

NOVEL CONCEPTS OF DOUBT BIPOLAR FUZZY H-IDEALS OF *BCK/BCI*-ALGEBRAS

ANAS AL-MASARWAH AND ABD GHAFUR AHMAD

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi, Selangor, Malaysia
almasarwah85@gmail.com; ghafur@ukm.edu.my

Received February 2018; revised June 2018

ABSTRACT. *In this paper, we introduce the notion of doubt bipolar fuzzy H-ideals of BCK/BCI-algebras and investigate some interesting properties. We characterize strong doubt positive t -level cut set, strong doubt negative s -level cut set, homomorphism and equivalence relation by considering doubt bipolar fuzzy H-ideals of BCK/BCI-algebras. Characterization theorem of characteristic doubt bipolar fuzzy H-ideals is also discussed. Particularly, the notion of Cartesian product of two doubt bipolar fuzzy H-ideals by using max-min operations is introduced, and some related properties are studied. Ordinary H-ideals are linked with doubt bipolar fuzzy H-ideals by means of doubt level cut set of Cartesian product of two bipolar-valued fuzzy sets.*

Keywords: *BCK/BCI*-algebra, Cartesian product, Doubt bipolar fuzzy H-ideal, Characteristic doubt bipolar fuzzy H-ideal

1. Introduction. The notion of fuzzy set and fuzzy relation on a set was given by Zadeh [1, 2]. Since then, the concept of fuzzy set and fuzzy relation provides a natural framework for generalizing some of the basic notions of algebra, for example, logic, set theory, group theory, semigroup theory, ring theory, semiring theory, hemiring theory, and modules. Also, there are several applications of fuzzy set theory and fuzzy relation in various fields, for example, decision making, artificial intelligence, expert system, computer science, and operation research. Lee [3] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to the interval $[-1, 1]$. In a bipolar-valued fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property.

Imai and Iséki [4] introduced two classes of abstract algebra, namely *BCK*-algebras and *BCI*-algebras. It is known that the class of *BCK*-algebras is a proper subclass of the class of *BCI*-algebras. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1971 by Rosenfeld [5]. Afterwards many researchers had worked on the structures of fuzzy sets in *BCK/BCI*-algebras and in other algebraic structures. Xi [6] applied the concept of fuzzy sets to *BCK*-algebras. Also, Jun [7] and Ahmad [8] applied it to *BCI*-algebras. Khalid and Ahmad [9] introduced fuzzy H-ideals in *BCI*-algebras and studied their properties. Huang [10] gave another definition of fuzzy *BCI*-algebras and some results about it. In 1994, Jun [11] established the definition

of a doubt fuzzy subalgebra and a doubt fuzzy ideal in BCK/BCI -algebras to avoid the confusion created in Huang's definition of fuzzy BCI -algebras [10] and gave some results about it. Zhan and Tan [12, 13] introduced the notion of doubt fuzzy H-ideals in BCK -algebras and studied their properties.

Recently, based on the results of bipolar-valued fuzzy sets, more and more researchers have devoted themselves to studying some bipolar fuzzy algebraic structures. Lee [14] applied the concept of bipolar-valued fuzzy set theory to BCK/BCI -algebras, and introduced the notions of bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI -algebras. Recently, the notion of bipolar-valued fuzzy set theory was applied to BCK/BCI -algebras [15, 16] and other algebraic structures such as semigroups [17], semirings [18], hemirings [19, 20], K -algebras [21], BF -algebras [22, 23], Lie algebras [24] and Lie superalgebras [25]. For the general development of BCK/BCI -algebras, the H-ideal theory plays an important role. Thus, there are a number of works on BCK/BCI -algebras and related algebraic systems but to the best of our knowledge no work is available on novel concepts of doubt bipolar fuzzy H-ideals in BCK/BCI -algebras. For this reason we are motivated to develop these theories for BCK/BCI -algebras.

In this paper, after introductory section, some basic notions are discussed in Section 2. In Section 3, we introduce the notion of doubt bipolar fuzzy H-ideals of BCK/BCI -algebras and investigate some interesting properties. We characterize strong doubt positive t -level cut set, strong doubt negative s -level cut set and homomorphism by considering doubt bipolar fuzzy H-ideals of BCK/BCI -algebras. In Section 4, the characterizations of the maps from the set of doubt bipolar fuzzy H-ideals to the set of H-ideals are investigated through the equivalence relation. In Section 5, we define Cartesian product of doubt bipolar fuzzy H-ideals of BCK/BCI -algebras. We give some interesting results about Cartesian product of doubt bipolar fuzzy H-ideals. In addition, we prove that the Cartesian product of two bipolar-valued fuzzy sets becomes doubt bipolar fuzzy H-ideals if and only if for any $(t, s) \in [0, 1] \times [-1, 0]$, doubt positive t -level cut set and doubt negative s -level cut set are H-ideals of a BCK/BCI -algebra $X \times Y$. At last, conclusions are drawn in Section 6.

2. Preliminaries. We first recall some elementary aspects which are used to present the paper. Throughout this paper, X always denotes a BCK/BCI -algebra without any specifications. A BCK/BCI -algebra is an important class of logical algebras introduced by Imai and Iséki [4, 26] and was extensively investigated by several researchers. This algebra is defined as follows.

By a BCI -algebra we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \in X$:

- 1) $((x * y) * (x * z)) * (z * y) = 0$,
- 2) $(x * (x * y)) * y = 0$,
- 3) $x * x = 0$,
- 4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

If a BCI -algebra X satisfies $0 * x = 0$, then X is called a BCK -algebra. A mapping $f : X \rightarrow X'$ of BCK/BCI -algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. If f is one to one (onto), then f is called monomorphism (epimorphism). A partial ordering \leq on a BCK/BCI -algebra X can be defined by $x \leq y$ if and only if $x * y = 0$. Any BCK/BCI -algebra X satisfies the following axioms for all $x, y, z \in X$:

- 1) $(x * y) * z = (x * z) * y$,
- 2) $x * y \leq x$,
- 3) $(x * y) * z \leq (x * z) * (y * z)$,

4) $x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x$.

Definition 2.1. [6] A non-empty subset S of a BCK/BCI -algebra X is called an ideal of X if

- 1) $0 \in S$,
- 2) $x * y \in S$ and $y \in S$ then $x \in S$ for all $x, y \in X$.

Definition 2.2. [9] A non-empty subset S of a BCK/BCI -algebra X is called an H -ideal of X if

- 1) $0 \in S$,
- 2) $x * (y * z) \in S$ and $y \in S$ then $x * z \in S$ for all $x, y, z \in X$.

We refer the reader to [27, 28, 29] for further information regarding BCK/BCI -algebras.

Definition 2.3. [9] A fuzzy set $A = \{(x, \mu_A(x)) | x \in X\}$ in X is called a fuzzy H -ideal of X if

- 1) $\mu_A(0) \geq \mu_A(x)$,
- 2) $\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$ for all $x, y, z \in X$.

Definition 2.4. [12] A fuzzy set $A = \{(x, \mu_A(x)) | x \in X\}$ in X is called a doubt fuzzy H -ideal of X if

- 1) $\mu_A(0) \leq \mu_A(x)$,
- 2) $\mu_A(x * z) \leq \max\{\mu_A(x * (y * z)), \mu_A(y)\}$ for all $x, y, z \in X$.

The proposed work is done on a bipolar-valued fuzzy set. The formal definition of a bipolar-valued fuzzy set is given below.

Definition 2.5. [3] Let X be a non-empty set. A bipolar-valued fuzzy set A in X is an object having the form

$$A = \{(x, \mu_A^P(x), \mu_A^N(x)) | x \in X\}$$

where $\mu_A^P : X \rightarrow [0, 1]$ and $\mu_A^N : X \rightarrow [-1, 0]$ are mappings.

We use the positive membership degree $\mu_A^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set A , and the negative membership degree $\mu_A^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued fuzzy set A . If $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A . If $\mu_A^P(x) = 0$ and $\mu_A^N(x) \neq 0$, it is the situation that x does not satisfy the property of A but somewhat satisfies the counter property of A . It is possible for an element x to be such that $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X .

For the sake of simplicity, we shall use the symbol $A = (\mu_A^P, \mu_A^N)$ for the bipolar-valued fuzzy set $A = \{(x, \mu_A^P(x), \mu_A^N(x)) | x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

3. Doubt Bipolar Fuzzy H-Ideals. In this section, we introduce a new generalized doubt fuzzy H -ideal of a BCK/BCI -algebra called, doubt bipolar fuzzy H -ideal. We discuss the concepts of strong doubt positive t -level cut set and strong doubt negative s -level cut set of a bipolar fuzzy set in BCK/BCI -algebras and study some of their properties. Also, we investigate the homomorphism preimage and doubt image of doubt bipolar fuzzy H -ideals in BCK/BCI -algebras under some conditions.

Definition 3.1. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of X . Then A is called a doubt bipolar fuzzy H -ideal of X if it satisfies the following conditions for all $x, y, z \in X$:

- 1) $\mu_A^P(0) \leq \mu_A^P(x)$ and $\mu_A^N(0) \geq \mu_A^N(x)$,
- 2) $\mu_A^P(x * z) \leq \max \{ \mu_A^P(x * (y * z)), \mu_A^P(y) \}$,
- 3) $\mu_A^N(x * z) \geq \min \{ \mu_A^N(x * (y * z)), \mu_A^N(y) \}$.

Example 3.1. Let $X = \{0, a, b, c\}$ be a BCK -algebra with the following Cayley table:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set in X defined by:

$$\mu_A^P(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.4, & \text{if } x = a, c \\ 0.3, & \text{if } x = b, \end{cases} \text{ and } \mu_A^N(x) = \begin{cases} -0.1, & \text{if } x = 0 \\ -0.2, & \text{if } x = a, b, c. \end{cases}$$

Then by routine calculation, we know that $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H -ideal of X .

An interesting consequence of doubt bipolar fuzzy H -ideals of BCK/BCI -algebras is the following.

Proposition 3.1. Let A be a nonempty subset of X . A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is defined by

$$\mu_A^P = \begin{cases} n_1, & x \in A \\ n_2, & x \notin A, \end{cases} \text{ and } \mu_A^N = \begin{cases} m_1, & x \in A \\ m_2, & x \notin A, \end{cases}$$

where $0 \leq n_1 \leq n_2 \leq 1$ and $-1 \leq m_2 \leq m_1 \leq 0$ is a doubt bipolar fuzzy H -ideal of X if and only if A is an H -ideal of X .

Proof: The proof is obvious and is omitted. □

The research about the relationships of doubt fuzzy subalgebras and crisp subalgebras by level cut sets is usual but important, as it is a tie which can connect abstract algebraic structures and fuzzy ones. However, now we encounter a significant challenge that the traditional level cut sets are not suitable for the framework of doubt bipolar fuzzy H -ideals of BCK/BCI -algebras because of the characterization of bipolarity. As a consequence, we defined (strong) doubt positive t -level cut set and (strong) doubt negative s -level cut set.

Definition 3.2. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set on X . For any $(t, s) \in [0, 1] \times [-1, 0]$, the sets

$$A_t^P = \{x \in X \mid \mu_A^P(x) \leq t\} \text{ and } {}^s A_t^P = \{x \in X \mid \mu_A^P(x) < t\}$$

are called a doubt positive t -level cut set and a strong doubt positive s -level cut set of $A = (\mu_A^P, \mu_A^N)$, respectively.

The sets

$$A_s^N = \{x \in X \mid \mu_A^N(x) \geq s\} \text{ and } {}^s A_s^N = \{x \in X \mid \mu_A^N(x) > s\}$$

are called a doubt negative t -level cut set and a strong doubt negative s -level cut set of $A = (\mu_A^P, \mu_A^N)$, respectively.

The sets

$$A_{(t,s)} = \{x \in X \mid \mu_A^P(x) \leq t, \mu_A^N(x) \geq s\} \text{ and } {}^s A_{(t,s)} = \{x \in X \mid \mu_A^P(x) < t, \mu_A^N(x) > s\}$$

are called a doubt (t, s) -level cut set and a strong doubt (t, s) -level cut set of $A = (\mu_A^P, \mu_A^N)$, respectively. Note that

$$A_{(t,s)} = A_t^P \cap A_s^N \text{ and } {}^sA_{(t,s)} = {}^sA_t^P \cap {}^sA_s^N.$$

For every $\gamma \in [0, 1]$, the sets $A_\gamma = A_\gamma^P \cap A_{-\gamma}^N$ and ${}^sA_\gamma \cap {}^sA_{-\gamma}^N$ are called a doubt γ -level cut set and a strong doubt γ -level cut set of $A = (\mu_A^P, \mu_A^N)$, respectively.

Proposition 3.2. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of X and let $x \in X$. Then $\mu_A^P(x) = t$, $\mu_A^N(x) = s$ if and only if $x \in A_t^P$, $x \notin A_u^P$ and $x \in A_s^N$, $x \notin A_v^N$ for all $u < t$, $v > s$.

Proof: Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of X and let $x \in X$. Assume $\mu_A^P(x) = t$ so that $x \in A_t^P$. If possible, let $x \in A_u^P$ for $u < t$. Then $\mu_A^P(x) \leq u < t$. This contradicts the fact that $\mu_A^P(x) = t$, concluding that $x \notin A_u^P$ for all $u < t$. Also, assume $\mu_A^N(x) = s$, then $x \in A_s^N$. If possible, let $x \in A_v^N$ for $v > s$. Then $\mu_A^N(x) \geq v > s$. This contradicts the fact that $\mu_A^N(x) = s$, concluding that $x \notin A_v^N$ for all $v > s$.

Conversely, let $x \in A_t^P$, $x \notin A_u^P$ for all $u < t$. Now, $x \in A_t^P \Rightarrow \mu_A^P(x) \leq t$. Since $x \notin A_u^P$ for all $u < t$. Therefore, $\mu_A^P(x) = t$. Also, let $x \in A_s^N$, $x \notin A_v^N$ for all $v > s$. Now, $x \in A_s^N \Rightarrow \mu_A^N(x) \geq s$. Since $x \notin A_v^N$ for all $v > s$, $\mu_A^N(x) = s$. \square

From Definition 3.2, in the next two theorems, we can easily obtain the relation between doubt bipolar fuzzy H-ideals and H-ideals of BCK/BCI -algebras.

Theorem 3.1. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set over X and let $(t, s) \in [0, 1] \times [-1, 0]$. If $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X , then the nonempty strong doubt (t, s) -level cut set of A is an H-ideal of X .

Proof: Assume that ${}^sA_{(t,s)} \neq \emptyset$ for $(t, s) \in [0, 1] \times [-1, 0]$. Clearly, $0 \in {}^sA_{(t,s)}$. Let $x, y, z \in X$ such that $x * (y * z) \in {}^sA_{(t,s)}$ and $y \in {}^sA_{(t,s)}$. Then $\mu_A^P(x * y) < t$, $\mu_A^N(x * y) > s$, $\mu_A^P(y) < t$ and $\mu_A^N(y) > s$. It follows from Definition 3.1 that

$$\mu_A^P(x * z) \leq \max \{ \mu_A^P(x * y), \mu_A^P(y) \} < \max \{ t, t \} = t$$

and

$$\mu_A^N(x * z) \geq \min \{ \mu_A^N(x * y), \mu_A^N(y) \} > \min \{ s, s \} = s,$$

so $x * z \in {}^sA_{(t,s)}$. Therefore, ${}^sA_{(t,s)}$ is an H-ideal of X . \square

Theorem 3.2. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set over X and assume that $\emptyset \neq {}^sA_t^P$ and $\emptyset \neq {}^sA_s^N$ are H-ideals of X for all $(t, s) \in [0, 1] \times [-1, 0]$. Then $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X .

Proof: Assume that ${}^sA_t^P \neq \emptyset$ and ${}^sA_s^N \neq \emptyset$ are H-ideals of X for all $t \in [0, 1]$ and $s \in [-1, 0]$. Suppose that there exists $a \in X$ such that $\mu_A^P(0) > \mu_A^P(a)$ and $\mu_A^N(0) < \mu_A^N(a)$. Taking

$$t_o = \frac{1}{2} (\mu_A^P(0) + \mu_A^P(a)), \quad s_o = \frac{1}{2} (\mu_A^N(0) + \mu_A^N(a)),$$

implies that $\mu_A^P(a) < t_o < \mu_A^P(0)$ and $\mu_A^N(a) > s_o > \mu_A^N(0)$. This shows that $0 \notin {}^sA_t^P$ and $0 \notin {}^sA_s^N$, which leads to a contradiction. Therefore, $\mu_A^P(0) \leq \mu_A^P(x)$ and $\mu_A^N(0) \geq \mu_A^N(x)$ for all $x \in X$. Now, suppose that there are $a, b, c \in X$ such that $\mu_A^P(a * c) > \max \{ \mu_A^P(a * (b * c)), \mu_A^P(b) \}$. Then by taking

$$t_1 = \frac{1}{2} (\mu_A^P(a * c) + \max \{ \mu_A^P(a * (b * c)), \mu_A^P(b) \}),$$

we have $\max \{ \mu_A^P(a * (b * c)), \mu_A^P(b) \} < t_1 < \mu_A^P(a * c)$. Hence, $a * c \notin {}^sA_{t_1}^P$, $a * (b * c) \in {}^sA_{t_1}^P$ and $b \in {}^sA_{t_1}^P$, that is ${}^sA_{t_1}^P$ is not H-ideal of X , which is a contradiction. Therefore, $\mu_A^P(x * y) \leq \max \{ \mu_A^P(x * (y * z)), \mu_A^P(y) \}$ for all $x, y, z \in X$.

Finally, assume that $p, q, r \in X$ such that $\mu_A^N(p * r) < \min \{ \mu_A^N(p * (q * r)), \mu_A^N(q) \}$. Taking

$$s_1 = \frac{1}{2} (\mu_A^N(p * r) + \min \{ \mu_A^N(p * (q * r)), \mu_A^N(q) \}),$$

then $\mu_A^N(p * r) < s_1 < \min \{ \mu_A^N(p * (q * r)), \mu_A^N(q) \}$. Therefore, $p * (q * r) \in {}^sA_{s_1}^N$ and $q \in {}^sA_{s_1}^N$ but $p * r \notin {}^sA_{s_1}^N$. Again it is a contradiction. Thus, $\mu_A^N(x * z) \geq \min \{ \mu_A^N(x * (y * z)), \mu_A^N(y) \}$ for all $x, y, z \in X$. Hence, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X . \square

If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set in a BCK/BCI-algebra X and if f is a self mapping of X , we define mappings $\mu_A^P[f] : X \rightarrow [0, 1]$ by $\mu_A^P[f](x) = \mu_A^P(f(x))$ and $\mu_A^N[f] : X \rightarrow [-1, 0]$ by $\mu_A^N[f](x) = \mu_A^N(f(x))$ for all $x \in X$, respectively.

Theorem 3.3. *If $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X and if f is an epimorphism of X , then $(\mu_A^P[f], \mu_A^N[f])$ is a doubt bipolar fuzzy H-ideal of X .*

Proof: Let $\mu_A^P[f](0) = \mu_A^P(f(0)) = \mu_A^P(0) \leq \mu_A^P(y)$ and $\mu_A^N[f](0) = \mu_A^N(f(0)) = \mu_A^N(0) \geq \mu_A^N(y)$ for any $y \in X$. Since f is an epimorphism of X , then there exists $x \in X$ such that $f(x) = y$. Thus, $\mu_A^P[f](0) \leq \mu_A^P(f(x)) = \mu_A^P[f](x)$ and $\mu_A^N[f](0) \geq \mu_A^N(f(x)) = \mu_A^N[f](x)$. As y is an arbitrary element of X , the above result is true for any $x \in X$. Moreover, for any $x, y, z \in X$, we have

$$\begin{aligned} \mu_A^P[f](x * z) &= \mu_A^P(f(x * z)) = \mu_A^P(f(x) * f(z)) \\ &\leq \max \{ \mu_A^P(f(x) * (f(y) * f(z))), \mu_A^P(f(y)) \} \\ &= \max \{ \mu_A^P(f(x) * f(y * z)), \mu_A^P(f(y)) \} \\ &= \max \{ \mu_A^P(f(x * (y * z))), \mu_A^P(f(y)) \} \\ &= \max \{ \mu_A^P[f](x * (y * z)), \mu_A^P[f](y) \} \end{aligned}$$

and

$$\begin{aligned} \mu_A^N[f](x * z) &= \mu_A^N(f(x * z)) = \mu_A^N(f(x) * f(z)) \\ &\geq \min \{ \mu_A^N(f(x) * (f(y) * f(z))), \mu_A^N(f(y)) \} \\ &= \min \{ \mu_A^N(f(x) * f(y * z)), \mu_A^N(f(y)) \} \\ &= \min \{ \mu_A^N(f(x * (y * z))), \mu_A^N(f(y)) \} \\ &= \min \{ \mu_A^N[f](x * (y * z)), \mu_A^N[f](y) \}. \end{aligned}$$

Hence, $(\mu_A^P[f], \mu_A^N[f])$ is a doubt bipolar fuzzy H-ideal of X . \square

Definition 3.3. *An H-ideal C of a BCK/BCI-algebra X is said to be characteristic if $f(C) = C$ for all $f \in \text{Aut}(X)$, where $\text{Aut}(X)$ is the set of all automorphisms of a BCK/BCI-algebra X . A doubt bipolar fuzzy H-ideal $A = (\mu_A^P, \mu_A^N)$ of X is called characteristic if*

$$\mu_A^P(f(x)) = \mu_A^P(x) \text{ and } \mu_A^N(f(x)) = \mu_A^N(x)$$

for all $x \in X$ and $f \in \text{Aut}(X)$.

For characteristic doubt bipolar fuzzy H-ideals in BCK/BCI-algebras, we have the following theorem.

Theorem 3.4. *A doubt bipolar fuzzy H-ideal is characteristic if and only if each of its doubt level cut set is a characteristic H-ideal.*

Proof: Let a doubt bipolar fuzzy H-ideal $A = (\mu_A^P, \mu_A^N)$ be characteristic, $t \in Im(\mu_A^P)$, $f \in Aut(X)$ and let $x \in A_t^P$. Then $\mu_A^P(f(x)) = \mu_A^P(x) \leq t$, which means that $f(x) \in A_t^P$. Thus, $f(A_t^P) \subseteq A_t^P$. Since for each $x \in A_t^P$ there exists $y \in X$ such that $f(y) = x$, we have $\mu_A^P(y) = \mu_A^P(f(y)) = \mu_A^P(x) \leq t$, we conclude $y \in A_t^P$. Consequently, $x = f(y) \in f(A_t^P)$. Therefore, $A_t^P \subseteq f(A_t^P)$; hence, $A_t^P = f(A_t^P)$. Similarly $A_s^N = f(A_s^N)$. This proves that A_t^P and A_s^N are characteristic.

Conversely, if all doubt level cuts of $A = (\mu_A^P, \mu_A^N)$ are characteristic H-ideals of X , then for $x \in X$, $f \in Aut(X)$ and $\mu_A^P(x) = t > s = \mu_A^N(x)$, by Proposition 3.2, we have $x \in A_t^P$, $x \notin A_s^N$ and $x \in A_s^N$, $x \notin A_t^P$. Thus, $f(x) \in f(A_t^P) = A_t^P$ and $f(x) \in f(A_s^N) = A_s^N$, i.e., $\mu_A^P(f(x)) \leq t$ and $\mu_A^N(f(x)) \geq s$. For $\mu_A^P(f(x)) = t_1 < t$, $\mu_A^N(f(x)) = s_1 > s$, we have $f(x) \in A_{t_1}^P = f(A_{t_1}^P)$, $f(x) \in A_{s_1}^N = f(A_{s_1}^N)$, so $x \in A_{t_1}^P$, $x \in A_{s_1}^N$. This is a contradiction. Thus, $\mu_A^P(f(x)) = \mu_A^P(x)$ and $\mu_A^N(f(x)) = \mu_A^N(x)$. Hence, $A = (\mu_A^P, \mu_A^N)$ is characteristic. \square

Now, we discuss the properties of the preimage and doubt image of doubt bipolar fuzzy H-ideals by homomorphism of BCK/BCI-algebras.

Definition 3.4. If f is a self mapping of a BCK/BCI-algebra X and $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set in $f(X)$, then the bipolar fuzzy set $A_f = (\mu_{A_f}^P, \mu_{A_f}^N)$ in X defined by $\mu_{A_f}^P = \mu_A^P \circ f$ and $\mu_{A_f}^N = \mu_A^N \circ f$ (i.e., $\mu_{A_f}^P(x) = \mu_A^P(f(x))$ and $\mu_{A_f}^N(x) = \mu_A^N(f(x))$ for all $x \in X$) is called the preimage of A under f .

Theorem 3.5. An onto homomorphic preimage of a doubt bipolar fuzzy H-ideal is a doubt bipolar fuzzy H-ideal.

Proof: Let $f : X \rightarrow X'$ be an onto homomorphism of BCK/BCI-algebras, $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of X' and $A_f = (\mu_{A_f}^P, \mu_{A_f}^N)$ be preimage of A under f . For any $x' \in X'$ there exists $x \in X$ such that $f(x) = x'$. We have

$$\begin{aligned} \mu_{A_f}^P(0) &= \mu_A^P(f(0)) = \mu_A^P(0') \leq \mu_A^P(x') = \mu_A^P(f(x)) = \mu_{A_f}^P(x), \text{ and} \\ \mu_{A_f}^N(0) &= \mu_A^N(f(0)) = \mu_A^N(0') \geq \mu_A^N(x') = \mu_A^N(f(x)) = \mu_{A_f}^N(x), \end{aligned}$$

where $0'$ is the zero element of X' . So $\mu_{A_f}^P(0) \leq \mu_{A_f}^P(x)$ and $\mu_{A_f}^N(0) \geq \mu_{A_f}^N(x)$ for all $x \in X$. Moreover, for any $x, y, z \in X$, we have

$$\begin{aligned} \mu_{A_f}^P(x * z) &= \mu_A^P(f(x * z)) = \mu_A^P(f(x) * f(z)) \\ &\leq \max \{ \mu_A^P(f(x) * (f(y) * f(z))), \mu_A^P(f(y)) \} \\ &= \max \{ \mu_A^P(f(x) * f(y * z)), \mu_A^P(f(y)) \} \\ &= \max \{ \mu_A^P(f(x * (y * z))), \mu_A^P(f(y)) \} \\ &= \max \{ \mu_{A_f}^P(x * (y * z)), \mu_{A_f}^P(y) \} \end{aligned}$$

and

$$\begin{aligned} \mu_{A_f}^N(x * z) &= \mu_A^N(f(x * z)) = \mu_A^N(f(x) * f(z)) \\ &\geq \min \{ \mu_A^N(f(x) * (f(y) * f(z))), \mu_A^N(f(y)) \} \\ &= \min \{ \mu_A^N(f(x) * f(y * z)), \mu_A^N(f(y)) \} \\ &= \min \{ \mu_A^N(f(x * (y * z))), \mu_A^N(f(y)) \} \\ &= \min \{ \mu_{A_f}^N(x * (y * z)), \mu_{A_f}^N(y) \}. \end{aligned}$$

Hence, $A_f = (\mu_{A_f}^P, \mu_{A_f}^N)$ is a doubt bipolar fuzzy H-ideal of X . \square

Definition 3.5. If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set in X and f is a mapping on X , then the doubt image of A under f , denoted by $f(A)$, is a bipolar fuzzy set of $f(X)$ defined by

$$f(A) = (f_{\inf}(\mu_A^P), f_{\sup}(\mu_A^N)),$$

where

$$\begin{aligned} f_{\inf}(\mu_A^P)(y) &= \inf_{x \in f^{-1}(y)} (\mu_A^P(x)) \text{ and} \\ f_{\sup}(\mu_A^N)(y) &= \sup_{x \in f^{-1}(y)} (\mu_A^N(x)), \text{ for each } y \in f(X). \end{aligned}$$

Definition 3.6. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy set in a BCK/BCI-algebra X , then we say that μ_A^P has inf property if for any subset S of X there exists $s_1 \in S$ such that $\mu_A^P(s_1) = \inf_{r \in S} \mu_A^P(r)$ and we say that μ_A^N has sup property if for any subset T of X there exists $t_1 \in T$ such that $\mu_A^N(t_1) = \sup_{k \in T} \mu_A^N(k)$.

Theorem 3.6. An onto homomorphic doubt image of a doubt bipolar fuzzy H -ideal with μ_A^P has inf property and μ_A^N has sup property is a doubt bipolar fuzzy H -ideal.

Proof: Let $f : X \rightarrow X'$ be an onto homomorphism of BCK/BCI-algebras, $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X with sup and inf properties and $f(A)$ be the doubt image of A under f . Since $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H -ideal of X , we have $\mu_A^P(0) \leq \mu_A^P(x)$ and $\mu_A^N(0) \geq \mu_A^N(x)$ for all $x \in X$. Note that $0 \in f^{-1}(0')$ where 0 and $0'$ are the zeros of X and X' , respectively. Thus, $f_{\inf}(\mu_A^P)(0') = \inf_{t \in f^{-1}(0')} (\mu_A^P(t)) = \mu_A^P(0) \leq \mu_A^P(x)$ and $f_{\sup}(\mu_A^N)(0') = \sup_{t \in f^{-1}(0')} (\mu_A^N(t)) = \mu_A^N(0) \geq \mu_A^N(x)$ for all $x \in X$. This implies that $f_{\inf}(\mu_A^P)(0') \leq \inf_{t \in f^{-1}(x')} (\mu_A^P(t)) = f_{\inf}(\mu_A^P)(x')$ and $f_{\sup}(\mu_A^N)(0') \geq \sup_{t \in f^{-1}(x')} (\mu_A^N(t)) = f_{\sup}(\mu_A^N)(x')$ for any $x' \in X'$.

Moreover, for any $x', y', z' \in X'$, let $x_o \in f^{-1}(x')$, $y_o \in f^{-1}(y')$ and $z_o \in f^{-1}(z')$ such that

$$\begin{aligned} \mu_A^P(x_o * z_o) &= \inf_{t \in f^{-1}(x' * z')} (\mu_A^P(t)), \\ \mu_A^N(x_o * z_o) &= \sup_{t \in f^{-1}(x' * z')} (\mu_A^N(t)), \\ \mu_A^P(y_o) &= \inf_{t \in f^{-1}(y')} (\mu_A^P(t)), \\ \mu_A^N(y_o) &= \sup_{t \in f^{-1}(y')} (\mu_A^N(t)), \end{aligned}$$

$$\begin{aligned} \mu_A^P(x_o * (y_o * z_o)) &= f_{\inf}(\mu_A^P)(f(x_o * (y_o * z_o))) \\ &= f_{\inf}(\mu_A^P)(x' * (y' * z')) \\ &= \inf_{(x_o * (y_o * z_o)) \in f^{-1}(x' * (y' * z'))} \mu_A^P(x_o * (y_o * z_o)) \\ &= \inf_{t \in f^{-1}(x' * (y' * z'))} \mu_A^P(t) \end{aligned}$$

and

$$\begin{aligned} \mu_A^N(x_o * (y_o * z_o)) &= f_{\sup}(\mu_A^N)(f(x_o * (y_o * z_o))) \\ &= f_{\sup}(\mu_A^N)(x' * (y' * z')) \end{aligned}$$

$$\begin{aligned}
 &= \sup_{(x_o*(y_o*z_o)) \in f^{-1}(x'*(y'*z'))} \mu_A^N(x_o * (y_o * z_o)) \\
 &= \sup_{t \in f^{-1}(x'*(y'*z'))} \mu_A^N(t).
 \end{aligned}$$

Then,

$$\begin{aligned}
 f_{\inf}(\mu_A^P)(x' * z') &= \inf_{t \in f^{-1}(x' * z')} \mu_A^P(t) = \mu_A^P(x_o * z_o) \\
 &\leq \max \{ \mu_A^P(x_o * (y_o * z_o)), \mu_A^P(y_o) \} \\
 &= \max \left\{ \inf_{t \in f^{-1}(x'*(y'*z'))} \mu_A^P(t), \inf_{t \in f^{-1}(y')} \mu_A^P(t) \right\} \\
 &= \max \{ f_{\inf}(\mu_A^P)(x' * (y' * z')), f_{\inf}(\mu_A^P)(y') \}
 \end{aligned}$$

and

$$\begin{aligned}
 f_{\sup}(\mu_A^N)(x' * z') &= \sup_{t \in f^{-1}(x' * z')} \mu_A^N(t) = \mu_A^N(x_o * z_o) \\
 &\geq \min \{ \mu_A^P(x_o * (y_o * z_o)), \mu_A^N(y_o) \} \\
 &= \min \left\{ \sup_{t \in f^{-1}(x'*(y'*z'))} \mu_A^N(t), \sup_{t \in f^{-1}(y')} \mu_A^N(t) \right\} \\
 &= \min \{ f_{\sup}(\mu_A^N)(x' * (y' * z')), f_{\sup}(\mu_A^N)(y') \}.
 \end{aligned}$$

Hence, the image $f(A)$ of A under f is a doubt bipolar fuzzy H-ideal of X' . □

4. Equivalence Relations on Doubt Bipolar Fuzzy H-Ideals. In this section, we continue to discuss the relation between doubt bipolar fuzzy H-ideals and H-ideals of BCK/BCI -algebras by another means: equivalence relations.

Let $DBFHI(X)$ be the collection of all doubt bipolar fuzzy H-ideals of X . For any $(t, s) \in [0, 1] \times [-1, 0]$, define two binary relations P^t and N^s on $DBFHI(X)$ as follows:

$$(A, B) \in P^t \Leftrightarrow A_t^P = B_t^P \text{ and } (A, B) \in N^s \Leftrightarrow A_s^N = B_s^N,$$

for all $A = (\mu_A^P, \mu_A^N) \in DBFHI(X)$, $B = (\mu_B^P, \mu_B^N) \in DBFHI(X)$. It is easy to know P^t and N^s are equivalence relations on $DBFHI(X)$.

For all $A = (\mu_A^P, \mu_A^N) \in DBFHI(X)$, let $[A]_{P^t}$ (resp., $[A]_{N^s}$) be the equivalence class of $A = (\mu_A^P, \mu_A^N)$ modular P^t (resp., N^s). That is, $DBFHI(X)/P^t = \{[A]_{P^t} | A = (\mu_A^P, \mu_A^N) \in DBFHI(X)\}$ (resp., $DBFHI(X)/N^s = \{[A]_{N^s} | A = (\mu_A^P, \mu_A^N) \in DBFHI(X)\}$). Let $HI(X)$ be the family of all H-ideals of X . Define maps

$$\begin{aligned}
 g_t &: DBFHI(X) \rightarrow HI(X) \cup \{\phi\}, A \rightarrow A_t^P, \\
 h_s &: DBFHI(X) \rightarrow HI(X) \cup \{\phi\}, A \rightarrow A_s^N,
 \end{aligned}$$

for all $A = (\mu_A^P, \mu_A^N) \in DBFHI(X)$. Then g_t and h_s are clearly well-defined.

In the light of the definition of equivalence relations on $DBFHI(X)$, we can obtain the following properties.

Theorem 4.1. *The maps g_t and h_s are surjective for any $(t, s) \in (0, 1) \times (-1, 0)$.*

Proof: Clearly, a bipolar fuzzy set $\mathbf{1} = (\mu_1^P, \mu_1^N)$ is a doubt bipolar fuzzy H-ideal of X , where $\mu_1^P = 1$ and $\mu_1^N = -1$ for all $x \in X$. Then we have $g_t(\mathbf{1}) = \mathbf{1}_t^P = \{x \in X | \mu_1^P(x) \leq t\} = \phi$ and $h_s(\mathbf{1}) = \mathbf{1}_s^N = \{x \in X | \mu_1^N(x) \geq s\} = \phi$. For any M in $HI(X)$, consider a bipolar fuzzy set $M_{\sim} = (\mu_{M_{\sim}}^P, \mu_{M_{\sim}}^N)$ in X , where

$$\mu_{M_{\sim}}^P : X \rightarrow [0, 1], \quad \mu_{M_{\sim}}^P(x) = \begin{cases} 0, & x \in M \\ 1, & x \notin M \end{cases}$$

and

$$\mu_{M_\sim}^N : X \rightarrow [-1, 0], \quad \mu_{M_\sim}^N(x) = \begin{cases} 0, & x \in M \\ -1, & x \notin M. \end{cases}$$

By Proposition 3.1, $M_\sim = (\mu_{M_\sim}^P, \mu_{M_\sim}^N) \in DBFHI(X)$. Now, we get

$$g_t(M_\sim) = \mu_{M_\sim t}^P = \{x \in X \mid \mu_{M_\sim}^P(x) \leq t\} = \{x \in X \mid \mu_{M_\sim}^P(x) = 0\} = M$$

and

$$h_s(M_\sim) = \mu_{M_\sim s}^N = \{x \in X \mid \mu_{M_\sim}^N(x) \geq s\} = \{x \in X \mid \mu_{M_\sim}^N(x) = 0\} = M.$$

Therefore, g_t and h_s are surjective. □

Example 4.1. Consider a BCK-algebra $X = \{0, a, b, c\}$ which is given in Example 3.1 and we consider H-ideal $M = \{0\} \in HI(X)$. Define a bipolar fuzzy set $M_\sim = (\mu_{M_\sim}^P, \mu_{M_\sim}^N)$ in X as follows:

$$\mu_{M_\sim}^P(x) = \begin{cases} 0, & x \in M \\ 1, & x \notin M, \end{cases} \quad \text{and} \quad \mu_{M_\sim}^N(x) = \begin{cases} 0, & x \in M \\ -1, & x \notin M. \end{cases}$$

Clearly, $M_\sim = (\mu_{M_\sim}^P, \mu_{M_\sim}^N) \in DBFHI(X)$. We have $g_t(M_\sim) = \mu_{M_\sim t}^P = \{x \in X \mid \mu_{M_\sim}^P(x) \leq t\} = \{0\}$ and $h_s(M_\sim) = \mu_{M_\sim s}^N = \{x \in X \mid \mu_{M_\sim}^N(x) \geq s\} = \{0\}$. Thus, the maps g_t and h_s are surjective for all $(t, s) \in (0, 1) \times (-1, 0)$.

Now, a natural question arises here: are there any relationships between the quotient sets and the set of all H-ideals of X . In the following, we will concentrate on giving the answers.

Theorem 4.2. The quotient sets $DBFHI(X)/P^t$ and $DBFHI(X)/N^s$ are equipotent to $HI(X) \cup \{\phi\}$ for all $(t, s) \in (0, 1) \times (-1, 0)$.

Proof: For all $(t, s) \in [0, 1] \times [-1, 0]$ and $A = (\mu_A^P, \mu_A^N) \in DBFHI(X)$. Let

$$g_t^* : DBFHI(X)/P^t \rightarrow HI(X) \cup \{\phi\}, A \rightarrow A_t^P$$

and

$$h_s^* : DBFHI(X)/N^s \rightarrow HI(X) \cup \{\phi\}, A \rightarrow A_s^N$$

be defined by

$$g_t^*([A]_{P^t}) = g_t(A) \text{ and } h_s^*([A]_{N^s}) = h_s(A), \text{ respectively.}$$

For any $A = (\mu_A^P, \mu_A^N)$, $B = (\mu_B^P, \mu_B^N) \in DBFHI(X)$, if $A_t^P = B_t^P$ and $A_s^N = B_s^N$, then $(A, B) \in P^t$ and $(A, B) \in N^s$, which means $[A]_{P^t} = [B]_{P^t}$ and $[A]_{N^s} = [B]_{N^s}$. Thus, g_t^* and h_s^* are injective.

For any nonempty M in $HI(X)$, consider the doubt bipolar fuzzy H-ideal $M_\sim = (\mu_{M_\sim}^P, \mu_{M_\sim}^N)$ which is given in the proof in Theorem 4.1, we have $g_t^*([M_\sim]_{P^t}) = g_t(M_\sim) = M_{\sim t}^P = M$ and $h_s^*([M_\sim]_{N^s}) = h_s(M_\sim) = M_{\sim s}^N = M$.

For any doubt bipolar fuzzy H-ideal $\mathbf{1} = (\mu_{\mathbf{1}}^P, \mu_{\mathbf{1}}^N)$ of X , we have $g_t^*([\mathbf{1}]_{P^t}) = g_t(\mathbf{1}) = \mathbf{1}_t^P = \{x \in X \mid \mu_{\mathbf{1}}^P(x) \leq t\} = \phi$ and $h_s^*([\mathbf{1}]_{N^s}) = h_s(\mathbf{1}) = \mathbf{1}_s^N = \{x \in X \mid \mu_{\mathbf{1}}^N(x) \geq s\} = \phi$. Hence, g_t^* and h_s^* are surjective. This completes the proof. □

For any $0 < \gamma < 1$, we define another relation X^γ on $DBFHI(X)$ as follows:

$$(A, B) \in X^\gamma \Leftrightarrow A_\gamma = B_\gamma,$$

where $A_\gamma = A_\gamma^P \cap A_{-\gamma}^N$ and $B_\gamma = B_\gamma^P \cap B_{-\gamma}^N$. Then the relation X^γ is also an equivalence relation on $DBFHI(X)$.

Theorem 4.3. Let $0 < \gamma < 1$. Then the map $\Phi_\gamma : DBFHI(X) \rightarrow I(X) \cup \{\phi\}$ defined by $\Phi_\gamma(A) = A_\gamma$ is surjective.

Proof: Let $0 < \gamma < 1$, for $\mathbf{1} = (\mu_1^P, \mu_1^N) \in DBFHI(X)$, we have $\Phi_\gamma(\mathbf{1}) = \mathbf{1}_\gamma^P \cap \mathbf{1}_{-\gamma}^N = \phi$. For any nonempty M in $DBFHI(X)$, considering a doubt bipolar fuzzy H-ideal $M_\sim = (\mu_{M_\sim}^P, \mu_{M_\sim}^N)$ which is given in the proof of Theorem 4.1, we obtain $\Phi_\gamma(M_\sim) = M_{\sim\gamma} = M_{\sim\gamma}^P \cap M_{\sim-\gamma}^N = \{x \in X \mid \mu_{M_\sim}^P(x) \leq \gamma\} \cap \{x \in X \mid \mu_{M_\sim}^N(x) \geq -\gamma\} = M$. Therefore, Φ_γ is surjective. \square

Theorem 4.4. *Let $0 < \gamma < 1$. Then the quotient set $DBFHI(X)/X^\gamma$ is equipotent to $I(X) \cup \{\phi\}$.*

Proof: Suppose that $0 < \gamma < 1$ and $\Phi_\gamma^*: DBFHI(X)/X^\gamma \rightarrow I(X) \cup \{\phi\}$ is a map defined by $\Phi_\gamma^*([A]_{X^\gamma}) = \Phi_\gamma(A)$ for all $[A]_{X^\gamma} \in DBFHI(X)/X^\gamma$. Let $\Phi_\gamma^*([A]_{X^\gamma}) = \Phi_\gamma^*([B]_{X^\gamma})$ for every $[A]_{X^\gamma}$ and $[B]_{X^\gamma} \in DBFHI(X)/X^\gamma$, then $\phi_\gamma(A) = \phi_\gamma(B)$, i.e., $A_\gamma = B_\gamma$. It implies that $(A, B) \in X^\gamma$. Thus, $[A]_{X^\gamma} = [B]_{X^\gamma}$ and Φ_γ^* is injective.

Moreover, for any nonempty M in $I(X)$, we consider the doubt bipolar fuzzy H-ideal $M_\sim = (\mu_{M_\sim}^P, \mu_{M_\sim}^N)$ which is given in the proof of Theorem 4.1, then we have $\Phi_\gamma^*([M_\sim]_{X^\gamma}) = \Phi_\gamma(M_\sim) = M_{\sim\gamma} = \{x \in X \mid \mu_{M_\sim}^P(x) \leq \gamma\} \cap \{x \in X \mid \mu_{M_\sim}^N(x) \geq -\gamma\} = M$. On the other hand, for $\mathbf{1} = (\mu_1^P, \mu_1^N) \in DBFHI(X)$, we have $\Phi_\gamma^*([\mathbf{1}]_{X^\gamma}) = \phi_\gamma(\mathbf{1}) = \mathbf{1}_\gamma^P \cap \mathbf{1}_{-\gamma}^N = M$. Therefore, Φ_γ^* is surjective. This completes the proof. \square

5. Cartesian Product of Doubt Bipolar Fuzzy H-Ideals. In this section, we give the definition of the Cartesian product of two doubt bipolar fuzzy H-ideals of two BCK/BCI -algebras X and Y . Also, we provide some of their properties. In what follows, X and Y are BCK/BCI -algebras, so we use $(X \times Y; *, (0, 0))$ to denote a BCK/BCI -algebra unless otherwise specified. For the sake of brevity, we call $X \times Y$ a BCK/BCI -algebras.

Definition 5.1. *Let $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ be two BCK/BCI -algebras. The Cartesian product of $X \times Y$ is defined to be the set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$. In $X \times Y$ we define the product $*_{X \times Y}$ as follows:*

$$(x, y) *_{X \times Y} (u, v) = (x *_X u, y *_Y v) \text{ for all } (x, y), (u, v) \in X \times Y.$$

One can easily verify that the Cartesian product of two BCK/BCI -algebras is again a BCK/BCI -algebra. Now, we write the following definition.

Definition 5.2. *Let X be a BCK/BCI -algebra and let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be two doubt bipolar fuzzy H-ideals of X . The Cartesian product of A and B is defined by $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$, where $\mu_{A \times B}^P : X \times X \rightarrow [0, 1]$ is given by*

$$\mu_{A \times B}^P(x, y) = \max \{ \mu_A^P(x), \mu_A^P(y) \}$$

and $\mu_{A \times B}^N : X \times X \rightarrow [-1, 0]$ is given by

$$\mu_{A \times B}^N(x, y) = \min \{ \mu_A^N(x), \mu_A^N(y) \}$$

for all $(x, y) \in X \times X$.

In the following, we extend the above definition to the Cartesian product of doubt bipolar fuzzy H-ideals of any BCK/BCI -algebras X and Y .

Definition 5.3. *Let X and Y be two BCK/BCI -algebras and let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be two doubt bipolar fuzzy H-ideals of X and Y , respectively. The Cartesian product of A and B is defined by $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$, where $\mu_{A \times B}^P : X \times Y \rightarrow [0, 1]$ is given by*

$$\mu_{A \times B}^P(x, y) = \max \{ \mu_A^P(x), \mu_A^P(y) \}$$

and $\mu_{A \times B}^N : X \times Y \rightarrow [-1, 0]$ is given by

$$\mu_{A \times B}^N(x, y) = \min \{ \mu_A^N(x), \mu_A^N(y) \}$$

for all $(x, y) \in X \times Y$.

Definition 5.4. A bipolar fuzzy set $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ of a BCK/BCI-algebra $X \times Y$ is called a doubt bipolar fuzzy H-ideal of $X \times Y$ if it satisfies the following conditions for all $(x, y), (u, v), (w, z) \in X \times Y$:

- 1) $\mu_{A \times B}^P(0, 0) \leq \mu_{A \times B}^P(x, y)$ and $\mu_{A \times B}^N(0, 0) \geq \mu_{A \times B}^N(x, y)$,
- 2) $\mu_{A \times B}^P((x, y) * (w, z)) \leq \max \{ \mu_{A \times B}^P((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^P(u, v) \}$,
- 3) $\mu_{A \times B}^N((x, y) * (w, z)) \geq \min \{ \mu_{A \times B}^N((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^N(u, v) \}$.

Example 5.1. Consider a BCK-algebra $X = \{0, a, b, c\}$ and a doubt bipolar fuzzy H-ideal $A = (\mu_A^P, \mu_A^N)$ of X which are given in Example 3.1. Define a doubt bipolar fuzzy H-ideal $B = (\mu_B^P, \mu_B^N)$ in X as follows:

$$\mu_B^P(x) = \begin{cases} 0.3, & \text{if } x = 0 \\ 0.5, & \text{if } x = a \\ 0.6, & \text{if } x = b, c, \end{cases} \quad \text{and} \quad \mu_B^N(x) = \begin{cases} -0.3, & \text{if } x = 0 \\ -0.4, & \text{if } x = a, b, c. \end{cases}$$

Let $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$, where $\mu_{A \times B}^P = \max \{ \mu_A^P, \mu_B^P \}$ and $\mu_{A \times B}^N = \min \{ \mu_A^N, \mu_B^N \}$ are defined as:

$$\mu_{A \times B}^P(x) = \begin{cases} 0.3, & \text{if } (x, y) = (0, 0), (b, 0) \\ 0.4, & \text{if } (x, y) = (a, 0), (c, 0) \\ 0.5, & \text{if } (x, y) = (0, a), (a, a), (b, a), (c, a) \\ 0.6, & \text{otherwise,} \end{cases}$$

and

$$\mu_{A \times B}^N(x) = \begin{cases} -0.3, & \text{if } (x, y) = (0, 0), (a, 0), (b, 0), (c, 0) \\ -0.4, & \text{otherwise.} \end{cases}$$

By routine calculations, we know that $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ is a doubt bipolar fuzzy H-ideal of $X \times X$.

Theorem 5.1. Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be two doubt bipolar fuzzy H-ideals of BCK/BCI-algebras X and Y , respectively. Then $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ is a doubt bipolar fuzzy H-ideal of $X \times Y$.

Proof: For any $(x, y) \in X \times Y$, we have $\mu_{A \times B}^P(0, 0) = \max \{ \mu_A^P(0), \mu_B^P(0) \} \leq \max \{ \mu_A^P(x), \mu_B^P(y) \} = \mu_{A \times B}^P(x, y)$ and $\mu_{A \times B}^N(0, 0) = \min \{ \mu_A^N(0), \mu_B^N(0) \} \geq \min \{ \mu_A^N(x), \mu_B^N(y) \} = \mu_{A \times B}^N(x, y)$. Now, for any $(x, y), (u, v), (w, z) \in X \times Y$, we have

$$\begin{aligned} \mu_{A \times B}^P((x, y) * (w, z)) &= \mu_{A \times B}^P(x * w, y * z) \\ &= \max \{ \mu_A^P(x * w), \mu_B^P(y * z) \} \\ &\leq \max \{ \max \{ \mu_A^P(x * (u, w)), \mu_A^P(u) \}, \max \{ \mu_B^P(y * (v, z)), \mu_B^P(v) \} \} \\ &= \max \{ \max \{ \mu_A^P(x * (u, w)), \mu_B^P(y * (v, z)) \}, \max \{ \mu_A^P(u), \mu_B^P(v) \} \} \\ &= \max \{ \mu_{A \times B}^P \{ (x * (u * w)), (y * (v * z)) \}, \mu_{A \times B}^P(u, v) \} \\ &= \max \{ \mu_{A \times B}^P((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^P(u, v) \} \end{aligned}$$

and

$$\begin{aligned} \mu_{A \times B}^N((x, y) * (w, z)) &= \mu_{A \times B}^N(x * w, y * z) \\ &= \min \{ \mu_A^N(x * w), \mu_B^N(y * z) \} \end{aligned}$$

$$\begin{aligned} &\geq \min \{ \min \{ \mu_A^N(x * (u, w)), \mu_A^N(u) \}, \min \{ \mu_B^N(y * (v, z)), \mu_B^N(v) \} \} \\ &= \min \{ \min \{ \mu_A^N(x * (u, w)), \mu_B^N(y * (v, z)) \}, \min \{ \mu_A^N(u), \mu_B^N(v) \} \} \\ &= \min \{ \mu_{A \times B}^N \{ (x * (u * w)), (y * (v * z)) \}, \mu_{A \times B}^N(u, v) \} \\ &= \min \{ \mu_{A \times B}^N((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^N(u, v) \}. \end{aligned}$$

Hence, $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ is a doubt bipolar fuzzy H-ideal of a BCK/BCI -algebra $X \times Y$. □

Proposition 5.1. *Let $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ be a doubt bipolar fuzzy H-ideal of $X \times Y$. If $(x, y) \leq (u, v)$, then $\mu_A^P(x, y) \leq \mu_A^P(u, v)$ and $\mu_A^N(x, y) \geq \mu_A^N(u, v)$ for all $(x, y), (u, v) \in X \times Y$.*

Proof: Let $(x, y), (u, v) \in X \times Y$ such that $(x, y) \leq (u, v)$. Then $(x, y) * (u, v) = (0, 0)$.
Now

$$\begin{aligned} \mu_A^P(x, y) &= \mu_A^P((x, y) * (0, 0)) \leq \max \{ \mu_A^P((x, y) * ((u, v) * (0, 0))), \mu_A^P(u, v) \} \\ &= \max \{ \mu_A^P((x, y) * (u, v)), \mu_A^P(u, v) \} \\ &= \max \{ \mu_A^P(0, 0), \mu_A^P(u, v) \} = \mu_A^P(u, v). \end{aligned}$$

Therefore, $\mu_A^P(x, y) \leq \mu_A^P(u, v)$ for all $(x, y), (u, v) \in X \times Y$. Again,

$$\begin{aligned} \mu_A^N(x, y) &= \mu_A^N((x, y) * (0, 0)) \geq \min \{ \mu_A^N((x, y) * ((u, v) * (0, 0))), \mu_A^N(u, v) \} \\ &= \min \{ \mu_A^N((x, y) * (u, v)), \mu_A^N(u, v) \} \\ &= \min \{ \mu_A^N(0, 0), \mu_A^N(u, v) \} = \mu_A^N(u, v). \end{aligned}$$

Therefore, $\mu_A^N(x, y) \geq \mu_A^N(u, v)$ for all $(x, y), (u, v) \in X \times Y$. □

Proposition 5.2. *Let $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ be a doubt bipolar fuzzy H-ideal of $X \times Y$ such that*

$$\mu_{A \times B}^P((x, y) * (u, v)) \leq \mu_{A \times B}^P(u, v) \text{ and } \mu_{A \times B}^N((x, y) * (u, v)) \geq \mu_{A \times B}^N(u, v)$$

for all $(x, y), (u, v) \in X \times Y$, then $A \times B$ is constant.

Proof: Note that in a BCK/BCI -algebra $X \times Y$, $(x, y) * (0, 0) = (x, y)$ for all $(x, y) \in X \times Y$, and by using assumption we have

$$\mu_{A \times B}^P(x, y) = \mu_{A \times B}^P((x, y) * (0, 0)) \leq \mu_{A \times B}^P(0, 0)$$

and

$$\mu_{A \times B}^N(x, y) = \mu_{A \times B}^N((x, y) * (0, 0)) \geq \mu_{A \times B}^N(0, 0).$$

It follows from Definition 5.4, $\mu_{A \times B}^P(x, y) = \mu_{A \times B}^P(0, 0)$ and $\mu_{A \times B}^N(x, y) = \mu_{A \times B}^N(0, 0)$ for all $(x, y), (u, v) \in X \times Y$. Therefore, $X \times Y$ is constant. □

Proposition 5.3. *Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be two doubt bipolar fuzzy H-ideals of BCK/BCI -algebras X and Y , respectively. If $A \times B$ is a doubt bipolar fuzzy H-ideal of $X \times Y$, the following are true:*

- 1) $\mu_A^P(0) \leq \mu_B^P(y)$ and $\mu_B^P(0) \leq \mu_A^P(x)$ for all $x \in X, y \in Y$,
- 2) $\mu_A^N(0) \geq \mu_B^N(y)$ and $\mu_B^N(0) \geq \mu_A^N(x)$ for all $x \in X, y \in Y$.

Proof: 1) Assume that $\mu_A^P(0) > \mu_B^P(y)$ and $\mu_B^P(0) > \mu_A^P(x)$ for some $x \in X, y \in Y$. Then

$$\begin{aligned} \mu_{A \times B}^P(x, y) &= \max \{ \mu_A^P(x), \mu_B^P(y) \} \\ &< \max \{ \mu_A^P(0), \mu_B^P(0) \} = \mu_{A \times B}^P(0, 0), \end{aligned}$$

which is a contradiction. Therefore, $\mu_A^P(0) \leq \mu_B^P(y)$ and $\mu_B^P(0) \leq \mu_A^P(x)$ for all $x \in X, y \in Y$.

2) Assume that $\mu_A^N(0) < \mu_B^N(y)$ and $\mu_B^N(0) < \mu_A^N(x)$ for some $x \in X, y \in Y$. Then

$$\begin{aligned} \mu_{A \times B}^N(x, y) &= \min \{ \mu_A^N(x), \mu_B^N(y) \} \\ &> \min \{ \mu_A^N(0), \mu_B^N(0) \} = \mu_{A \times B}^N(0, 0), \end{aligned}$$

which is also a contradiction. Therefore, $\mu_A^N(0) \geq \mu_B^N(y)$ and $\mu_B^N(0) \geq \mu_A^N(x)$ for all $x \in X, y \in Y$. \square

Theorem 5.2. *Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be two bipolar fuzzy subsets of X and Y , respectively, such that $A \times B$ is a doubt bipolar fuzzy H-ideal of $X \times Y$. Then either A is a doubt bipolar fuzzy H-ideal of X or B is a doubt bipolar fuzzy H-ideal of Y .*

Proof: Since $A \times B$ is a doubt bipolar fuzzy H-ideal of $X \times Y$, then for all $(x, y), (u, v), (w, z) \in X \times Y$, we have $\mu_{A \times B}^P((x, y) * ((u, v) * (w, z))) \leq \max \{ \mu_{A \times B}^P((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^P(u, v) \}$. By putting $y = z = v = 0$, we have

$$\mu_{A \times B}^P((x, 0) * (w, 0)) \leq \max \{ \mu_{A \times B}^P((x, 0) * ((u, 0) * (w, 0))), \mu_{A \times B}^P(u, 0) \}. \tag{1}$$

Also, we have

$$\begin{aligned} \mu_{A \times B}^P((x, 0) * (w, 0)) &= \mu_{A \times B}^P((x * w), (0 * 0)) \\ &= \max \{ \mu_A^P(x * w), \mu_B^P(0 * 0) \} \\ &= \mu_A^P(x * w) \end{aligned} \tag{2}$$

and

$$\begin{aligned} \mu_{A \times B}^P((x, 0) * ((u, 0) * (w, 0))) &= \mu_{A \times B}^P((x, 0) * ((u * w), (0 * 0))) \\ &= \mu_{A \times B}^P((x * (u * w)), (0 * (0, 0))) \\ &= \max \{ \mu_A^P(x * (u * w)), \mu_B^P(0 * (0, 0)) \} \\ &= \mu_A^P(x * (u * w)). \end{aligned} \tag{3}$$

Again, by using Proposition 5.3, we have

$$\mu_{A \times B}^P(u, 0) = \max \{ \mu_A^P(u), \mu_B^P(0) \} = \mu_A^P(u). \tag{4}$$

So, from (1), (2), (3) and (4) we get, $\mu_A^P(x * w) \leq \max \{ \mu_A^P(x * (u, w)), \mu_A^P(u) \}$.

Similarly, we can prove $\mu_A^N(x * w) \geq \min \{ \mu_A^N(x * (u, w)), \mu_A^N(u) \}$. Hence, A is a doubt bipolar fuzzy H-ideal of X . \square

Proposition 5.4. *Let $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ be a doubt bipolar fuzzy H-ideal of a BCK-algebra $X \times Y$, then $\mu_{A \times B}^P((0, 0) * ((0, 0) * (x, y))) \leq \mu_{A \times B}^P(x, y)$ and $\mu_{A \times B}^N((0, 0) * ((0, 0) * (x, y))) \geq \mu_{A \times B}^N(x, y)$ for all $(x, y) \in X \times Y$.*

Proof: Note that

$$\begin{aligned} \mu_{A \times B}^P((0, 0) * ((0, 0) * (x, y))) &\leq \max \{ \mu_{A \times B}^P((0, 0) * ((x, y) * ((0, 0) * (x, y)))) , \mu_{A \times B}^P(x, y) \} \\ &= \max \{ \mu_{A \times B}^P((0, 0) * ((x, y) * (0, 0))) , \mu_{A \times B}^P(x, y) \} \\ &= \max \{ \mu_{A \times B}^P((0, 0) * (x, y)) , \mu_{A \times B}^P(x, y) \} \\ &= \max \{ \mu_{A \times B}^P(0, 0) , \mu_{A \times B}^P(x, y) \} \\ &= \mu_{A \times B}^P(x, y) \text{ for all } (x, y) \in X \times Y. \end{aligned}$$

Therefore, $\mu_{A \times B}^P((0, 0) * ((0, 0) * (x, y))) \leq \mu_{A \times B}^P(x, y)$ for all $(x, y) \in X \times Y$. Again,

$$\mu_{A \times B}^N((0, 0) * ((0, 0) * (x, y))) \geq \min \{ \mu_{A \times B}^N((0, 0) * ((x, y) * ((0, 0) * (x, y)))) , \mu_{A \times B}^N(x, y) \}$$

$$\begin{aligned}
 &= \min \{ \mu_{A \times B}^N((0, 0) * ((x, y) * (0, 0))), \mu_{A \times B}^N(x, y) \} \\
 &= \min \{ \mu_{A \times B}^N((0, 0) * (x, y)), \mu_{A \times B}^N(x, y) \} \\
 &= \min \{ \mu_{A \times B}^N(0, 0), \mu_{A \times B}^N(x, y) \} \\
 &= \mu_{A \times B}^N(x, y) \text{ for all } (x, y) \in X \times Y.
 \end{aligned}$$

Therefore, $\mu_{A \times B}^N((0, 0) * ((0, 0) * (x, y))) \geq \mu_{A \times B}^N(x, y)$ for all $(x, y) \in X \times Y$. □

Corollary 5.1. *Let $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ be a doubt bipolar fuzzy H-ideal of $X \times Y$. Then the sets $D_{\mu_{A \times B}^P} = \{(x, y) \in X \times Y | \mu_{A \times B}^P(x, y) = \mu_{A \times B}^P(0, 0)\}$ and $D_{\mu_{A \times B}^N} = \{(x, y) \in X \times Y | \mu_{A \times B}^N(x, y) = \mu_{A \times B}^N(0, 0)\}$ are H-ideals of $X \times Y$.*

Proof: Let $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ be a doubt bipolar fuzzy H-ideal of $X \times Y$. Obviously, $(0, 0) \in D_{\mu_{A \times B}^P}$ and $(0, 0) \in D_{\mu_{A \times B}^N}$. Now, let $(x, y), (u, v), (w, z) \in D_{\mu_{A \times B}^P}$ such that $(x, y) * ((u, v) * (w, z)), (u, v) \in D_{\mu_{A \times B}^P}$. Then $\mu_{A \times B}^P((x, y) * ((u, v) * (w, z))) = \mu_{A \times B}^P(0, 0) = \mu_{A \times B}^P(u, v)$. Now $\mu_{A \times B}^P((x, y) * (w, z)) \leq \max \{ \mu_{A \times B}^P((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^P(u, v) \} = \mu_{A \times B}^P(0, 0)$. Again, since $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ is a doubt bipolar fuzzy H-ideal of $X \times Y$, so $\mu_{A \times B}^P(0, 0) \leq \mu_{A \times B}^P((x, y) * (w, z))$. Therefore, $\mu_{A \times B}^P(0, 0) = \mu_{A \times B}^P((x, y) * (w, z))$. It follows that $(x, y) * (w, z) \in D_{\mu_{A \times B}^P}$ for all $(x, y), (u, v), (w, z) \in X \times Y$. Therefore, $D_{\mu_{A \times B}^P}$ is an H-ideal of $X \times Y$. Similarly, we can prove that $D_{\mu_{A \times B}^N}$ is an H-ideal of $X \times Y$. □

Definition 5.5. *Let $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ be a doubt bipolar fuzzy H-ideal of $X \times Y$, and $(t, s) \in [0, 1] \times [-1, 0]$. Then the doubt positive t-level cut set and the doubt negative s-level cut set of $A \times B$ are as follows:*

$$(A \times B)_t^P = \{(x, y) \in X \times Y | \mu_{A \times B}^P(x, y) \leq t\}$$

and

$$(A \times B)_s^N = \{(x, y) \in X \times Y | \mu_{A \times B}^N(x, y) \geq s\}.$$

Theorem 5.3. *Let $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ be a bipolar fuzzy set of $X \times Y$. Then $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ is a doubt bipolar fuzzy H-ideal of $X \times Y$ if and only if $(A \times B)_t^P \neq \emptyset$ and $(A \times B)_s^N \neq \emptyset$ are H-ideals of $X \times Y$ for all $(t, s) \in [0, 1] \times [-1, 0]$.*

Proof: Assume that $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ is a doubt bipolar fuzzy H-ideal of $X \times Y$ and $(t, s) \in [0, 1] \times [-1, 0]$ such that $(A \times B)_t^P \neq \emptyset$ and $(A \times B)_s^N \neq \emptyset$. Let $(a, b) \in (A \times B)_t^P$ and $(c, d) \in (A \times B)_s^N$. Then we have $\mu_{A \times B}^P(a, b) \leq t$ and $\mu_{A \times B}^N(c, d) \geq s$. So we deduce that $\mu_{A \times B}^P(0, 0) \leq \mu_{A \times B}^P(a, b) \leq t$ and $\mu_{A \times B}^N(0, 0) \geq \mu_{A \times B}^N(c, d) \geq s$. This shows that $(0, 0) \in (A \times B)_t^P$ and $(0, 0) \in (A \times B)_s^N$. Let $(x, y), (u, v), (w, z) \in X \times Y$ such that $(x, y) * ((u, v) * (w, z)) \in (A \times B)_t^P$ and $(u, v) \in (A \times B)_t^P$, and let $(x', y'), (u', v'), (w', z') \in X \times Y$ such that $(x', y') * ((u', v') * (w', z')) \in (A \times B)_s^N$ and $(u', v') \in (A \times B)_s^N$. Then $\mu_{A \times B}^P((x, y) * ((u, v) * (w, z))) \leq t$, $\mu_{A \times B}^P(u, v) \leq t$, $\mu_{A \times B}^N((x', y') * ((u', v') * (w', z'))) \geq s$ and $\mu_{A \times B}^N(u', v') \geq s$. Since $A \times B$ is a doubt bipolar fuzzy H-ideal of $X \times Y$, it follows that

$$\begin{aligned}
 \mu_{A \times B}^P((x, y) * (w, z)) &\leq \max \{ \mu_{A \times B}^P((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^P(u, v) \} \\
 &\leq \max\{t, t\} = t
 \end{aligned}$$

and

$$\begin{aligned}
 \mu_{A \times B}^N((x', y') * (w', z')) &\geq \min \{ \mu_{A \times B}^N((x', y') * ((u', v') * (w', z'))), \mu_{A \times B}^N(u', v') \} \\
 &\geq \min\{s, s\} = s,
 \end{aligned}$$

so $(x, y) * (w, z) \in (A \times B)_t^P$ and $(x', y') * (w', z') \in (A \times B)_s^N$. Therefore, $(A \times B)_t^P$ and $(A \times B)_s^N$ are H-ideals of $X \times Y$.

Conversely, assume that $(A \times B)_t^P \neq \emptyset$ and $(A \times B)_t^N \neq \emptyset$ are H-ideals of $X \times Y$ for all $(t, s) \in [0, 1] \times [-1, 0]$. Let $(x, y) \in X \times Y$ such that $\mu_{A \times B}^P(0, 0) > \mu_{A \times B}^P(x, y)$ and $\mu_{A \times B}^N(0, 0) < \mu_{A \times B}^N(x, y)$. By taking

$$t_o = \frac{1}{2} [\mu_{A \times B}^P(0, 0) + \mu_{A \times B}^P(x, y)], \quad s_o = \frac{1}{2} [\mu_{A \times B}^N(0, 0) + \mu_{A \times B}^N(x, y)],$$

we get $\mu_{A \times B}^P(0, 0) > t_o > \mu_{A \times B}^P(x, y)$ and $\mu_{A \times B}^N(0, 0) < s_o < \mu_{A \times B}^N(x, y)$. Therefore, $(x, y) \in (A \times B)_{t_o}^P$, $(0, 0) \notin (A \times B)_{t_o}^P$, $(x, y) \in (A \times B)_{s_o}^N$ and $(0, 0) \notin (A \times B)_{s_o}^N$. This is a contradiction. Hence, $\mu_{A \times B}^P(0, 0) \leq \mu_{A \times B}^P(x, y)$ and $\mu_{A \times B}^N(0, 0) \geq \mu_{A \times B}^N(x, y)$ for all $(x, y) \in X \times Y$. Again, we assume that $(x, y), (u, v), (w, z) \in X \times Y$ such that

$$\mu_{A \times B}^P((x, y) * (w, z)) > \max \{ \mu_{A \times B}^P((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^P(u, v) \}$$

and

$$\mu_{A \times B}^N((x, y) * (w, z)) < \min \{ \mu_{A \times B}^N((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^N(u, v) \}.$$

Then by taking

$$t_1 = \frac{1}{2} [\mu_{A \times B}^P((x, y) * (w, z)) + \max \{ \mu_{A \times B}^P((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^P(u, v) \}],$$

$$s_1 = \frac{1}{2} [\mu_{A \times B}^N((x, y) * (w, z)) + \max \{ \mu_{A \times B}^N((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^N(u, v) \}],$$

we have

$$\mu_{A \times B}^P((x, y) * (w, z)) > t_1 > \max \{ \mu_{A \times B}^P((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^P(u, v) \},$$

$$\mu_{A \times B}^N((x, y) * (w, z)) < s_1 < \min \{ \mu_{A \times B}^N((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^N(u, v) \}.$$

This shows that, $(x, y) * ((u, v) * (w, z)) \in (A \times B)_{t_1}^P$, $(u, v) \in (A \times B)_{t_1}^P$, but $(x, y) * (w, z) \notin (A \times B)_{t_1}^P$, which is a contradiction. Also, $(x, y) * ((u, v) * (w, z)) \in (A \times B)_{s_1}^N$, $(u, v) \in (A \times B)_{s_1}^N$, but $(x, y) * (w, z) \notin (A \times B)_{s_1}^N$, again this is a contradiction. Therefore,

$$\mu_{A \times B}^P((x, y) * (w, z)) \leq \max \{ \mu_{A \times B}^P((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^P(u, v) \}$$

and

$$\mu_{A \times B}^N((x, y) * (w, z)) \geq \min \{ \mu_{A \times B}^N((x, y) * ((u, v) * (w, z))), \mu_{A \times B}^N(u, v) \}$$

for all $(x, y), (u, v), (w, z) \in X \times Y$. Hence, $A \times B = (\mu_{A \times B}^P, \mu_{A \times B}^N)$ is a doubt bipolar fuzzy H-ideal of $X \times Y$. \square

6. Conclusions. To investigate the structure of an algebraic system, it is clear that doubt bipolar fuzzy H-ideals with special properties play an important role. In this paper, we introduced the notion of doubt bipolar fuzzy H-ideals of *BCK/BCI*-algebras and investigated related properties. We characterized strong doubt positive t -level cut set, strong doubt negative s -level cut set, homomorphism and equivalence relation by considering doubt bipolar fuzzy H-ideals of *BCK/BCI*-algebras. Also, we considered the structure of *BCK/BCI*-algebras and defined Cartesian product of two doubt bipolar fuzzy H-ideals. We presented some interesting results about Cartesian product of two doubt bipolar fuzzy H-ideals of *BCK/BCI*-algebras. In addition, we proved that the Cartesian product of two bipolar fuzzy sets becomes doubt bipolar fuzzy H-ideals if and only if for any $(t, s) \in [0, 1] \times [-1, 0]$, doubt positive t -level cut set and doubt negative s -level cut set are H-ideals of a *BCK/BCI*-algebra $X \times Y$. In our opinion, these definitions and main results can be similarly extended to some other fuzzy algebraic systems such as groups, semigroups, rings, nearings, semirings (hemirings), lattices and Lie algebras.

REFERENCES

- [1] L. A. Zadeh, Fuzzy sets, *Information and Control*, vol.8, no.3, pp.338-353, 1965.
- [2] L. A. Zadeh, Similarity relations and fuzzy orderings, *Information Sciences*, vol.3, no.2, pp.177-200, 1971.
- [3] K. M. Lee, Bipolar-valued fuzzy sets and their operations, *Proc. of Int. Conf. on Intelligent Technologies*, Bangkok, Thailand, pp.307-312, 2000.
- [4] Y. Imai and K. Iséki, On axiom systems of propositional calculi, *Proc. of the Japan Academy*, vol.42, pp.19-22, 1966.
- [5] A. Rosenfeld, Fuzzy groups, *Journal of Mathematical Analysis and Applications*, vol.35, pp.512-517, 1971.
- [6] O. G. Xi, Fuzzy BCK-algebras, *Mathematica Japonica*, vol.36, no.5, pp.935-942, 1991.
- [7] Y. B. Jun, Closed fuzzy ideals in BCI-algebras, *Mathematica Japonica*, vol.38, no.1, pp.199-202, 1993.
- [8] B. Ahmad, Fuzzy BCI-algebras, *J. Fuzzy Math.*, vol.1, no.2, pp.445-452, 1993.
- [9] H. M. Khalid and B. Ahmad, Fuzzy H-ideals in BCI-algebras, *Fuzzy Sets and Systems*, vol.101, no.1, pp.153-158, 1999.
- [10] F. Y. Huang, Another definition of fuzzy BCI-algebras, *Selected Papers on BCK/BCI-Algebras*, vol.1, pp.91-92, 1990.
- [11] Y. B. Jun, Doubt fuzzy BCK/BCI-algebras, *Soochow Journal of Mathematics*, vol.20, no.3, pp.351-358, 1994.
- [12] J. Zhan and Z. Tan, Characterization of doubt fuzzy H-ideals in BCK-algebras, *Soochow Journal of Mathematics*, vol.29, no.3, pp.293-298, 2003.
- [13] J. Zhan and Z. Tan, Fuzzy H-ideals in BCK-algebras, *Southeast Asian Bull. Math.*, vol.29, no.6, pp.1165-1173, 2005.
- [14] K. J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, *Bulletin of the Malaysian Mathematical Sciences Society*, vol.32, no.3, pp.361-373, 2009.
- [15] M. A. Alghamdi, N. M. Muthana and N. O. Alshehri, Novel concepts of bipolar fuzzy BCK-submodules, *Discrete Dynamics in Nature and Society*, vol.2017, 2017.
- [16] C. Jana, M. Pal and A. B. Saied, $(\in, \in \vee q)$ -bipolar fuzzy BCK/BCI-algebras, *Missouri J. Math. Sci.*, vol.29, no.2, pp.139-160, 2017.
- [17] C. S. Kim, J. G. Kang and J. M. Kang, Ideal theory of sub-semigroups based on the bipolar valued fuzzy set theory, *Annals of Fuzzy Mathematics and Informatics*, vol.2, pp.193-206, 2011.
- [18] M. Zhou and S. Li, Application of bipolar fuzzy sets in semirings, *Journal of Mathematical Research with Applications*, vol.34, no.1, pp.61-72, 2014.
- [19] M. Zhou and S. Li, Applications of bipolar fuzzy theory to hemirings, *International Journal of Innovative Computing, Information and Control*, vol.10, no.2, pp.767-781, 2014.
- [20] K. Hayat, T. Mahmood and B. Y. Cao, On bipolar anti fuzzy H-ideals in hemirings, *Fuzzy Inf. Eng.*, vol.9, pp.1-19, 2017.
- [21] M. Akram, A. B. Saeid, K. P. Shum and B. L. Meng, Bipolar fuzzy K-algebras, *International Journal of Fuzzy System*, vol.12, no.3, pp.252-258, 2010.
- [22] S. Sabarinathan, P. Muralikrishna and D. C. Kumar, Bipolar valued fuzzy H-ideals of BF-algebras, *International Journal of Pure and Applied Mathematics*, vol.112, no.5, pp.87-92, 2017.
- [23] S. Sabarinathan, D. C. Kumar and P. Muralikrishna, Bipolar valued fuzzy α -ideal of BF-algebra, *Circuits and Systems*, vol.7, pp.3054-3092, 2016.
- [24] M. Akram and N. O. Alshehri, Bipolar fuzzy Lie ideals, *Utilitas Mathematica*, vol.87, pp.265-278, 2012.
- [25] M. Akram, W. Chen and Y. Lin, Bipolar fuzzy Lie superalgebras, *Quasigroups Related Systems*, vol.20, pp.139-156, 2012.
- [26] K. Iséki, An algebra related with a propositional calculus, *Proc. of the Japan Academy*, vol.42, pp.26-29, 1966.
- [27] K. Iséki, On BCI-algebras, *Mathematics Seminar Notes (Kobe University)*, vol.8, pp.125-130, 1980.
- [28] T. Lei and C. Xi, p-Radical in BCI-algebras, *Mathematica Japonica*, vol.30, pp.511-517, 1987.
- [29] Y. S. Huang, *BCI-Algebra*, Science Press, Beijing, China, 2006.