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ANTI-PERIODIC DYNAMICS IN DELAYED NEURAL NETWORKS WITH UNIDIRECTIONAL COUPLING

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Abstract. *Anti-periodic phenomenon of neural networks plays an important role in designing neural networks to serve human beings. Thus, anti-periodic dynamics of neural networks has attracted much attention. This paper is concerned with the existence and exponential stability of anti-periodic solutions of delayed neural networks with unidirectional coupling. Using some analysis skills and Lyapunov method, a series of sufficient conditions for the existence and exponential stability of anti-periodic solutions to delayed neural networks with unidirectional coupling are presented. Our results are new and complement some previously known ones. The obtained results have theoretical value and practical significance and can be widely applied to neural information processing, artificial intelligence, diagnosis of disease and so on. The analysis method of this paper can be applied to studying many other anti-periodic dynamics of similar neural networks.*
Keywords: Neural network, Anti-periodic solution, Exponential stability, Delay

1. **Introduction.** In the last three decades, considerable attention has been paid to different types of artificial neural networks for their essential applications, such as pattern recognition, optimization, signal and image processing, classification, parallel computation, and associative memory [1-3]. In these applications, the dynamical behaviors such as the existence, uniqueness, Hopf bifurcation and global asymptotic stability or global exponential stability of the equilibrium point, periodic solution and almost periodic solutions for neural networks play a key role (see [4-16]). Recently, a great deal of neural networks models [34-40] have been proposed and investigated extensively to understand the dynamical behaviors of neurons. In contrast, however, very few results are available on a generic, in depth, existence and exponential stability of anti-periodic solutions of neural networks. Moreover, the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations [17-31]. Thus, it is worthwhile to investigate the existence and stability of anti-periodic solutions for neural networks. In 2009, He et al. [33] have investigated the codimensional two bifurcation of the following delayed neural networks with undirectional coupling

$$\begin{cases} \dfrac{du_1(t)}{dt} = -u_1(t) + a_{12}f(u_2(t-\tau)) + \alpha f(u_4(t-\tau)), \\ \dfrac{du_2(t)}{dt} = -u_2(t) + a_{21}f(u_1(t-\tau)), \\ \dfrac{du_3(t)}{dt} = -u_3(t) + a_{12}f(u_4(t-\tau)) + \alpha f(u_2(t-\tau)), \\ \dfrac{du_4(t)}{dt} = -u_4(t) + a_{21}f(u_3(t-\tau)), \end{cases} \tag{1}$$

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where $u_i(t)$ denotes the state of the ith neuron, α_{ij}, $i,j=1,2$, represents the connection weights within the individual network and in the literature, for $a_{ij}>0$ (<0), the j-th neuron is called excitatory (inhibitory otherwise), α measures the coupling strength between two networks, τ stands for the time delay due to the finite switching speed of amplifiers and $f:R\to R$ is the amplification function. In detail, one can see [33]. Considering that any system parameters are naturally subject to fluctuation in time, it is more realistic that neural networks should take account of the changing environment. Thus, we can modify neural network model (1) as the following form:

$$\begin{cases} \dfrac{du_1(t)}{dt} = -u_1(t) + a_{12}(t)f(u_2(t-\tau)) + \alpha(t)f(u_4(t-\tau)), \\ \dfrac{du_2(t)}{dt} = -u_2(t) + a_{21}(t)f(u_1(t-\tau)), \\ \dfrac{du_3(t)}{dt} = -u_3(t) + a_{12}(t)f(u_4(t-\tau)) + \alpha(t)f(u_2(t-\tau)), \\ \dfrac{du_4(t)}{dt} = -u_4(t) + a_{21}(t)f(u_3(t-\tau)). \end{cases} \tag{2}$$

The purpose of this paper is to focus on the existence and exponential stability of anti-periodic solution of system (2). Although many papers (see [17-31,41-54]) deal with the anti-periodic solutions of neural networks, most authors handle this aspect by applying contraction mapping fixed point theorem and differential inequality. In this paper, we will establish a series of sufficient conditions of existence and global exponential stability of anti-periodic solutions of delayed neural network with unidirectional coupling by the fundamental solution matrix, Lyapunov function and constructing fundamental function sequences based on the solution of networks. Recently, few papers consider the anti-periodic solutions of neural networks by fundamental solution matrix. Moreover, the construction of Lyapunov function is different from that in earlier works [17-31,41-54]. Our results not only can be applied directly to many concrete examples of neural networks, but also extend, to a certain extent, some previously known ones. In addition, an example is given to illustrate the effectiveness of our main results. Our results are a good complement of He et al. [33].

The rest of this paper is organized as follows. In Section 2, we give some notations and preliminary results. In Section 3, we present our main results on the existence and global exponential stability of anti-periodic solution of the delayed neural network with unidirectional coupling. In Section 4, we present an example and numerical simulations to illustrate the effectiveness of our main theoretical findings. In Section 5, we give a simple conclusion.

2. **Preliminary Results.** In this section, we shall present some notations and introduce some lemmas which are used in the following sections.

For any vector $V=(v_1,v_2,\ldots,v_n)^T$ and matrix $D=(d_{ij})_{n\times n}$, we define the norm as

$$||v|| = \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}, \quad ||D|| = \left(\sum_{i=1}^{n} d_{ij}^2\right)^{\frac{1}{2}},$$

respectively.

We assume that system (2) always satisfies the following initial conditions:

$$u_{i0} = \varphi_i(s), \quad s\in[-\tau,0]. \tag{3}$$

Let $u(t)=(u_1(t),u_2(t),u_3(t),u_4(t))^T$ be the solution of system (2) with initial conditions (3). We say the solution $u(t)=(u_1(t),u_2(t),u_3(t),u_4(t))^T$ is T-anti-periodic on R^4 if $u_i(t+T)=-u_i(t)$ for all $t\in R$ and $i=1,2,\ldots,4$, where T is a positive constant.

In order to obtain our main results in this paper, we make the assumptions as follows.
(H1) There exists a constant $L > 0$ such that

$$|f(u) - f(v)| \le L|u - v|$$

for all $u, v \in R$.
(H2) For all $t, u \in R$,

$$a_{12}(t+T)f(u) = -a_{12}(t)f(-u), \quad a_{21}(t+T)f(u) = -a_{21}(t)f(-u),$$
$$\alpha(t+T)f(u) = -\alpha(t)f(-u),$$

where T is a positive constant.

Definition 2.1. *The solution $u^*(t)$ of system (2) is said to be globally exponentially stable if there exist constants $\beta > 0$ and $M > 1$ such that*

$$\sum_{i=1}^{4} |u_i(t) - u_i^*(t)| \le M e^{-\beta t} ||\varphi - \varphi^*||^2$$

for each solution $u(t)$ of system (2).

Next, we present three important lemmas which are necessary for proving our main results in Section 3.

Lemma 2.1. *Let*

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and then we have

$$|| \exp At|| \le 2e^{-t}$$

for all $t \ge 0$.

Proof: Since

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

it follows that

$$\exp At = \begin{pmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}.$$

According to the definition of matrix norm, we get

$$|| \exp At|| = \left(e^{-2t} + e^{-2t} + e^{-2t} + e^{-2t}\right)^{\frac{1}{2}} \le 2e^{-t}.$$

Lemma 2.2. *Assume that*

$$\text{(H3)} \qquad \begin{cases} -2 + \bar{a}_{12}L^{2\varepsilon_1} + \bar{\alpha}L^{2\varepsilon_2} + \bar{a}_{21}L^{2(1-\varepsilon_3)} < 0, \\ -2 + \bar{a}_{21}L^{2\varepsilon_3} + \bar{a}_{12}L^{2(1-\varepsilon_1)} + \bar{\alpha}L^{2(1-\varepsilon_5)} < 0, \\ -2 + \bar{a}_{12}L^{2\varepsilon_4} + \bar{\alpha}L^{2\varepsilon_5} + \bar{a}_{21}L^{2(1-\varepsilon_6)} < 0, \\ -2 + \bar{a}_{21}L^{2\varepsilon_6} + \bar{a}_{12}L^{2(1-\varepsilon_4)} + \bar{\alpha}L^{2(1-\varepsilon_2)} < 0, \end{cases}$$

where $0 \le \varepsilon_i \le 1$ $(i = 1,2,3,4,5,6)$ *are any constants. Then there exists* $\beta > 0$ *such that*

$$\beta - 2 + \bar{a}_{12}L^{2\varepsilon_1} + \bar{\alpha}L^{2\varepsilon_2} + \bar{a}_{21}L^{2(1-\varepsilon_3)}e^{\beta\tau} \le 0,$$
$$\beta - 2 + \bar{a}_{21}L^{2\varepsilon_3} + \left(\bar{a}_{12}L^{2(1-\varepsilon_1)} + \bar{\alpha}L^{2(1-\varepsilon_5)}\right)e^{\beta\tau} \le 0,$$
$$\beta - 2 + \bar{a}_{12}L^{2\varepsilon_4} + \bar{\alpha}L^{2\varepsilon_5} + \bar{a}_{21}L^{2(1-\varepsilon_6)}e^{\beta\tau} \le 0,$$
$$\beta - 2 + \bar{a}_{21}L^{2\varepsilon_6} + \left(\bar{a}_{12}L^{2(1-\varepsilon_4)} + \bar{\alpha}L^{2(1-\varepsilon_2)}\right)e^{\beta\tau} \le 0.$$

Proof: Let

$$\begin{aligned}
\varrho_1(\beta) &= \beta - 2 + \bar{a}_{12}L^{2\varepsilon_1} + \bar{\alpha}L^{2\varepsilon_2} + \bar{a}_{21}L^{2(1-\varepsilon_3)}e^{\beta\tau},\\
\varrho_2(\beta) &= \beta - 2 + \bar{a}_{21}L^{2\varepsilon_3} + \left(\bar{a}_{12}L^{2(1-\varepsilon_1)} + \bar{\alpha}L^{2(1-\varepsilon_5)}\right)e^{\beta\tau},\\
\varrho_3(\beta) &= \beta - 2 + \bar{a}_{12}L^{2\varepsilon_4} + \bar{\alpha}L^{2\varepsilon_5} + \bar{a}_{21}L^{2(1-\varepsilon_6)}e^{\beta\tau},\\
\varrho_4(\beta) &= \beta - 2 + \bar{a}_{21}L^{2\varepsilon_6} + \left(\bar{a}_{12}L^{2(1-\varepsilon_4)} + \bar{\alpha}L^{2(1-\varepsilon_2)}\right)e^{\beta\tau}.
\end{aligned}$$

Obviously, $\varrho_i(\beta)$ $(i = 1,2,3,4)$ is continuously differential function. We can easily check that

$$\begin{aligned}
\frac{d\varrho_1(\beta)}{d\beta} &= 1 + \tau\bar{a}_{21}L^{2(1-\varepsilon_3)}e^{\beta\tau} > 0,\ \lim_{\beta\to+\infty}\varrho_1(\beta) = +\infty,\ \varrho_1(0) < 0,\\
\frac{d\varrho_2(\beta)}{d\beta} &= 1 + \tau\left(\bar{a}_{12}L^{2(1-\varepsilon_1)} + \bar{\alpha}L^{2(1-\varepsilon_5)}\right)e^{\beta\tau} > 0,\ \lim_{\beta\to+\infty}\varrho_2(\beta) = +\infty,\ \varrho_2(0) < 0,\\
\frac{d\varrho_3(\beta)}{d\beta} &= 1 + \tau\bar{a}_{21}L^{2(1-\varepsilon_6)}e^{\beta\tau} > 0,\ \lim_{\beta\to+\infty}\varrho_3(\beta) = +\infty,\ \varrho_3(0) < 0,\\
\frac{d\varrho_4(\beta)}{d\beta} &= 1 + \tau\left(\bar{a}_{12}L^{2(1-\varepsilon_4)} + \bar{\alpha}L^{2(1-\varepsilon_2)}\right)e^{\beta\tau} > 0,\ \lim_{\beta\to+\infty}\varrho_4(\beta) = +\infty,\ \varrho_4(0) < 0.
\end{aligned}$$

By using the intermediate value theorem, there exist constants $\beta_i^* > 0$ $(i = 1,2,3,4)$ such that

$$\varrho_i(\beta_i^*) = 0, \quad i = 1,2,3,4.$$

Let $\beta_0 = \min\{\beta_1^*, \beta_2^*, \beta_3^*, \beta_4^*\}$, and then it follows that $\beta_0 > 0$ and

$$\varrho_i(\beta_0) \le 0, \quad i = 1,2,3,4.$$

This completes the proof of Lemma 2.2.

Lemma 2.3. *Assume that* (H1) *is satisfied. Then for any solution* $(u_1(t), u_2(t), u_3(t), u_4(t))^T$ *of system (2) there exists a constant*

$$\chi^* = 2||\varphi||^2 + \frac{4}{\alpha}(\bar{a}_{12} + \bar{a}_{21} + \bar{\alpha})M$$

such that

$$|u_i(t)| \le \chi^*, \quad i = 1,2,3,4$$

for all $t > 0$.

Proof: Let

$$z(t) = \begin{pmatrix} u_1(t)\\ u_2(t)\\ u_3(t)\\ u_4(t)\end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix},$$

$$F(u_1(t), u_2(t), u_3(t), u_4(t)) = \begin{pmatrix} a_{12}(t)f(u_2(t-\tau)) + \alpha(t)f(u_4(t-\tau))\\ a_{21}(t)f(u_1(t-\tau))\\ a_{12}(t)f(u_4(t-\tau)) + \alpha(t)f(u_2(t-\tau))\\ a_{21}(t)f(u_3(t-\tau))\end{pmatrix},$$

and then system (2) can be written as the following equivalent form

$$z'(t) \leq Az(t) + F(u_1(t), u_2(t), u_3(t), u_4(t)). \tag{4}$$

Solving the inequality (4), we have

$$z(t) \leq e^{At}z(0) + \int_0^t e^{A(t-s)}[F(u_1(s), u_2(s), u_3(s), u_4(s))]ds.$$

It follows from Lemma 2.1 that

$$\begin{aligned}
||z(t)|| &\leq 2e^{-\alpha t}||z(0)|| + \sqrt{2}\int_0^t e^{\alpha(t-s)}[||F(u_1(s), u_2(s), u_3(s), u_4(s))||]ds \\
&\leq 2||\varphi||^2 + \frac{4}{\alpha}\left(1 - e^{-t}\right)(\bar{a}_{12} + \bar{a}_{21} + \bar{\alpha})M \\
&\leq 2||\varphi||^2 + \frac{4}{\alpha}\left(\bar{a}_{12} + \bar{a}_{21} + \bar{\alpha}\right)M.
\end{aligned} \tag{5}$$

Let

$$\chi^* = 2||\varphi||^2 + \frac{4}{\alpha}\left(\bar{a}_{12} + \bar{a}_{21} + \bar{\alpha}\right)M.$$

Then it follows that $|u_i(t)| \leq \chi^*$, $i = 1, 2, 3, 4$, for all $t > 0$. This completes the proof of Lemma 2.3.

3. **Main Results.**

In this section, we present our main result that there exists the exponentially stable anti-periodic solution of (2).

Theorem 3.1. *Assume that* (H1), (H2) *and* (H3) *hold true. Then any solution* $u^*(t)$ *of system (2) is globally exponentially stable.*

Proof: Let $x_i(t) = u_i(t) - u_i^*(t)$, $i = 1, 2, 3, 4$. It follows from system (2) that

$$\begin{cases}
\dfrac{dx_1(t)}{dt} = -x_1(t) + a_{12}(t)\left[f(u_2(t-\tau)) - f(u_2^*(t-\tau))\right] \\
\qquad\qquad + \alpha(t)[f(u_4(t-\tau)) - f(u_4^*(t-\tau))], \\
\dfrac{dx_2(t)}{dt} = -x_2(t) + a_{21}(t)[f(u_1(t-\tau)) - f(u_1^*(t-\tau))], \\
\dfrac{dx_3(t)}{dt} = -x_3(t) + a_{12}(t)[f(u_4(t-\tau)) - f(u_4^*(t-\tau))] \\
\qquad\qquad + \alpha(t)[f(u_2(t-\tau)) - f(u_2^*(t-\tau))], \\
\dfrac{dx_4(t)}{dt} = -x_4(t) + a_{21}(t)[f(u_3(t-\tau)) - f(u_3^*(t-\tau))],
\end{cases} \tag{6}$$

which leads to

$$\begin{cases}
\dfrac{1}{2}\dfrac{dx_1^2(t)}{dt} = -x_1^2(t) + x_1(t)a_{12}(t)\left[f(u_2(t-\tau)) - f(u_2^*(t-\tau))\right] \\
\qquad\qquad + x_1(t)\alpha(t)[f(u_4(t-\tau)) - f(u_4^*(t-\tau))], \\
\dfrac{1}{2}\dfrac{dx_2^2(t)}{dt} = -x_2^2(t) + x_2(t)a_{21}(t)[f(u_1(t-\tau)) - f(u_1^*(t-\tau))], \\
\dfrac{1}{2}\dfrac{dx_3^2(t)}{dt} = -x_3^2(t) + x_3(t)a_{12}(t)[f(u_4(t-\tau)) - f(u_4^*(t-\tau))] \\
\qquad\qquad + x_3(t)\alpha(t)[f(u_2(t-\tau)) - f(u_2^*(t-\tau))], \\
\dfrac{1}{2}\dfrac{dx_4^2(t)}{dt} = -x_4^2(t) + x_4(t)a_{21}(t)[f(u_3(t-\tau)) - f(u_3^*(t-\tau))].
\end{cases} \tag{7}$$

In view of the condition (H1), we get

$$\begin{cases} \dfrac{dx_1^2(t)}{dt} \le -2x_1^2(t) + \bar{a}_{12}\left[L^{2\varepsilon_1}x_1^2(t) + L^{2(1-\varepsilon_1)}x_2^2(t-\tau)\right] \\ \qquad\qquad + \bar{\alpha}\left[L^{2\varepsilon_2}x_1^2(t) + L^{2(1-\varepsilon_2)}x_4^2(t-\tau)\right], \\ \dfrac{dx_2^2(t)}{dt} \le -2x_2^2(t) + \bar{a}_{21}\left[L^{2\varepsilon_3}x_2^2(t) + L^{2(1-\varepsilon_3)}x_1^2(t-\tau)\right], \\ \dfrac{dx_3^2(t)}{dt} \le -2x_3^2(t) + \bar{a}_{12}\left[L^{2\varepsilon_4}x_3^2(t) + L^{2(1-\varepsilon_4)}x_4^2(t-\tau)\right] \\ \qquad\qquad + \bar{\alpha}\left[L^{2\varepsilon_5}x_3^2(t) + L^{2(1-\varepsilon_5)}x_2^2(t-\tau)\right], \\ \dfrac{dx_4^2(t)}{dt} \le -2x_4^2(t) + \bar{a}_{21}\left[L^{2\varepsilon_6}x_4^2(t) + L^{2(1-\varepsilon_6)}x_3^2(t-\tau)\right], \end{cases} \tag{8}$$

where $0 \le \varepsilon_i \le 1$ $(i = 1,2,3,4,5,6)$. Now we consider the following Lyapunov function

$$\begin{aligned} V(t) = {} & e^{\beta t}\left[x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t)\right] \\ & + \bar{a}_{12}L^{2(1-\varepsilon_1)}\int_{t-\tau}^{t} e^{\beta(s+\tau)}x_2^2(s)ds + \bar{\alpha}L^{2(1-\varepsilon_2)}\int_{t-\tau}^{t} e^{\beta(s+\tau)}x_4^2(s)ds \\ & + \bar{a}_{21}L^{2(1-\varepsilon_3)}\int_{t-\tau}^{t} e^{\beta(s+\tau)}x_1^2(s)ds + \bar{a}_{12}L^{2(1-\varepsilon_4)}\int_{t-\tau}^{t} e^{\beta(s+\tau)}x_4^2(s)ds \\ & + \bar{\alpha}L^{2(1-\varepsilon_5)}\int_{t-\tau}^{t} e^{\beta(s+\tau)}x_2^2(s)ds + \bar{a}_{21}L^{2(1-\varepsilon_6)}\int_{t-\tau}^{t} e^{\beta(s+\tau)}x_3^2(s)ds, \end{aligned} \tag{9}$$

where β is given by Lemma 2.2. Differentiating $V(t)$ along solutions to system (2), together with (8), we have

$$\begin{aligned} \frac{dV(t)}{dt} \le {} & \beta e^{\beta t}\left[x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t)\right] \\ & + e^{\beta t}\left\{-2x_1^2(t) + \bar{a}_{12}\left[L^{2\varepsilon_1}x_1^2(t) + L^{2(1-\varepsilon_1)}x_2^2(t-\tau)\right]\right. \\ & \left. + \bar{\alpha}\left[L^{2\varepsilon_2}x_1^2(t) + L^{2(1-\varepsilon_2)}x_4^2(t-\tau)\right]\right\} \\ & + e^{\beta t}\left\{-2x_2^2(t) + \bar{a}_{21}\left[L^{2\varepsilon_3}x_2^2(t) + L^{2(1-\varepsilon_3)}x_1^2(t-\tau)\right]\right\} \\ & + e^{\beta t}\left\{-2x_3^2(t) + \bar{a}_{12}\left[L^{2\varepsilon_4}x_3^2(t) + L^{2(1-\varepsilon_4)}x_4^2(t-\tau)\right]\right. \\ & \left. + \bar{\alpha}\left[L^{2\varepsilon_5}x_3^2(t) + L^{2(1-\varepsilon_5)}x_2^2(t-\tau)\right]\right\} \\ & + e^{\beta t}\left\{-2x_4^2(t) + \bar{a}_{21}\left[L^{2\varepsilon_6}x_4^2(t) + L^{2(1-\varepsilon_6)}x_3^2(t-\tau)\right]\right\} \\ & + \bar{a}_{12}L^{2(1-\varepsilon_1)}\left[e^{\beta(t+\tau)}x_2^2(t) - e^{\beta t}x_2^2(t-\tau)\right] \\ & + \bar{\alpha}L^{2(1-\varepsilon_2)}\left[e^{\beta(t+\tau)}x_4^2(t) - e^{\beta t}x_4^2(t-\tau)\right] \\ & + \bar{a}_{21}L^{2(1-\varepsilon_3)}\left[e^{\beta(t+\tau)}x_1^2(t) - e^{\beta t}x_1^2(t-\tau)\right] \\ & + \bar{a}_{12}L^{2(1-\varepsilon_4)}\left[e^{\beta(t+\tau)}x_4^2(t) - e^{\beta t}x_4^2(t-\tau)\right] \\ & + \bar{\alpha}L^{2(1-\varepsilon_5)}\left[e^{\beta(t+\tau)}x_2^2(t) - e^{\beta t}x_2^2(t-\tau)\right] \\ & + \bar{a}_{21}L^{2(1-\varepsilon_6)}\left[e^{\beta(t+\tau)}x_3^2(t) - e^{\beta t}x_3^2(t-\tau)\right] \\ = {} & e^{\beta t}\left[\beta - 2 + \bar{a}_{12}L^{2\varepsilon_1} + \bar{\alpha}L^{2\varepsilon_2} + \bar{a}_{21}L^{2(1-\varepsilon_3)}e^{\beta\tau}\right]x_1^2(t) \\ & + e^{\beta t}\left[\beta - 2 + \bar{a}_{21}L^{2\varepsilon_3} + \left(\bar{a}_{12}L^{2(1-\varepsilon_1)} + \bar{\alpha}L^{2(1-\varepsilon_5)}\right)e^{\beta\tau}\right]x_2^2(t) \\ & + e^{\beta t}\left[\beta - 2 + \bar{a}_{12}L^{2\varepsilon_4} + \bar{\alpha}L^{2\varepsilon_5} + \bar{a}_{21}L^{2(1-\varepsilon_6)}e^{\beta\tau}\right]x_3^2(t) \\ & + e^{\beta t}\left[\beta - 2 + \bar{a}_{21}L^{2\varepsilon_6} + \left(\bar{a}_{12}L^{2(1-\varepsilon_4)} + \bar{\alpha}L^{2(1-\varepsilon_2)}\right)e^{\beta\tau}\right]x_4^2(t). \end{aligned} \tag{10}$$

It follows from Lemma 2.2 that $\frac{dV(t)}{dt} \leq 0$, which implies that $V(t) \leq V(0)$ for all $t > 0$. Thus,

$$\begin{aligned}
& e^{\beta t}\left[x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t)\right] \\
\leq & \left[x_1^2(0) + x_2^2(0) + x_3^2(0) + x_4^2(0)\right] \\
& + \bar{a}_{12} L^{2(1-\varepsilon_1)} \int_{-\tau}^{0} e^{\beta(s+\tau)} x_2^2(s)ds + \bar{\alpha} L^{2(1-\varepsilon_2)} \int_{-\tau}^{0} e^{\beta(s+\tau)} x_4^2(s)ds \\
& + \bar{a}_{21} L^{2(1-\varepsilon_3)} \int_{-\tau}^{0} e^{\beta(s+\tau)} x_1^2(s)ds + \bar{a}_{12} L^{2(1-\varepsilon_4)} \int_{-\tau}^{0} e^{\beta(s+\tau)} x_4^2(s)ds \\
& + \bar{\alpha} L^{2(1-\varepsilon_5)} \int_{-\tau}^{0} e^{\beta(s+\tau)} x_2^2(s)ds + \bar{a}_{21} L^{2(1-\varepsilon_6)} \int_{-\tau}^{0} e^{\beta(s+\tau)} x_3^2(s)ds \\
\leq & ||\varphi - \varphi^*||^2 + \theta \frac{1}{\beta} e^{\beta\tau} ||\varphi - \varphi^*||^2 \\
= & \left(1 + \theta \frac{1}{\beta} e^{\beta\tau}\right) ||\varphi - \varphi^*||^2, \qquad (11)
\end{aligned}$$

where

$$\theta = \left\{\bar{a}_{12} L^{2(1-\varepsilon_1)} + \bar{\alpha} L^{2(1-\varepsilon_5)}, \bar{a}_{21} L^{2(1-\varepsilon_3)}, \bar{a}_{21} L^{2(1-\varepsilon_6)}, \bar{a}_{12} L^{2(1-\varepsilon_4)} + \bar{\alpha} L^{2(1-\varepsilon_2)}\right\}.$$

Let

$$M = 1 + \theta \frac{1}{\beta} e^{\beta\tau} > 1.$$

Then Equation (11) can be rewritten as

$$x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) \leq M e^{-\beta t} ||\varphi - \varphi^*||^2$$

for all $t > 0$. Then

$$\sum_{i=1}^{4} |u_i(t) - u_i^*(t)|^2 \leq M e^{-\beta t} ||\varphi - \varphi^*||^2$$

for all $t > 0$. Thus, the solution $u(t)$ of system (2) is globally exponentially stable.

Theorem 3.2. *Assume that* (H1)-(H3) *are satisfied. Then system (2) has exactly one T-anti-periodic solution which is globally stable.*

Proof: It follows from system (2) and (H2) that for each $k \in N$, we have

$$\begin{aligned}
& \frac{d}{dt}\left[(-1)^{k+1} u_1(t + (k+1)T)\right] \\
= & (-1)^{k+1}\big[-u_1(t + (k+1)T) + a_{12}(t + (k+1)T) f(u_2(t + (k+1)T - \tau)) \\
& + \alpha(t + (k+1)T) f(u_4(t + (k+1)T - \tau))\big] \\
= & -(-1)^{k+1} u_1(t + (k+1)T) + a_{12}(t) f\left((-1)^{k+1} u_2(t + (k+1)T - \tau)\right) \\
& + \alpha(t) f\left((-1)^{k+1} u_4(t + (k+1)T - \tau)\right), \qquad (12)
\end{aligned}$$

$$\begin{aligned}
& \frac{d}{dt}\left[(-1)^{k+1} u_2(t + (k+1)T)\right] \\
= & (-1)^{k+1}\big[-u_2(t + (k+1)T) + a_{21}(t + (k+1)T) f(u_1(t + (k+1)T - \tau))\big] \\
= & -(-1)^{k+1} u_2(t + (k+1)T) + a_{21}(t) f\left((-1)^{k+1} u_1(t + (k+1)T - \tau)\right), \qquad (13)
\end{aligned}$$

$$
\begin{aligned}
&\frac{d}{dt}\left[(-1)^{k+1}u_3(t+(k+1)T)\right]\\
&= (-1)^{k+1}\big[-u_3(t+(k+1)T)+a_{12}(t+(k+1)T)f(u_4(t+(k+1)T-\tau))\\
&\quad +\alpha(t+(k+1)T)f(u_2(t+(k+1)T-\tau))\big]\\
&= -(-1)^{k+1}u_3(t+(k+1)T)+a_{12}(t)f\left((-1)^{k+1}u_4(t+(k+1)T-\tau)\right)\\
&\quad +\alpha(t)f\left((-1)^{k+1}u_2(t+(k+1)T-\tau)\right), \qquad (14)\\
&\frac{d}{dt}\left[(-1)^{k+1}u_4(t+(k+1)T)\right]\\
&= (-1)^{k+1}\big[-u_4(t+(k+1)T)+a_{21}(t+(k+1)T)f(u_3(t+(k+1)T-\tau))\big]\\
&= -(-1)^{k+1}u_4(t+(k+1)T)+a_{21}(t)f\left((-1)^{k+1}u_3(t+(k+1)T-\tau)\right). \qquad (15)
\end{aligned}
$$

Let $\bar{u}(t) = \left((-1)^{k+1}u_1(t+(k+1)T), (-1)^{k+1}u_2(t+(k+1)T), (-1)^{k+1}u_3(t+(k+1)T), (-1)^{k+1}u_4(t+(k+1)T)\right)^T$. Obviously, for any $k \in N$, $\bar{u}(t)$ is also the solution of system (2). If the initial function $\varphi_i(s)$ $(i = 1,2,3,4)$ is bounded, it follows from Theorem 3.1 that there exists a constant $\gamma > 1$ such that

$$
\begin{aligned}
&\left|(-1)^{k+1}u_i(t+(k+1)T)-(-1)^k u_i(t+kT)\right|\\
&\le Me^{-\beta(t+kT)}\sup_{-\tau\le s\le 0}\sum_{i=1}^{4}|u_i(t+T)+u_i(s)|^2\\
&\le \gamma e^{-\beta(t+kT)}, \qquad (16)
\end{aligned}
$$

where $t+kT>0$, $i=1,2,3,4$. Since for any $k \in N$ we have

$$
(-1)^{k+1}u_i(t+(k+1)T) = u_i(t)+\sum_{j=0}^{k}\left[(-1)^{j+1}u_i(t+(j+1)T)-(-1)^j u_i(t+jT)\right]. \qquad (17)
$$

Then

$$
(-1)^{k+1}u_i(t+(k+1)T) \le |u_i(t)|+\sum_{j=0}^{k}|(-1)^{j+1}u_i(t+(j+1)T)-(-1)^j u_i(t+jT)|. \qquad (18)
$$

By Lemma 2.3, we know that the solutions of system (2) are bounded. In view of (16) and (18), we can easily know that $\{(-1)^{k+1}u_i(t+(k+1)T)\}$ uniformly converges to a continuous function $u^*(t) = (u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t))^T$ on any compact set of R^4.

Now we show that $u^*(t)$ is T-anti-periodic solution of (2). Firstly, $u^*(t)$ is T-anti-periodic, since

$$
\begin{aligned}
u^*(t+T) &= \lim_{k\to\infty}(-1)^k u(t+T+kT)\\
&= -\lim_{(k+1)\to\infty}(-1)^{k+1}u(t+(k+1)T) = -u^*(t). \qquad (19)
\end{aligned}
$$

Thus, we can conclude that $u_i^*(t)$ is T-anti-periodic on R.

In the sequel, we prove that $u^*(t)$ is a solution of (2). Because of the continuity of the right-hand side of (2), we can conclude that $\{((-1)^{k+1}u_i(t+(k+1)T))'\}$ uniformly converges to a continuous function on any compact subset of R, respectively. Letting

$k \to \infty$, we can easily get

$$\begin{cases} \dfrac{du_1^*(t)}{dt} = -u_1^*(t) + a_{12}(t)f(u_2^*(t-\tau)) + \alpha(t)f(u_4^*(t-\tau)), \\ \dfrac{du_2^*(t)}{dt} = -u_2^*(t) + a_{21}(t)f(u_1^*(t-\tau)), \\ \dfrac{du_3^*(t)}{dt} = -u_3^*(t) + a_{12}(t)f(u_4^*(t-\tau)) + \alpha(t)f(u_2^*(t-\tau)), \\ \dfrac{du_4^*(t)}{dt} = -u_4^*(t) + a_{21}(t)f(u_3^*(t-\tau)). \end{cases} \tag{20}$$

Therefore, $u^*(t)$ is a solution of (2). Finally, by applying Theorem 3.1, it is easy to check that $u^*(t)$ is globally exponentially stable. This completes the proof of Theorem 3.2.

4. **An Example.** In this section, we give an example to illustrate our main results derived in previous sections. Consider the following delayed neural networks with unidirectional coupling

$$\begin{cases} \dfrac{du_1(t)}{dt} = -u_1(t) + (0.5\sin t)f(u_2(t-\tau)) - (2.5\sin t)f(u_4(t-\tau)), \\ \dfrac{du_2(t)}{dt} = -u_2(t) - (0.5\sin t)f(u_1(t-\tau)), \\ \dfrac{du_3(t)}{dt} = -u_3(t) + (0.5\sin t)f(u_4(t-\tau)) - (2.5\sin t)f(u_2(t-\tau)), \\ \dfrac{du_4(t)}{dt} = -u_4(t) - (0.5\sin t)f(u_3(t-\tau)). \end{cases} \tag{21}$$

Set $f(u) = \frac{1}{2}(|u+1| - |u-1|)$, $\tau = 0.03$. Then $L = M = 1$. It is easy to verify that all the conditions (H1)-(H3) hold. Thus, system (21) has exactly one π-anti-periodic solution which is globally exponentially stable. The results are illustrated in Figures 1-4. Figure 1 shows time history graph of $t - u_1(t)$; Figure 2 shows time history graph of $t - u_2(t)$; Figure 3 shows time history graph of $t - u_3(t)$; Figure 4 shows time history graph of $t - u_4(t)$.

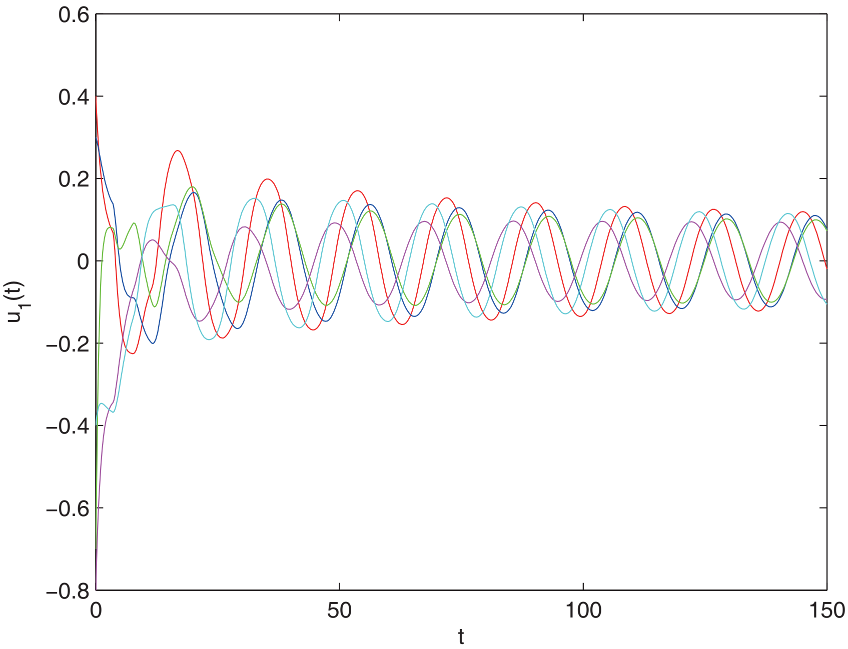

FIGURE 1. (color online) Transient response of state variables $u_1(t)$

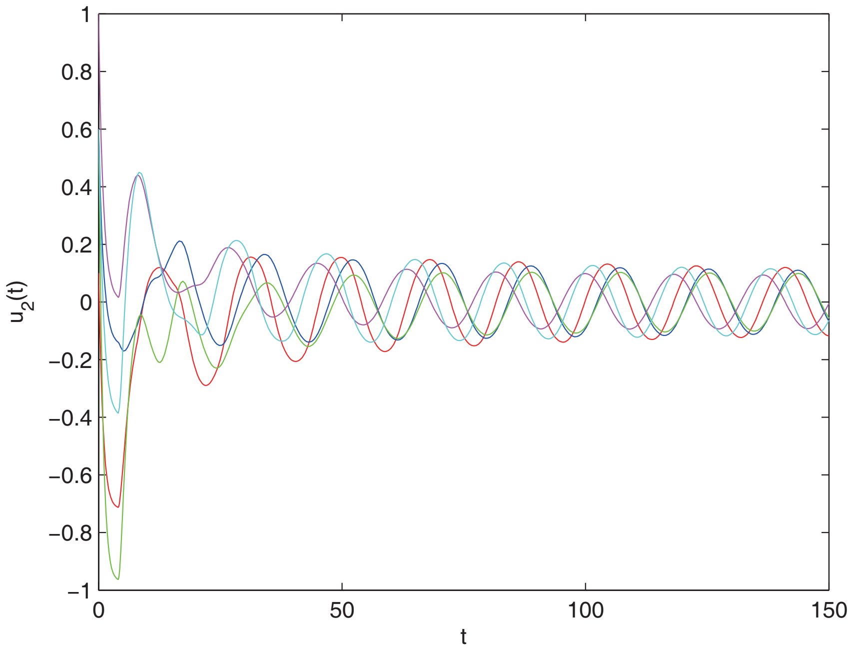

FIGURE 2. (color online) Transient response of state variables $u_2(t)$

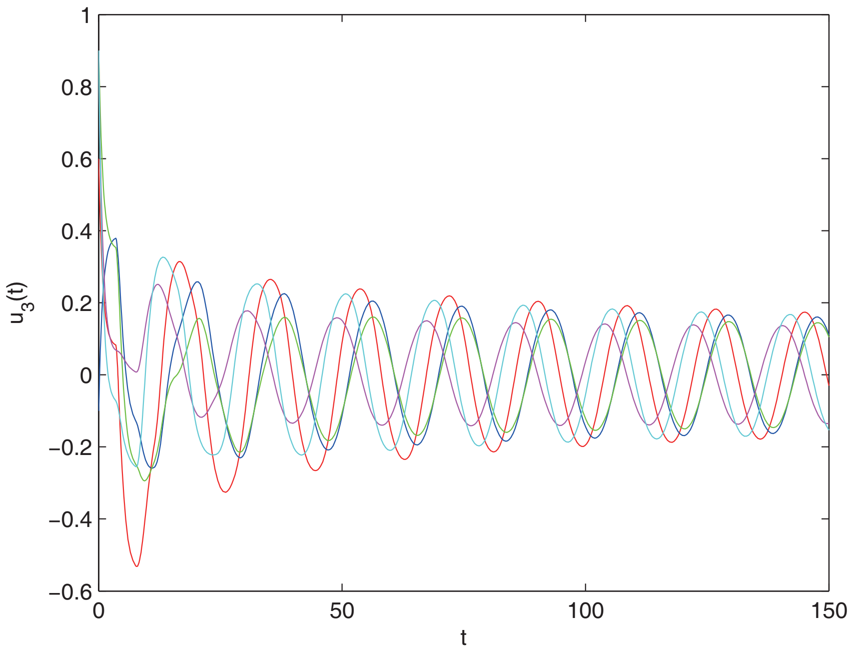

FIGURE 3. (color online) Transient response of state variables $u_3(t)$

From Figures 1-4, we can see that u_1, u_2, u_3, u_4 will keep anti-periodic oscillation with the increase of time t.

5. **Conclusions.** The anti-periodic oscillation plays an important role in characterizing the behavior of nonlinear differential equations. So it has been widely studied by numerous scholars in recent decades. In this article, we have considered the delayed neural networks with unidirectional coupling. Applying some analysis skills and Lyapunov method, we establish a set of sufficient conditions to ensure the existence and exponential stability of anti-periodic solutions of the delayed neural networks with unidirectional coupling. The

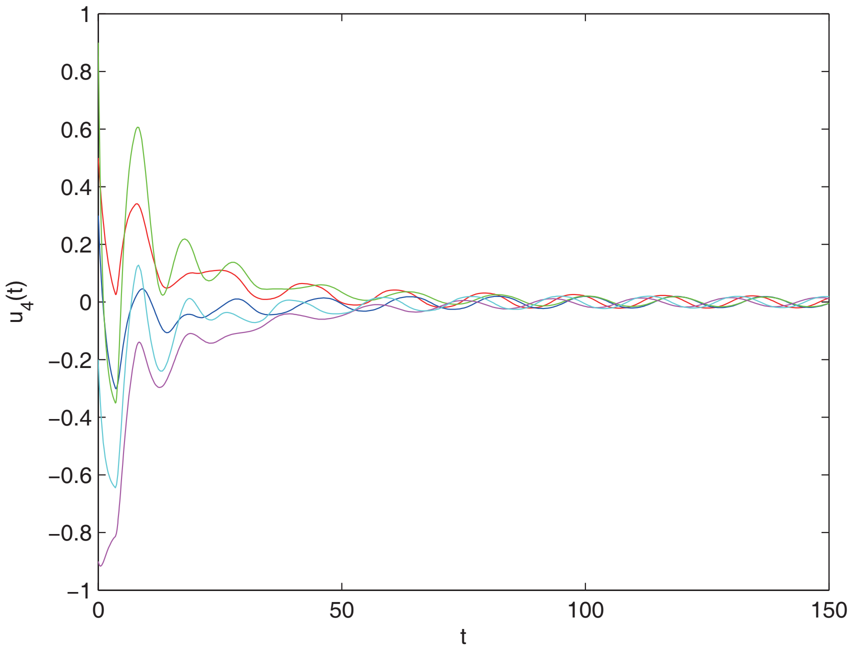

FIGURE 4. (color online) Transient response of state variables $u_4(t)$

derived results have theoretical significance in neural information processing, artificial intelligence, etc. In addition, the idea of this paper can be used to discuss many other network models.

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