, \beta) d\alpha d\beta = 0$ and $\int_a^b \int_a^\beta p_2(\alpha, \beta) q_1(\alpha, \beta) d\alpha d\beta = 0$, the following inequality holds*

$$
\begin{aligned}
&\int_a^b \int_a^\beta y^T(\alpha) R y(\alpha) d\alpha d\beta \\
&\geq \frac{2}{(b-a)^2} \left(\int_a^b \int_a^\beta y(\alpha) d\alpha d\beta \right)^T R \left(\int_a^b \int_a^\beta y(\alpha) d\alpha d\beta \right) + \frac{1}{P_1} F_1^T R F_1 \\
&\quad + \frac{1}{P_2} F_2^T R F_2 + \frac{1}{Q_1} \left[G_1 - \frac{\int_a^b \int_a^\beta p_1(\alpha, \beta) q_1(\alpha, \beta) d\alpha d\beta}{P_1} F_1 \right]^T R \Bigg[G_1 \\
&\quad - \frac{\int_a^b \int_a^\beta p_1(\alpha, \beta) q_1(\alpha, \beta) d\alpha d\beta}{P_1} F_1 \Bigg],
\end{aligned} \tag{19}
$$

where

$$\begin{aligned}
F_i &= \int_a^b \int_a^\beta p_i(u,\beta) y(u) du d\beta, \quad i = 1, 2, \\
P_i &= \int_a^b \int_a^\beta p_i^2(u,\beta) du d\beta, \quad i = 1, 2, \\
G_1 &= \int_a^b \int_a^\beta q_1(u,\beta) y(u) du d\beta, \\
Q_1 &= \int_a^b \int_a^\beta q_1^2(u,\beta) du d\beta.
\end{aligned} \tag{20}$$

Specially, taking

$$\begin{aligned}
& p_1(\alpha,\beta) = 3(a-\alpha) + (b-a), \\
& p_2(\alpha,\beta) = 5(b-a)^2(a-\alpha)^2 + 4(b-a)^3(a-\alpha) + \frac{(b-a)^4}{2}, \\
& q_1(\alpha,\beta) = \begin{cases} 1, & a \le \alpha \le \frac{a+b}{2}, \quad \frac{a+b}{2} \le \beta \le b, \\ 3, & \frac{a+b}{2} \le \alpha \le \beta, \quad \frac{a+b}{2} \le \beta \le b, \\ -5, & a \le \alpha \le \beta, \quad a \le \beta \le \frac{a+b}{2}, \end{cases}
\end{aligned} \tag{21}$$

we have

$$\begin{aligned}
& \int_a^b \int_a^\beta p_i(\alpha,\beta) d\alpha d\beta = 0, \quad i = 1, 2, \\
& \int_a^b \int_a^\beta q_1(\alpha,\beta) d\alpha d\beta = 0, \\
& \int_a^b \int_a^\beta p_1(\alpha,\beta) p_2(\alpha,\beta) d\alpha d\beta = 0, \\
& \int_a^b \int_a^\beta p_2(\alpha,\beta) q_1(\alpha,\beta) d\alpha d\beta = 0, \\
& \int_a^b \int_a^\beta p_1(\alpha,\beta) q_1(\alpha,\beta) d\alpha d\beta = -\frac{5(b-a)^3}{8},
\end{aligned} \tag{22}$$

$$\begin{aligned}
P_1 &= \int_a^b \int_a^\beta p_1^2(\alpha,\beta) d\alpha d\beta = \frac{(b-a)^4}{4}, \\
P_2 &= \int_a^b \int_a^\beta p_2^2(\alpha,\beta) d\alpha d\beta = \frac{(b-a)^{10}}{24}, \\
Q_1 &= \int_a^b \int_a^\beta q_1^2(\alpha,\beta) d\alpha d\beta = \frac{9(b-a)^2}{2},
\end{aligned} \tag{23}$$

$$\begin{aligned}
F_1 &= \int_a^b \int_a^\beta p_1(\alpha,\beta) y(\alpha) d\alpha d\beta \\
&= 3 \int_a^b \int_a^\beta (a - b + b - \beta + \beta - \alpha) y(\alpha) d\alpha d\beta + (b-a) \int_a^b \int_a^\beta y(\alpha) d\alpha d\beta \\
&= 6 \int_a^b \int_a^\beta \int_a^\alpha y(r) dr d\alpha d\beta - 2(b-a) \int_a^b \int_a^\beta y(\alpha) d\alpha d\beta
\end{aligned} \tag{24}$$

$$
\begin{aligned}
&= (b-a)^2 \int_a^b y(\alpha)d\alpha - 4(b-a)\int_a^b \int_\beta^b y(\alpha)d\alpha d\beta \\
&\quad + 6\int_a^b \int_\beta^b \int_\alpha^b y(r)drd\alpha d\beta = 2(b-a)\Omega_6,
\end{aligned}
$$

$$
\begin{aligned}
F_2 &= \int_a^b \int_a^\beta p_2(\alpha,\beta)y(\alpha)d\alpha d\beta \\
&= (b-a)^2 \Big\{ 5\Big[6\int_a^b \int_a^s \int_a^\beta \int_a^\alpha y(r)drd\alpha d\beta ds \\
&\quad - 4b\int_a^b \int_a^s \int_a^\beta y(r)drd\beta ds + b^2 \int_a^b \int_a^\beta y(r)drd\beta\Big] \\
&\quad + (6a+4b)\Big[2\int_a^b \int_a^s \int_a^\beta y(r)drd\beta ds - (b-a)\int_a^b \int_a^\beta y(\alpha)d\alpha d\beta\Big] \\
&\quad + \left(\frac{(b-a)^2}{2} - 5a^2\right)\int_a^b \int_a^\beta y(\alpha)d\alpha d\beta \Big\} \qquad (25)\\
&= 30(b-a)^2 \int_a^b \int_a^s \int_a^\beta \int_a^\alpha y(r)drd\alpha d\beta ds \\
&\quad - 12(b-a)^3 \int_a^b \int_b^s \int_a^\beta y(r)drd\beta ds + \frac{3}{2}(b-a)^4 \int_a^b \int_a^\beta y(r)drd\beta \\
&= \frac{(b-a)^5}{2}\int_a^b y(\alpha)d\alpha - \frac{9(b-a)^4}{2}\int_a^b \int_\beta^b y(\alpha)d\alpha d\beta \\
&\quad + 18(b-a)^3 \int_a^b \int_\beta^b \int_\alpha^b y(r)drd\alpha d\beta \\
&\quad - 30(b-a)^2 \int_a^b \int_s^b \int_\beta^b \int_\alpha^b y(r)drd\alpha d\beta ds = \frac{3(b-a)^4}{2}\Omega_7,
\end{aligned}
$$

$$
\begin{aligned}
G_1 &= \int_a^b \int_a^\beta q_1(\alpha,\beta)y(\alpha)d\alpha d\beta \\
&= \frac{b-a}{2}\int_a^{\frac{b+a}{2}} y(\alpha)d\alpha + 3\int_{\frac{b+a}{2}}^b \int_{\frac{b+a}{2}}^\beta y(\alpha)d\alpha d\beta - 5\int_a^{\frac{b+a}{2}} \int_a^\beta y(\alpha)d\alpha d\beta = \Omega_8, \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
v &= \frac{1}{Q_1}\left[G_1 - \frac{\int_a^b \int_a^\beta p_1(\alpha)q_1(\alpha)d\alpha d\beta}{P_1}F_1\right] \\
&= \frac{2}{9(b-a)^2}\left[\Omega_8 - \frac{-5(b-a)^3}{8}\left(\frac{4}{(b-a)^4}\right)(2(b-a)\Omega_6)\right] \\
&= \frac{2}{9(b-a)^2}(\Omega_8 + 5\Omega_6),
\end{aligned}
$$

where Ω_i, $i = 5,6,7,8$, are defined in Corollary 2.3.

From Theorem 2.2, we have

Corollary 2.3. *For a positive definite matrix $R > 0$ and an integrable vector-valued function $y(\alpha)$ defined on $[a,b]$, the following inequality holds*

$$\begin{aligned}\int_a^b\int_a^\beta y^T(\alpha)Ry(\alpha)d\alpha d\beta \geq\ & \frac{2}{(b-a)^2}\Omega_5^TR\Omega_5+\frac{16}{(b-a)^2}\Omega_6^TR\Omega_6+\frac{54}{(b-a)^2}\Omega_7^TR\Omega_7\\ &+\frac{2}{9(b-a)^2}(\Omega_8+5\Omega_6)^TR(\Omega_8+5\Omega_6),\end{aligned}\tag{27}$$

where

$$\Omega_5=(b-a)\int_a^b y(\alpha)d\alpha-\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta,$$
$$\Omega_6=\frac{1}{2}(b-a)\int_a^b y(\alpha)d\alpha-2\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta+\frac{3}{b-a}\int_a^b\int_\beta^b\int_\alpha^b y(r)drd\alpha d\beta,$$
$$\begin{aligned}\Omega_7=\ &\frac{1}{3}(b-a)\int_a^b y(\alpha)d\alpha-3\int_a^b\int_\beta^b y(\alpha)d\alpha d\beta+\frac{12}{b-a}\int_a^b\int_s^b\int_\beta^b y(\alpha)d\alpha d\beta ds\\ &-\frac{20}{(b-a)^2}\int_a^b\int_s^b\int_\beta^b\int_r^b y(\alpha)d\alpha dr d\beta ds,\end{aligned}$$
$$\Omega_8=\frac{b-a}{2}\int_a^{\frac{b+a}{2}} y(\alpha)d\alpha+3\int_{\frac{b+a}{2}}^b\int_{\frac{b+a}{2}}^\beta y(\alpha)d\alpha d\beta-5\int_a^{\frac{b+a}{2}}\int_a^\beta y(\alpha)d\alpha d\beta.$$

Let $y(\alpha)=\dot{x}(\alpha)$ in Corollary 2.3, we have

Corollary 2.4. *For a positive definite matrix $R>0$ and a differentiable vector-valued function $x(\alpha)$ defined on $[a,b]$, the following inequality holds:*

$$\begin{aligned}\int_a^b\int_a^\beta \dot{x}^T(\alpha)R\dot{x}(\alpha)d\alpha d\beta \geq\ & 2\Psi_5^TR\Psi_5+16\Psi_6^TR\Psi_6+54\Psi_7^TR\Psi_7\\ &+\frac{2}{9}(\Psi_8+5\Psi_6)^TR(\Psi_8+5\Psi_6),\end{aligned}\tag{28}$$

where

$$\Psi_5=-x(a)+\frac{1}{b-a}\int_a^b x(\alpha)d\alpha,$$
$$\Psi_6=-\frac{1}{2}x(a)+\frac{2}{b-a}\int_a^b x(\alpha)d\alpha-\frac{3}{(b-a)^2}\int_a^b\int_\beta^b x(\alpha)d\alpha d\beta,$$
$$\begin{aligned}\Psi_7=\ &-\frac{1}{3}x(a)+\frac{3}{b-a}\int_a^b x(\alpha)d\alpha-\frac{12}{(b-a)^2}\int_a^b\int_\beta^b x(\alpha)d\alpha d\beta\\ &+\frac{20}{(b-a)^3}\int_a^b\int_s^b\int_\beta^b x(\alpha)d\alpha d\beta ds,\end{aligned}$$
$$\Psi_8=2x(a)-x\left(\frac{b+a}{2}\right)-\frac{5}{b-a}\int_a^b x(\alpha)d\alpha+\frac{8}{b-a}\int_{\frac{b+a}{2}}^b x(\alpha)d\alpha.$$

Proof: Substituting $y(\alpha)=\dot{x}(\alpha)$ into Corollary 2.3, we can easily get that

$$\begin{aligned}\Omega_5 &= (b-a)\int_a^b \dot{x}(\alpha)d\alpha-\int_a^b\int_\beta^b \dot{x}(\alpha)d\alpha d\beta\\ &= (b-a)[x(b)-x(a)]-(b-a)x(b)+\int_a^b x(\alpha)d\alpha=(b-a)\Psi_5,\\ \Omega_6 &= \frac{1}{2}(b-a)\int_a^b \dot{x}(\alpha)d\alpha-2\int_a^b\int_\beta^b \dot{x}(\alpha)d\alpha d\beta+\frac{3}{b-a}\int_a^b\int_\beta^b\int_\alpha^b \dot{x}(r)drd\alpha d\beta\\ &= \frac{1}{2}(b-a)[x(b)-x(a)]-2(b-a)x(b)\end{aligned}$$

$$
\begin{aligned}
&+2\int_a^b x(\alpha)d\alpha + \frac{3}{b-a}\left[\frac{(b-a)^2}{2}x(b) - \int_a^b\int_\beta^b x(\alpha)d\alpha d\beta\right] = (b-a)\Psi_6, \\
\Omega_7 &= \frac{1}{3}(b-a)\int_a^b \dot{x}(\alpha)d\alpha - 3\int_a^b\int_\beta^b \dot{x}(\alpha)d\alpha d\beta \\
&+ \frac{12}{b-a}\int_a^b\int_\beta^b\int_\alpha^b \dot{x}(r)drd\alpha d\beta - \frac{20}{(b-a)^2}\int_a^b\int_s^b\int_\beta^b\int_\alpha^b \dot{x}(r)drd\alpha d\beta ds \\
&= \frac{1}{3}(b-a)[x(b)-x(a)] - 3(b-a)\left[x(b) - \frac{1}{b-a}\int_a^b x(\alpha)d\alpha\right] \\
&+ \frac{12}{b-a}\left[\frac{(b-a)^2}{2}x(b) - \int_a^b\int_\beta^b x(\alpha)d\alpha d\beta\right] \\
&- \frac{20}{(b-a)^2}\left[\frac{(b-a)^3}{6}x(b) - \int_a^b\int_s^b\int_\beta^b x(\alpha)d\alpha d\beta ds\right] = (b-a)\Psi_7,
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\Omega_8 + 5\Omega_6 &= \frac{b-a}{2}\int_a^{\frac{b+a}{2}} \dot{x}(\alpha)d\alpha + 3\int_{\frac{b+a}{2}}^b\int_{\frac{b+a}{2}}^\beta \dot{x}(\alpha)d\alpha d\beta - 5\int_a^{\frac{b+a}{2}}\int_a^\beta \dot{x}(\alpha)d\alpha d\beta \\
&= 2(b-a)x(a) - (b-a)x\left(\frac{b+a}{2}\right) - 5\int_a^b x(\alpha)d\alpha + 8\int_{\frac{b+a}{2}}^b x(\alpha)d\alpha \\
&+ 5(b-a)\Psi_6 = (b-a)[\Psi_8 + 5\Psi_6].
\end{aligned}
$$

This completes the proof of Corollary 2.4.

Lemma 2.1. [7] *Let $f_1, f_2, \ldots, f_N : R^m \to R$ have positive values in an open subset D of R^m. Then, the reciprocally convex combination of f_i over D satisfies*

$$
\min_{\left\{\alpha_i|\alpha_i>0,\sum\limits_i \alpha_i=1\right\}} \sum_i \frac{1}{\alpha_i}f_i(t) = \sum_i f_i(t) + \max_{g_{ij}(t)}\sum_{i\neq j} g_{ij}(t) \tag{30}
$$

subject to

$$
\left\{g_{ij} : R^m \to R, g_{ji}(t) \triangleq g_{ij}(t), \begin{pmatrix} f_i(t) & g_{i,j}(t) \\ g_{j,i}(t) & f_j(t)\end{pmatrix} \geq 0\right\}. \tag{31}
$$

Lemma 2.2. [33] *For a given matrix $R > 0$, the following inequality holds for all continuous differentiable function $x(t)$ in $[a,b] \in R^n$:*

$$
\begin{aligned}
-\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \leq &-\frac{1}{b-a}\Omega_9^T R\Omega_9 - \frac{3}{b-a}\Omega_{10}^T R\Omega_{10} \\
&- \frac{5}{b-a}\Omega_{11}^T R\Omega_{11} - \frac{7}{b-a}\Omega_{12}^T R\Omega_{12},
\end{aligned} \tag{32}
$$

where

$$
\begin{aligned}
&\Omega_9 = x(b) - x(a), \\
&\Omega_{10} = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds, \\
&\Omega_{11} = x(b) - x(a) + \frac{6}{b-a}\int_a^b x(s)ds - \frac{12}{(b-a)^2}\int_a^b\int_u^b x(s)dsdu, \\
&\Omega_{12} = x(b) + x(a) - \frac{12}{b-a}\int_a^b x(s)ds + \frac{60}{(b-a)^2}\int_a^b\int_u^b x(s)dsdu \\
&\quad - \frac{120}{(b-a)^3}\int_a^b\int_u^b\int_s^b x(\alpha)d\alpha dsdu.
\end{aligned}
$$

3. **Applications in Stability Analysis.** In this section, we will consider the following linear system with time-varying delay

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-h(t)), & t \geq 0 \\ x(t) = \varphi(t), & t \in [-h_2, 0] \end{cases} \tag{33}$$

where the initial condition $\varphi(t)$ is a continuously differentiable function and the delay $h(t)$ satisfies $0 \leq h_1 \leq h(t) \leq h_2$.

At first, the following notations are needed.

$$\begin{aligned}
&h_{12} = h_2 - h_1, \\
&e_i = [O_{n\times(i-1)n}, I, O_{n\times(15-i)n}]^T, \\
&e_s = [A, 0, B, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T, \\
&\alpha(t) = \left[x^T(t), \int_{t-h_1}^{t} x^T(s)ds, \int_{t-h_2}^{t-h_1} x^T(s)ds, \frac{2}{h_1}\int_{-h_1}^{0}\int_{t+s}^{t} x^T(u)duds,\right. \\
&\qquad \left.\frac{2}{h_1^2}\int_{-h_1}^{0}\int_{s}^{0}\int_{t+v}^{t} x^T(u)dudvds\right]^T, \\
&\xi(t) = \left[x^T(t), x^T(t-h_1), x^T(t-h(t)), x^T(t-h_2), \frac{1}{h_1}\int_{t-h_1}^{t} x^T(s)ds,\right. \\
&\qquad \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x^T(s)ds, \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x^T(s)ds, \frac{2}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t} x^T(u)duds, \\
&\qquad \frac{2}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1} x^T(u)duds, \frac{2}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)} x^T(u)duds, \\
&\qquad \frac{1}{h_1^3}\int_{-h_1}^{0}\int_{s}^{0}\int_{t+v}^{t} x^T(u)dudvds, \frac{1}{(h(t)-h_1)^3}\int_{-h(t)}^{-h_1}\int_{s}^{-h_1}\int_{t+v}^{t-h_1} x^T(u)dudvds, \\
&\qquad \frac{1}{(h_2-h(t))^3}\int_{-h_2}^{-h(t)}\int_{s}^{-h(t)}\int_{t+v}^{t-h(t)} x^T(u)dudvds, \\
&\qquad \left.\frac{1}{h_1}\int_{t-\frac{h_1}{2}}^{t} x^T(s)ds, x^T\left(t-\frac{h_1}{2}\right)\right]^T, \\
&R_{21} = diag\{R_2, 3R_2, 5R_2\}, \\
&R_{22} = diag\{R_2 + Z_3, 3(R_2 + Z_3), 5(R_2 + Z_3)\}, \\
&R_{23} = diag\{R_2 + Z_4, 3(R_2 + Z_4), 5(R_2 + Z_4)\}, \\
&Z_{31} = diag\{Z_3, 3Z_3, 5Z_3\}, \\
&Z_{41} = diag\{Z_4, 3Z_4, 5Z_4\}, \\
&R_{24} = \begin{pmatrix} R_{21} & S \\ S^T & R_{21} \end{pmatrix}, \quad R_{25} = \begin{pmatrix} R_{22} & S \\ S^T & R_{23} \end{pmatrix}, \\
&\Pi_1(h(t)) = [e_1, h_1e_5, (h(t)-h_1)e_6 + (h_2-h(t))e_7, h_1e_8, 2h_1e_{11}], \\
&\Pi_2 = [e_s, e_1 - e_2, e_2 - e_4, 2(e_1 - e_5), e_1 - e_8], \\
&\Pi_3 = [e_2 - e_3, e_2 + e_3 - 2e_6, e_2 - e_3 + 6e_6 - 6e_9], \\
&\Pi_4 = [e_3 - e_4, e_3 + e_4 - 2e_7, e_3 - e_4 + 6e_7 - 6e_{10}], \\
&\Pi_5 = [e_2 - e_3, e_2 + e_3 - 2e_6, e_2 - e_3 + 6e_6 - 6e_9, \\
&\qquad e_3 - e_4, e_3 + e_4 - 2e_7, e_3 - e_4 + 6e_7 - 6e_{10}], \\
&\Xi_1 = sym\left\{\Pi_1(h(t))P\Pi_2^T\right\},
\end{aligned}$$

$$
\begin{aligned}
\Xi_2 &= e_1 Q_1 e_1^T - e_2 Q_1 e_2^T + e_2 Q_2 e_2^T - e_4 Q_2 e_4^T + e_1 Q_3 e_1^T - e_{15} Q_3 e_{15}^T,\\
\Xi_3 &= e_s \left(h_1^2 R_1 + h_{12}^2 R_2\right) e_s^T - (e_1 - e_2) R_1 (e_1 - e_2)^T\\
&\quad - 3(e_1 + e_2 - 2e_5) R_1 (e_1 + e_2 - 2e_5)^T\\
&\quad - 5(e_1 - e_2 + 6e_5 - 6e_8) R_1 (e_1 - e_2 + 6e_5 - 6e_8)^T\\
&\quad - 7(e_1 + e_2 - 12e_5 + 30e_8 - 120e_{11}) R_1 (e_1 + e_2 - 12e_5 + 30e_8 - 120e_{11})^T\\
&\quad + \frac{h_1}{2} e_1 R_3 e_1^T - 2h_1 e_{14} R_3 e_{14}^T,\\
\Xi_4 &= e_s \left(\frac{h_1^2}{2} Z_1 + \frac{h_1^2}{2} Z_2\right) e_s^T - 2(e_1 - e_5) Z_1 (e_1 - e_5)^T\\
&\quad - 16\left(\frac{1}{2}e_1 + e_5 - \frac{3}{2}e_8\right) Z_1 \left(\frac{1}{2}e_1 + e_5 - \frac{3}{2}e_8\right)^T\\
&\quad - 54\left(\frac{1}{3}e_1 - e_5 + 4e_8 - 20e_{11}\right) Z_1 \left(\frac{1}{3}e_1 - e_5 + 4e_8 - 20e_{11}\right)^T\\
&\quad - \frac{2}{9}\left(\frac{1}{2}e_1 + e_{15} + 2e_5 + 8e_{14} - \frac{15}{2}e_8\right) Z_1 \left(\frac{1}{2}e_1 + e_{15} + 2e_5 + 8e_{14} - \frac{15}{2}e_8\right)^T\\
&\quad - 2(e_2 - e_5) Z_2 (e_2 - e_5)^T - 16\left(\frac{1}{2}e_2 - 2e_5 + \frac{3}{2}e_8\right) Z_2 \left(\frac{1}{2}e_2 - 2e_5 + \frac{3}{2}e_8\right)^T\\
&\quad - 54\left(\frac{1}{3}e_2 - 3e_5 + 6e_8 - 20e_{11}\right) Z_2 \left(\frac{1}{3}e_2 - 3e_5 + 6e_8 - 20e_{11}\right)^T\\
&\quad - \frac{2}{9}\left(\frac{1}{2}e_2 + e_{15} - 5e_5 - 8e_{14} + \frac{15}{2}e_8\right) Z_2 \left(\frac{1}{2}e_2 + e_{15} - 5e_5 - 8e_{14} + \frac{15}{2}e_8\right)^T,\\
\Xi_5 &= e_s \left(\frac{h_{12}^2}{2} Z_3 + \frac{h_{12}^2}{2} Z_4\right) e_s^T - 2(e_2 - e_6) Z_3 (e_2 - e_6)^T\\
&\quad - 16\left(\frac{1}{2}e_2 + e_6 - \frac{3}{2}e_9\right) Z_3 \left(\frac{1}{2}e_2 + e_6 - \frac{3}{2}e_9\right)^T\\
&\quad - 54\left(\frac{1}{3}e_2 - e_6 + 4e_9 - 20e_{12}\right) Z_3 \left(\frac{1}{3}e_2 - e_6 + 4e_9 - 20e_{12}\right)^T\\
&\quad - 2(e_3 - e_7) Z_3 (e_3 - e_7)^T - 16\left(\frac{1}{2}e_3 + e_7 - \frac{3}{2}e_{10}\right) Z_3 \left(\frac{1}{2}e_3 + e_7 - \frac{3}{2}e_{10}\right)^T\\
&\quad - 54\left(\frac{1}{3}e_3 - e_7 + 4e_{10} - 20e_{13}\right) Z_3 \left(\frac{1}{3}e_3 - e_7 + 4e_{10} - 20e_{13}\right)^T\\
&\quad - 2(e_3 - e_6) Z_4 (e_3 - e_6)^T - 16\left(\frac{1}{2}e_3 - 2e_6 + \frac{3}{2}e_9\right) Z_4 \left(\frac{1}{2}e_3 - 2e_6 + \frac{3}{2}e_9\right)^T\\
&\quad - 54\left(\frac{1}{3}e_3 - 3e_6 + 6e_9 - 20e_{12}\right) Z_4 \left(\frac{1}{3}e_3 - 3e_6 + 6e_9 - 20e_{12}\right)^T\\
&\quad - 2(e_4 - e_7) Z_4 (e_4 - e_7)^T - 16\left(\frac{1}{2}e_4 - 2e_7 + \frac{3}{2}e_{10}\right) Z_4 \left(\frac{1}{2}e_4 - 2e_7 + \frac{3}{2}e_{10}\right)^T\\
&\quad - 54\left(\frac{1}{3}e_4 - 3e_7 + 6e_{10} - 20e_{13}\right) Z_4 \left(\frac{1}{3}e_4 - 3e_7 + 6e_{10} - 20e_{13}\right)^T,\\
\Phi &= -\Pi_5 R_{24} \Pi_5^T,
\end{aligned}
$$

$$\Xi(h(t)) = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 + \Phi.$$

Theorem 3.1. *For given h_1 and h_2 satisfying $0 < h_1 \le h_2$, system (33) is asymptotically stable for $h_1 \le h(t) \le h_2$, if there are positive define matrices $P \in R^{5n\times 5n}$, $Q_1 \in R^{n\times n}$, $Q_2 \in R^{n\times n}$, $Q_3 \in R^{n\times n}$, $R_1 \in R^{n\times n}$, $R_2 \in R^{n\times n}$, $R_3 \in R^{n\times n}$, $Z_1 \in R^{n\times n}$, $Z_2 \in R^{n\times n}$, $Z_3 \in R^{n\times n}$, $Z_4 \in R^{n\times n}$, and any matrix $S \in R^{3n\times 3n}$ such that the following linear matrix inequalities (LMIs) hold*

$$\Xi(h_1) < 0, \quad \Xi(h_2) < 0, \quad R_{25} \ge 0. \tag{34}$$

Proof: Choose Lyapunov-Krasovskii functional candidate as follows

$$V(t) = \sum_{j=1}^{5} V_j(t), \tag{35}$$

where

$$\begin{aligned}
V_1(t) &= \alpha^T(t)P\alpha(t),\\
V_2(t) &= \int_{t-h_1}^{t} x^T(s)Q_1x(s)ds + \int_{t-h_2}^{t-h_1} x^T(s)Q_2x(s)ds + \int_{t-\frac{h_1}{2}}^{t} x^T(s)Q_3x(s)ds,\\
V_3(t) &= h_1\int_{-h_1}^{0}\int_{t+s}^{t} \dot{x}^T(u)R_1\dot{x}(u)duds + h_{12}\int_{-h_2}^{-h_1}\int_{t+s}^{t} \dot{x}^T(u)R_2\dot{x}(u)duds\\
&\quad + \int_{-\frac{h_1}{2}}^{0}\int_{t+s}^{t} x^T(u)R_3x(u)duds, \qquad (36)\\
V_4(t) &= \int_{-h_1}^{0}\int_{s}^{0}\int_{t+v}^{t} \dot{x}^T(u)Z_1\dot{x}(u)dudvds + \int_{-h_1}^{0}\int_{-h_1}^{s}\int_{t+v}^{t} \dot{x}^T(u)Z_2\dot{x}(u)dudvds,\\
V_5(t) &= \int_{-h_2}^{-h_1}\int_{s}^{-h_1}\int_{t+v}^{t} \dot{x}^T(u)Z_3\dot{x}(u)dudvds + \int_{-h_2}^{-h_1}\int_{-h_2}^{s}\int_{t+v}^{t} \dot{x}^T(u)Z_4\dot{x}(u)dudvds.
\end{aligned}$$

Taking the derivative of Lyapunov-Krasovskii functional $V(t)$ along the trajectory of system (33) yields

$$\dot{V}_1(t) = \xi^T(t)sym\left\{\Pi_1(h(t))P\Pi_2^T\right\}\xi(t) = \xi^T(t)\Xi_1(h(t))\xi(t), \tag{37}$$

$$\dot{V}_2(t) = \xi^T(t)\Xi_2\xi(t), \tag{38}$$

and

$$\begin{aligned}
\dot{V}_3(t) = {} & h_1^2\dot{x}^T(t)R_1\dot{x}(t) - h_1\int_{t-h_1}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds + h_{12}^2\dot{x}^T(t)R_2\dot{x}(t)\\
& - h_{12}\int_{t-h_2}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)ds + \frac{h_1}{2}x^T(t)R_3x(t) - \int_{t-\frac{h_1}{2}}^{t} x^T(s)R_3x(s)ds.
\end{aligned} \tag{39}$$

Using Lemma 2.2, we have

$$\begin{aligned}
&-h_1\int_{t-h_1}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds\\
&\le -[x(t) - x(t-h_1)]^T R_1[x(t) - x(t-h_1)] - 3\Big[x(t) + x(t-h_1)\\
&\quad - \frac{2}{h_1}\int_{t-h_1}^{t} x(u)du\Big]^T R_1\Big[x(t) + x(t-h_1) - \frac{2}{h_1}\int_{t-h_1}^{t} x(u)du\Big]
\end{aligned}$$

$$
\begin{aligned}
&-5\Big[x(t)-x(t-h_1)+\frac{6}{h_1}\int_{t-h_1}^{t}x(u)du-\frac{12}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\Big]^T R_1\Big[x(t)\\
&-x(t-h_1)+\frac{6}{h_1}\int_{t-h_1}^{t}x(u)du-\frac{12}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\Big] \qquad (40)\\
&-7\Big[x(t)+x(t-h_1)-\frac{12}{h_1}\int_{t-h_1}^{t}x(u)du+\frac{60}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\\
&-\frac{120}{h_1^3}\int_{-h_1}^{0}\int_{v}^{0}\int_{t+s}^{t}x(u)dudsdv\Big]^T R_1\Big[x(t)+x(t-h_1)-\frac{12}{h_1}\int_{t-h_1}^{t}x(u)du\\
&+\frac{60}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds-\frac{120}{h_1^3}\int_{-h_1}^{0}\int_{v}^{0}\int_{t+s}^{t}x(u)dudsdv\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&-h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(u)R_2\dot{x}(u)du\\
&=-h_{12}\int_{t-h(t)}^{t-h_1}\dot{x}^T(u)R_2\dot{x}(u)du-h_{12}\int_{t-h_2}^{t-h(t)}\dot{x}^T(u)R_2\dot{x}(u)du\\
&\le-\frac{h_{12}}{h(t)-h_1}\Big\{[x(t-h_1)-x(t-h(t))]^T R_2[x(t-h_1)-x(t-h(t))]\\
&+3\Big[x(t-h_1)+x(t-h(t))-\frac{2}{h(t)-h_1}\int_{t-h(t)}^{t-h_1}x(u)du\Big]^T R_2\Big[x(t-h_1)\\
&+x(t-h(t))-\frac{2}{h(t)-h_1}\int_{t-h(t)}^{t-h_1}x(u)du\Big]\\
&+5\Big[x(t-h_1)-x(t-h(t))+\frac{6}{h(t)-h_1}\int_{t-h(t)}^{t-h_1}x(u)du\\
&-\frac{12}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+v}^{t-h_1}x(u)dudv\Big]^T R_2\Big[x(t-h_1)-x(t-h(t))\\
&+\frac{6}{h(t)-h_1}\int_{t-h(t)}^{t-h_1}x(u)du-\frac{12}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+v}^{t-h_1}x(u)dudv\Big]\Big\} \qquad (41)\\
&-\frac{h_{12}}{h_2-h(t)}\Big\{[x(t-h(t))-x(t-h_2)]^T R_2[x(t-h(t))-x(t-h_2)]\\
&+3\Big[x(t-h_2)+x(t-h(t))-\frac{2}{h_2-h(t)}\int_{t-h_2}^{t-h(t)}x(u)du\Big]^T R_2\Big[x(t-h_2)\\
&+x(t-h(t))-\frac{2}{h_2-h(t)}\int_{t-h_2}^{t-h(t)}x(u)du\Big]\\
&+5\Big[x(t-h(t))-x(t-h_2)+\frac{6}{h_2-h(t)}\int_{t-h_2}^{t-h(t)}x(u)du\\
&-\frac{12}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+v}^{t-h(t)}x(u)dudv\Big]^T R_2\Big[x(t-h(t))-x(t-h_2)\\
&+\frac{6}{h_2-h(t)}\int_{t-h_2}^{t-h(t)}x(u)du-\frac{12}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+v}^{t-h(t)}x(u)dudv\Big]\Big\}.
\end{aligned}
$$

Using the Jensen inequality, we have

$$-\int_{t-\frac{h_1}{2}}^{t} x^T(s)R_3x(s)ds \le -\frac{2}{h_1}\bigg(\int_{t-\frac{h_1}{2}}^{t} x(s)ds\bigg)^T R_3\bigg(\int_{t-\frac{h_1}{2}}^{t} x(s)ds\bigg). \tag{42}$$

So

$$\dot{V}_3(t) \le \xi^T\bigg\{\Xi_3 - \frac{h_{12}}{h(t)-h_1}\Pi_3R_{21}\Pi_3^T - \frac{h_{12}}{h_2-h(t)}\Pi_4R_{21}\Pi_4^T\bigg\}\xi(t). \tag{43}$$

Calculate $\dot{V}_4(t)$ as follows

$$\begin{aligned}\dot{V}_4(t) =& \frac{h_1^2}{2}\dot{x}^T(t)Z_1\dot{x}(t) - \int_{-h_1}^{0}\int_{t+s}^{t}\dot{x}^T(u)Z_1\dot{x}(u)duds\\ &+\frac{h_1^2}{2}\dot{x}^T(t)Z_2\dot{x}(t) - \int_{-h_1}^{0}\int_{t-h_1}^{t+s}\dot{x}^T(u)Z_2\dot{x}(u)duds.\end{aligned} \tag{44}$$

Using Corollary 2.2, one has

$$\begin{aligned}&-\int_{-h_1}^{0}\int_{t+s}^{t}\dot{x}^T(u)Z_1\dot{x}(u)duds\\ \le& -2\bigg[x(t) - \frac{1}{h_1}\int_{t-h_1}^{t}x(u)du\bigg]^T Z_1\bigg[x(t) - \frac{1}{h_1}\int_{t-h_1}^{t}x(u)du\bigg]\\ &-16\bigg[\frac{1}{2}x(t) + \frac{1}{h_1}\int_{t-h_1}^{t}x(u)du - \frac{3}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\bigg]^T Z_1\bigg[\frac{1}{2}x(t)\\ &+\frac{1}{h_1}\int_{t-h_1}^{t}x(u)du - \frac{3}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\bigg]\\ &-54\bigg[\frac{1}{3}x(t) - \frac{1}{h_1}\int_{t-h_1}^{t}x(u)du + \frac{8}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\\ &-\frac{20}{h_1^3}\int_{-h_1}^{0}\int_{v}^{0}\int_{t+s}^{t}x(u)dudsdv\bigg]^T Z_1\bigg[\frac{1}{3}x(t) - \frac{1}{h_1}\int_{t-h_1}^{t}x(u)du\\ &+\frac{8}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds - \frac{20}{h_1^3}\int_{-h_1}^{0}\int_{v}^{0}\int_{t+s}^{t}x(u)dudsdv\bigg]\\ &-\frac{2}{9}\bigg[-2x(t) + x\Big(t-\frac{h_1}{2}\Big) - \frac{3}{h_1}\int_{t-h_1}^{t}x(u)du + \frac{8}{h_1}\int_{t-\frac{h_1}{2}}^{t}x(u)du\\ &-5\Big(-\frac{1}{2}x(t) - \frac{1}{h_1}\int_{t-h_1}^{t}x(u)du + \frac{3}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\Big)\bigg]^T Z_1\bigg[-2x(t)\\ &+x\Big(t-\frac{h_1}{2}\Big) - \frac{3}{h_1}\int_{t-h_1}^{t}x(u)du + \frac{8}{h_1}\int_{t-\frac{h_1}{2}}^{t}x(u)du\\ &-5\Big(-\frac{1}{2}x(t) - \frac{1}{h_1}\int_{t-h_1}^{t}x(u)du + \frac{3}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\Big)\bigg].\end{aligned} \tag{45}$$

According to Corollary 2.4, we have

$$\begin{aligned}&-\int_{-h_1}^{0}\int_{t-h_1}^{t+s}\dot{x}^T(u)Z_2\dot{x}(u)duds\\ \le& -2\bigg[x(t-h_1) - \frac{1}{h_1}\int_{t-h_1}^{t}x(u)du\bigg]^T Z_2\bigg[x(t-h_1) - \frac{1}{h_1}\int_{t-h_1}^{t}x(u)du\bigg]\end{aligned}$$

$$
\begin{aligned}
&-16\left[\frac{1}{2}x(t-h_1)-\frac{2}{h_1}\int_{t-h_1}^{t}x(u)du+\frac{3}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\right]^T Z_2\left[\frac{1}{2}x(t-h_1)\right.\\
&\left.-\frac{2}{h_1}\int_{t-h_1}^{t}x(u)du+\frac{3}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\right]\\
&-54\left[\frac{1}{3}x(t-h_1)-\frac{3}{h_1}\int_{t-h_1}^{t}x(u)du+\frac{12}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\right.\\
&\left.-\frac{20}{h_1^3}\int_{-h_1}^{0}\int_{v}^{0}\int_{t+s}^{t}x(u)dudsdv\right]^T Z_2\left[\frac{1}{3}x(t-h_1)-\frac{3}{h_1}\int_{t-h_1}^{t}x(u)du\right. \qquad (46)\\
&\left.+\frac{12}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds-\frac{20}{h_1^3}\int_{-h_1}^{0}\int_{v}^{0}\int_{t+s}^{t}x(u)dudsdv\right]\\
&-\frac{2}{9}\left[2x(t-h_1)-x\left(t-\frac{h_1}{2}\right)-\frac{5}{h_1}\int_{t-h_1}^{t}x(u)du+\frac{8}{h_1}\int_{t-\frac{h_1}{2}}^{t}x(u)du\right.\\
&\left.+5\left(-\frac{1}{2}x(t-h_1)+\frac{2}{h_1}\int_{t-h_1}^{t}x(u)du-\frac{3}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\right)\right]^T Z_2\left[2x(t-h_1)\right.\\
&-x\left(t-\frac{h_1}{2}\right)-\frac{5}{h_1}\int_{t-h_1}^{t}x(u)du+\frac{8}{h_1}\int_{t-\frac{h_1}{2}}^{t}x(u)du\\
&\left.+5\left(-\frac{1}{2}x(t-h_1)+\frac{2}{h_1}\int_{t-h_1}^{t}x(u)du-\frac{3}{h_1^2}\int_{-h_1}^{0}\int_{t+s}^{t}x(u)duds\right)\right].
\end{aligned}
$$

So

$$
\dot V_4(t)\le \xi^T(t)\Xi_4\xi(t). \qquad (47)
$$

For $\dot V_5(t)$, we have

$$
\begin{aligned}
\dot V_5(t)&=\frac{h_{12}^2}{2}\dot x^T(t)Z_3\dot x(t)-\int_{-h_2}^{-h_1}\int_{t+s}^{t-h_1}\dot x^T(u)Z_3\dot x(u)duds\\
&\quad+\frac{h_{12}^2}{2}\dot x^T(t)Z_4\dot x(t)-\int_{-h_2}^{-h_1}\int_{t-h_2}^{t+s}\dot x^T(u)Z_4\dot x(u)duds\\
&=\frac{h_{12}^2}{2}\dot x^T(t)Z_3\dot x(t)-\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1}\dot x^T(u)Z_3\dot x(u)duds\\
&\quad-\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)}\dot x^T(u)Z_3\dot x(u)duds \qquad (48)\\
&\quad-(h_2-h(t))\int_{t-h(t)}^{t-h_1}\dot x^T(s)Z_3\dot x(s)ds+\frac{h_{12}^2}{2}\dot x^T(t)Z_4\dot x(t)\\
&\quad-\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+s}\dot x^T(u)Z_4\dot x(u)duds-\int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+s}\dot x^T(u)Z_4\dot x(u)duds\\
&\quad-(h(t)-h_1)\int_{t-h_2}^{t-h(t)}\dot x^T(s)Z_4\dot x(s)ds.
\end{aligned}
$$

Corollary 2.2 implies

$$
-\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1}\dot x^T(u)Z_3\dot x(u)duds-\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)}\dot x^T(u)Z_3\dot x(u)duds
$$

$$
\begin{aligned}
\leq & -2\Big[x(t-h_1) - \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du\Big]^T Z_3 \Big[x(t-h_1) \\
& - \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du\Big] - 16\Big[\frac{1}{2}x(t-h_1) + \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du \\
& - \frac{3}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1} x(u)duds\Big]^T Z_3 \Big[\frac{1}{2}x(t-h_1) \\
& + \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du - \frac{3}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1} x(u)duds\Big] \\
& - 54\Big[\frac{1}{3}x(t-h_1) - \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du + \frac{8}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1} x(u)duds \\
& - \frac{20}{(h(t)-h_1)^3}\int_{-h(t)}^{-h_1}\int_{v}^{-h_1}\int_{t+s}^{t-h_1} x(u)dudsdv\Big]^T Z_3 \Big[\frac{1}{3}x(t-h_1) \\
& - \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du + \frac{8}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1} x(u)duds \\
& - \frac{20}{(h(t)-h_1)^3}\int_{-h(t)}^{-h_1}\int_{v}^{-h_1}\int_{t+s}^{t-h_1} x(u)dudsdv\Big] \\
& - 2\Big[x(t-h(t)) - \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du\Big]^T Z_3 \Big[x(t-h(t)) \\
& - \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du\Big] - 16\Big[\frac{1}{2}x(t-h(t)) + \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du \\
& - \frac{3}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)duds\Big]^T Z_3 \Big[\frac{1}{2}x(t-h(t)) \\
& + \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du - \frac{3}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)duds\Big] \\
& - 54\Big[\frac{1}{3}x(t-h(t)) - \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du \\
& + \frac{8}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)duds \\
& - \frac{20}{(h_2-h(t))^3}\int_{-h_2}^{-h(t)}\int_{v}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)dudsdv\Big]^T Z_3 \Big[\frac{1}{3}x(t-h(t)) \\
& - \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du + \frac{8}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)duds \\
& - \frac{20}{(h_2-h(t))^3}\int_{-h_2}^{-h(t)}\int_{v}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)dudsdv\Big].
\end{aligned} \tag{49}
$$

Using Corollary 2.4, we get

$$
\begin{aligned}
& -\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+s} \dot{x}^T(u)Z_4\dot{x}(u)duds - \int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+s} \dot{x}^T(u)Z_4\dot{x}(u)duds \\
\leq & -2\Big[x(t-h(t)) - \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du\Big]^T Z_4 \Big[x(t-h(t))
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du\Big] - 16\Big[\frac{1}{2}x(t-h(t)) - \frac{2}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du \\
&+ \frac{3}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1} x(u)duds\Big]^T Z_4\Big[\frac{1}{2}x(t-h(t)) \\
&- \frac{2}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du + \frac{3}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1} x(u)duds\Big] \\
&- 54\Big[\frac{1}{3}x(t-h(t)) - \frac{3}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du + \frac{12}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1} x(u)duds \\
&- \frac{20}{(h(t)-h_1)^3}\int_{-h(t)}^{-h_1}\int_{v}^{-h_1}\int_{t+s}^{t-h_1} x(u)dudsdv\Big]^T Z_4\Big[\frac{1}{3}x(t-h(t)) \\
&- \frac{3}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(u)du + \frac{12}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+s}^{t-h_1} x(u)duds \\
&- \frac{20}{(h(t)-h_1)^3}\int_{-h(t)}^{-h_1}\int_{v}^{-h_1}\int_{t+s}^{t-h_1} x(u)dudsdv\Big] \qquad (50) \\
&- 2\Big[x(t-h_2) - \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du\Big]^T Z_4\Big[x(t-h_2) \\
&- \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du\Big] - 16\Big[\frac{1}{2}x(t-h_2) - \frac{2}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du \\
&+ \frac{3}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)duds\Big]^T Z_4\Big[\frac{1}{2}x(t-h_2) \\
&- \frac{2}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du + \frac{3}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)duds\Big] \\
&- 54\Big[\frac{1}{3}x(t-h_2) - \frac{3}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du + \frac{12}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)duds \\
&- \frac{20}{(h_2-h(t))^3}\int_{-h_2}^{-h(t)}\int_{v}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)dudsdv\Big]^T Z_4\Big[\frac{1}{3}x(t-h_2) \\
&- \frac{3}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(u)du + \frac{12}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)duds \\
&- \frac{20}{(h_2-h(t))^3}\int_{-h_2}^{-h(t)}\int_{v}^{-h(t)}\int_{t+s}^{t-h(t)} x(u)dudsdv\Big],
\end{aligned}
$$

again

$$
\begin{aligned}
&- (h_2-h(t))\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)Z_3\dot{x}(s)ds - (h(t)-h_1)\int_{t-h_2}^{t-h(t)} \dot{x}^T(s)Z_4\dot{x}(s)ds \\
&\le -\xi^T(t)\Big[\Big(\frac{h_{12}}{h(t)-h_1}-1\Big)\Pi_3 Z_{31}\Pi_3^T - \Big(\frac{h_{12}}{h_2-h(t)}-1\Big)\Pi_4 Z_{41}\Pi_4^T\Big]\xi(t). \qquad (51)
\end{aligned}
$$

Thus,

$$
\dot{V}_5(t) \le \xi^T(t)\Big[\Xi_5 - \Big(\frac{h_{12}}{h(t)-h_1}-1\Big)\Pi_3 Z_{31}\Pi_3^T - \Big(\frac{h_{12}}{h_2-h(t)}-1\Big)\Pi_4 Z_{41}\Pi_4^T\Big]\xi(t). \qquad (52)
$$

Using Lemma 2.1 gives

$$\begin{aligned}&\xi^T(t)\bigg\{-\frac{h_{12}}{h(t)-h_1}\Pi_3R_{21}\Pi_3^T-\frac{h_{12}}{h_2-h(t)}\Pi_4R_{21}\Pi_4^T\\&-\left(\frac{h_{12}}{h(t)-h_1}-1\right)\Pi_3Z_{31}\Pi_3^T-\left(\frac{h_{12}}{h_2-h(t)}-1\right)\Pi_4Z_{41}\Pi_4^T\bigg\}\xi(t)\le\xi^T(t)\Phi\xi(t).\end{aligned}\tag{53}$$

Combining $\Xi(h_1)<0$ with $\Xi(h_2)<0$ gives $\Xi(h(t))<0$. Therefore,

$$\dot{V}(t)\le\xi^T(t)\Xi(h(t))\xi(t)<0,\quad\forall\xi(t)\neq 0.\tag{54}$$

The proof is completed.

Remark 3.1. *It has been proved that delay partitioning approach and introduction of some triple integral terms in Lyapunov-Krasovskii functional can significantly reduce the conservatism of stability criteria. In this paper, a new Lyapunov-Krasovskii functional with four triple integral terms is constructed. Lyapunov-Krasovskii functional terms* $\int_{t-\frac{h_1}{2}}^t x^T(s)Q_3x(s)ds$ *and* $\int_{-\frac{h_1}{2}}^0\int_{t+s}^t x^T(u)R_2x(u)duds$ *with the information of delay-partition are introduced.*

Remark 3.2. *Integral inequality method is an important method to reduce the conservativeness of stability criteria for the delayed systems. A novel double integral inequality (16) is obtained in Corollary 2.2, which can provide tighter bound than most of existing inequalities. Vectors* $x\left(t-\frac{h_1}{2}\right)$ *and* $\int_{t-\frac{h_1}{2}}^t x^T(u)du$ *will result from the use of this integral inequality, so* $\int_{t-\frac{h_1}{2}}^t x^T(s)Q_3x(s)ds$ *and* $\int_{-\frac{h_1}{2}}^0\int_{t+s}^t x^T(u)R_2x(u)duds$ *are introduced in the* $V_2(t)$ *and* $V_3(t)$*, respectively. Furthermore,* $\int_{t-\frac{h_1}{2}}^t x^T(s)Q_3x(s)ds$ *and* $\int_{-\frac{h_1}{2}}^0\int_{t+s}^t x^T(u)R_2x(u)duds$ *mean the application of the delay decomposition approach, which can reduce the conservatism of stability criterion.*

TABLE 1. Maximum bound h_2 with different h_1 (Example 4.1)

Method	0	0.3	0.7	1	2	Number of DVs
[7]	1.10	1.28	1.64	1.94	2.94	$3.5n^2+2.5n$
[27]	1.14	1.33	1.73	2.03	3.03	$21.5n^2+7.5n$
[28]	1.35	1.64	2.02	2.31	3.31	$9.5n^2+5.5n$
[31]	1.64	2.13	2.70	2.96	3.63	$21n^2+6n$
Corollary 3 [32]	1.62	2.12	2.74	2.97	3.61	$31.5n^2+5.5n$
Theorem 1 [32]	1.64	2.14	2.75	3.02	3.63	$32.5n^2+6.5n$
Theorem 3.1	1.64	2.14	2.75	3.02	3.63	$29n^2+5n$

4. Numerical Examples.

In this section, we give several numerical examples to show the effectiveness of the proposed approach.

Example 4.1. *Consider the following linear system as*

$$\dot{x}(t)=\begin{pmatrix}0.0 & 1.0\\-10.0 & -1.0\end{pmatrix}x(t)+\begin{pmatrix}0.0 & 0.1\\0.1 & 0.2\end{pmatrix}x(t-h(t)).$$

A detailed comparison to various methods from the existing literature is provided in Table 1. From Table 1, it is easy to see that the proposed approach in this paper can provide larger upper bounds h_2 *than those methods in [7,27,28,31] for various* $h_1>0$*,* *which means that our stability result has less conservativeness than the stability criteria obtained in [7,27,28,31]. It is well known that the choice of an appropriate Lyapunov-Krasovskii functional is the key point for deriving the delay-dependent stability criterion. If the Lyapunov-Krasovskii functional is reduced, the corresponding results might become more conservative. Although reciprocally convex approach is utilized in [7], the Lyapunov-Krasovskii functional only contains double-integral terms. Compared with [7], two triple-integral terms are added in the Lyapunov-Krasovskii functional in [27]. This may explain that the results in [27] are less conservative than those in [7]. A new Lyapunov-Krasovskii functional containing partially alternative four triple integral terms is proposed in [28]. The manipulation of the proposed Lyapunov-Krasovskii functional via the Jensen inequality lemma produces the combinations of the rational functions of the time-varying delay with first-order denominators. A less conservative asymptotical stability criterion is derived. However, the Jensen inequality entails some inherent conservatism. In [31], several auxiliary function-based integral inequalities have been developed , which encompass the Jensen inequality and the Wirtinger-based integral inequality. Two general integral inequalities are developed via orthogonal polynomials, respectively, in the upper and lower forms in [32]. These two integral inequalities imply the Jensen inequality, the Wirtinger-based inequality, the Bessel-Legendre inequality, the Wirtinger-based double integral inequality, and the auxiliary function-based integral inequalities. Based on these two integral inequalities, several less conservative stability conditions are obtained for systems with time-varying delay in [32]. In this paper, two different orthogonal systems are considered. One is the orthogonal system of bivariate polynomials. The other is the orthogonal system of bivariate functions, which need not be continuous. Based on the twice orthogonal approximation of vector, some new double integral inequalities are obtained, which can produce more tighter bounds than what the double integral inequalities in [31] produce. These improved double integral inequalities are used to handle the derivatives of triple integral terms in Lyapunov-Krasovskii functional. Therefore, Theorem 3.1 in this paper is effective in reducing the conservatism in [7,27,28,31]. From Table 1, it is easy to see that Theorem 3.1 in this paper provides a larger upper bound h_2 than Corollary 3 in [32] does for the same h_1. Theorem 3.1 also provides the same upper bound h_2 as Theorem 1 in [32] for the same h_1. Corollary 3 in [32] requires $31.5n^2 + 5.5n$ decision variables. Theorem 1 in [32] requires $32.5n^2 + 6.5n$ decision variables. However, Theorem 3.1 in this paper only needs $29n^2 + 5n$ decision variables. Therefore, our stability criterion is with less computation complexity.*

Example 4.2. *Consider the following linear system as*

$$\dot{x}(t) = \begin{pmatrix} 0.0 & 1.0 \\ -100.0 & -1.0 \end{pmatrix} x(t) + \begin{pmatrix} 0.0 & 0.1 \\ 0.1 & 0.2 \end{pmatrix} x(t - h(t)).$$

For various h_1, the comparison of allowable maximum delay bounds h_2 obtained by Theorem 3.1 in this paper with those derived in [7,27,28,31] is conducted in Table 2. Since the improved double integral inequalities obtained in Corollary 2.2 and Corollary 2.4 are fully used, the maximum allowable upper bounds h_2 obtained by Theorem 3.1 in this paper are larger than those derived by the methods in [7,27,28,31] for various $h_1 > 0$. From Table 2, it can be confirmed that Theorem 3.1 in this paper enhances the feasible region of the stability criteria in [7,27,28,31]. Theorem 3.1 in this paper is also less conservative than Corollary 3 in [32]. However, Corollary 3 in [32] requires more decision variables (DVs). It is worth mentioning that Theorem 3.1 in this paper requires less decision variables (DVs) than Theorem 1 in [32] while leading to the same result.

TABLE 2. Maximum bound h_2 with different h_1 (Example 4.2)

Method	0	0.1	0.2	0.3	0.4	0.5	Number of DVs
[7]	0.08	0.10	×	×	×	×	$3.5n^2 + 2.5n$
[27]	0.08	0.10	×	×	×	×	$21.5n^2 + 7.5n$
[28]	0.08	0.10	×	×	×	×	$9.5n^2 + 5.5n$
[31]	0.08	0.11	0.22	0.50	0.58	0.59	$21n^2 + 6n$
Corollary 3 [32]	0.08	0.11	0.26	0.58	0.65	×	$31.5n^2 + 5.5n$
Theorem 1 [32]	0.08	0.11	0.26	0.58	0.66	×	$32.5n^2 + 6.5n$
Theorem 3.1	0.08	0.11	0.26	0.58	0.66	0.68	$29n^2 + 5n$

Example 4.3. *Consider the following linear system with time-varying delay as*

$$\dot{x}(t) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} x(t - h(t)).$$

For $h_1 = 0.4$, the allowable maximum delay bound h_2 that guarantees the asymptotical stability of the above linear system obtained by Theorem 3.1 in this paper is 0.7.

Let $h(t) = 0.55 - 0.15 * \sin t$, $x(0) = [-1, 1]^T$, then $0.4 \le h(t) \le 0.7$. So this linear system is asymptotically stable by Theorem 3.1. The state responses of the linear system in Example 4.3 are shown in Figure 1, from which we can see the state of delayed linear system is asymptotically stable.

Let $x(0) = [-1, 1]^T$. When $h(t) \equiv 0.4, 0.5, 0.6, 0.7$, we depict the responses $x_1(t)$ and $x_2(t)$ of the delayed linear system in Example 4.3 in Figure 2 and Figure 3, respectively.

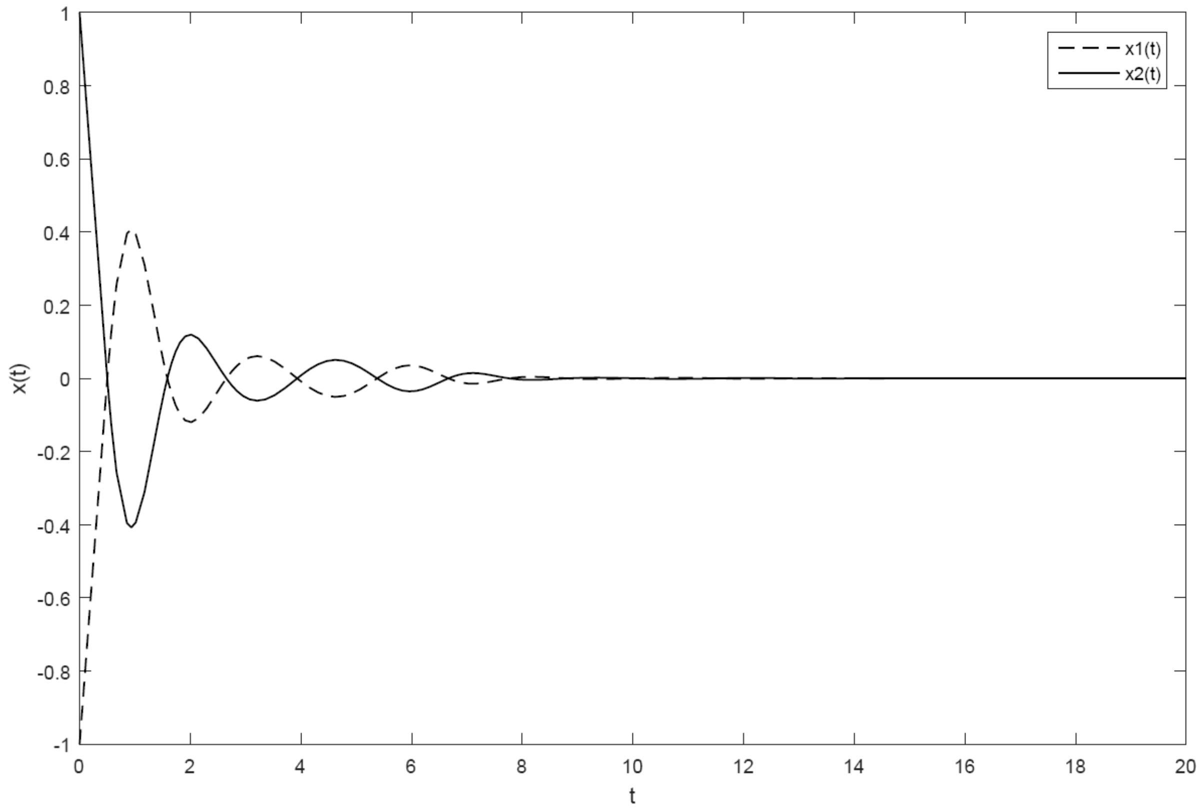

FIGURE 1. State trajectories of the system in Example 4.3

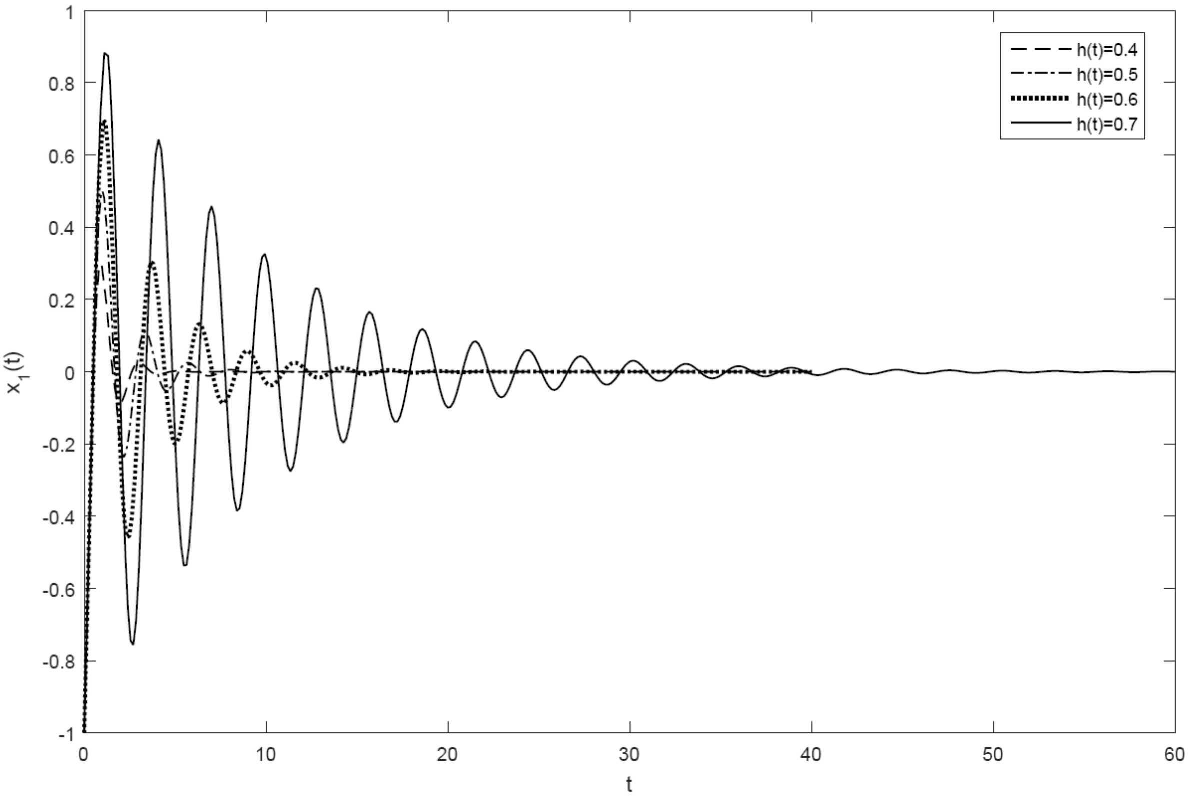

FIGURE 2. State trajectories $x_1(t)$ of the system in Example 4.3

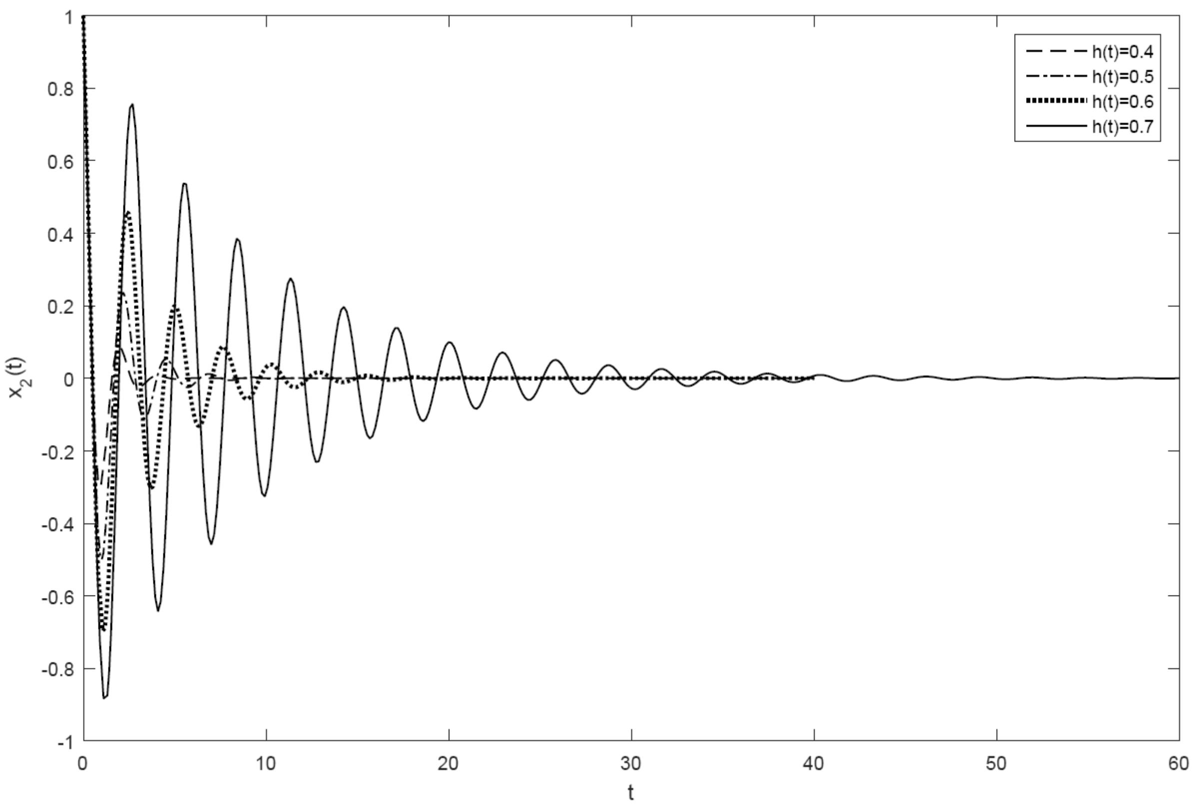

FIGURE 3. State trajectories $x_2(t)$ of the system in Example 4.3

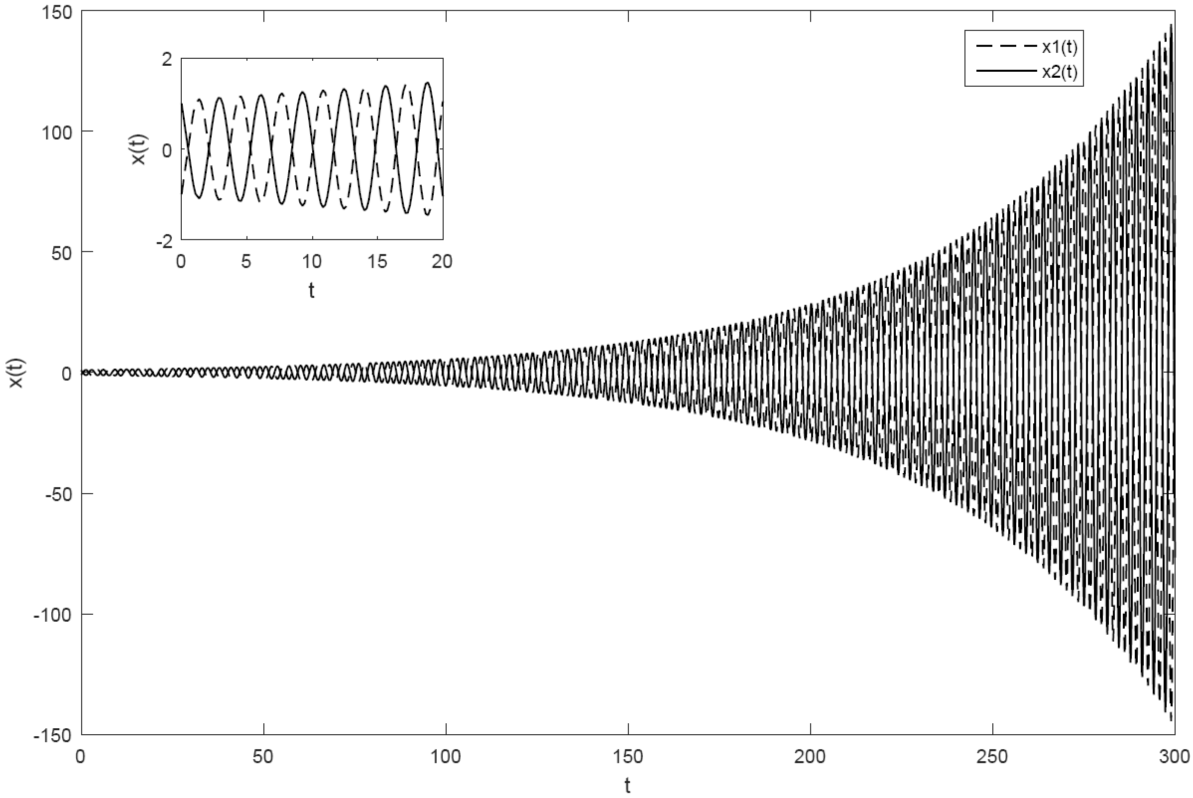

FIGURE 4. State trajectories of the system in Example 4.3 when $h(t) \equiv 0.8$

From Figure 2, we find that when the delay increases from 0.4 to 0.7, $x_1(t)$ *converges more slowly. Similar results can be obtained for* $x_2(t)$*.*

By setting $x(0) = [-1, 1]^T$*, Figure 4 shows the state trajectories of the system are divergent when* $h(t) \equiv 0.8$*. In fact, when* $h(t) \geq 0.8$*, the system is unstable by Theorem 3.1.*

5. **Conclusions.** In this paper, the triangulation of a bounded domain in the plane is proposed. Based on the triangulation of a bounded domain, an orthogonal system of bivariate functions is proposed, where the bivariate functions need not be polynomials. They may be discontinuous functions. Based on the twice orthogonal approximation of vector and minimization of the energy function, two novel double integral inequalities are derived. A new Lyapunov-Krasovskii functional with triple integral terms is constructed. Combining the improved double integral inequalities with delay-partitioning approach and the reciprocally convex approach, a less conservative asymptotical stability criterion with less decision variables is derived for system with time-varying delay. Three numerical examples have been given to demonstrate the effectiveness and advantages of the theoretical results. In the future work, the integral inequalities proposed in this paper will be applied to stability analysis of other dynamic systems such as sampled data system, fuzzy neural networks and stochastic systems.

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