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PIVOTAL INFERENCE FOR THE TWO-PARAMETER RAYLEIGH DISTRIBUTION BASED ON PROGRESSIVE FIRST-FAILURE CENSORING SCHEME

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Abstract. *In this paper, we consider the point estimators and the exact confidence intervals for the two-parameter Rayleigh distribution based on pivotal quantities with progressively first-failure censored data. The maximum likelihood estimation is introduced firstly. Then we derive the pivotal quantities to obtain the precise estimators of the location and scale parameters respectively under the progressive first-failure censoring. Also, we obtain the exact confidence intervals of the two parameters. We apply simulated data as well as a real data set to illustrate the proposed inference methods. The numerical results show that using pivotal quantities presented in this study, the inference of parameters for the two-parameter Rayleigh distribution with progressively first-failure censored data is effective.*
Keywords: Two-parameter Rayleigh distribution, Progressive first-failure censoring, Pivotal quantity, Point estimation, Exact interval inference

1. **Introduction.** In lifetime experiments, Rayleigh distribution is one of the most popular distributions since its failure function is monotonous which is regarded as a good feature. Lord Rayleigh (1880) introduced Rayleigh distribution, a special case of the well-known Weibull distribution taking 2 as the shape parameter (see [17]). Also, it has relations with chi-squared distribution and extreme value distribution. Many authors studied various aspects of the Rayleigh distribution. [10] considered the Bayesian estimation and prediction for the Rayleigh distribution based on Type-II censored data. By the natural conjugate family of priors, [6] concentrated on the prediction interval for the Rayleigh distribution. [5] compared the Bayes estimators for Rayleigh distribution under different loss function, squared error loss function and LINEX loss function. [14] obtained the maximum likelihood estimation and Bayesian estimation of the scale parameter for the Rayleigh distribution. Also, they addressed the highest posterior density (HPD) prediction interval. Rayleigh distribution is used widely in the industry and medical treatment, such as reliability experiments and survival analysis. In this case, more extensive distributions from the Rayleigh distribution are investigated. [12] mentioned some information of the two-parameter Rayleigh distribution.

The probability density function (PDF) of the two-parameter Rayleigh distribution is defined as follows, see Figure 1,

$$f(x;\mu,\sigma)=\frac{x-\mu}{\sigma^2}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\};\quad x>\mu,\ \sigma>0 \tag{1}$$

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and the cumulative distribution function (CDF) is given by

$$F(x; \mu, \sigma) = 1 - \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \tag{2}$$

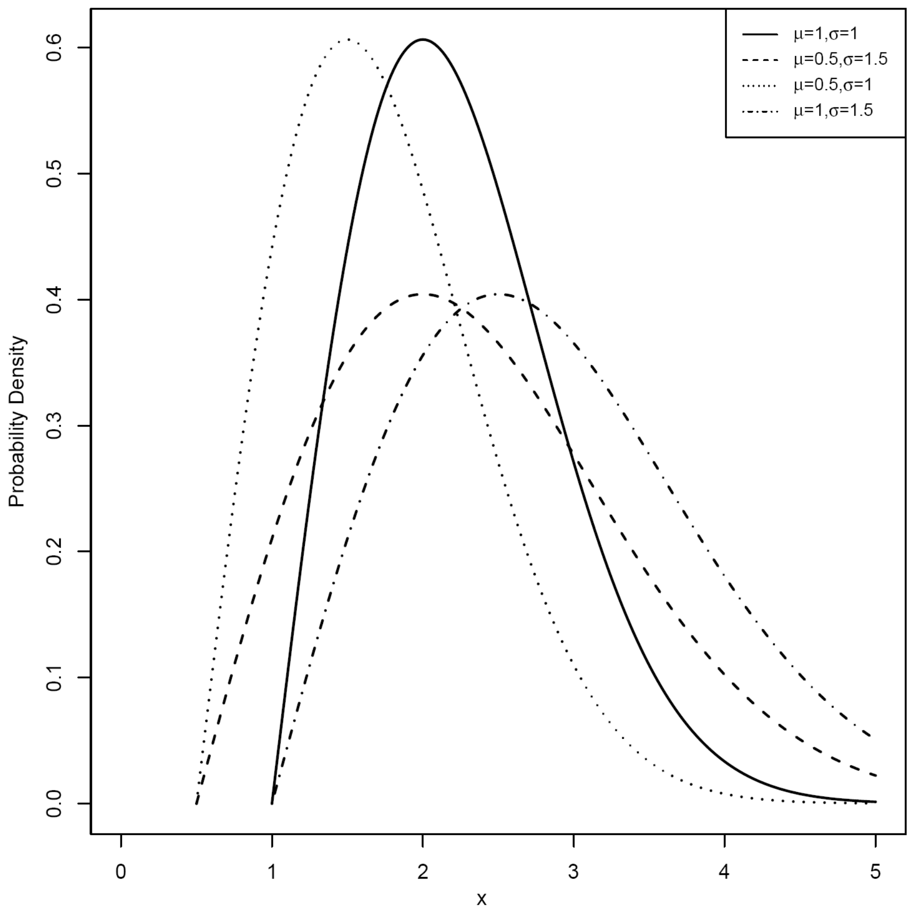

FIGURE 1. Probability density function of two-parameter Rayleigh distribution

Two-parameter Rayleigh distribution is the extended version of Rayleigh distribution. For it has two parameters, two-parameter Rayleigh distribution has more various forms and contains more information. In the manufacturing and service industry, many indicators obey two-parameter Rayleigh distribution, so the inference of the parameters is regarded as an important work for mass production. Furthermore, two-parameter Rayleigh distribution relates to the investigation of the wavelet, which is applied widely in the medical field based on threshold de-noising methods, such as [16] proposed. [7] did a lot of work on the inference for the two-parameter Rayleigh distribution. They proposed several different methods to derive the estimators of two parameters with complete samples, including maximum likelihood estimators, moments estimators, L-moment estimators, percentile based estimators and least squares estimators. Also, they considered Bayesian estimation as well, and concluded that the maximum likelihood estimators and Bayes estimators are more valid for all practical cases.

In practice, the life testing experiments often terminate before all the units fail. [11] introduced a censoring plan called the first-failure censoring. Then, [22] described the progressive first-failure censoring, an improved censoring plan. Instead of the complete sampling, the progressive first-failure censored sampling is often used in experiments to reduce time and cost. This censoring method has been studied by many authors. For example, [1] derived the maximum estimators and Bayes estimators for Rayleigh distribution under progressive first-failure censoring and [8] considered the estimators of the parameters for the generalized inverted exponential distribution based on the progressive first-failure censored data.

We can describe the progressive first-failure censoring scheme as follows. Suppose we have N units in total in the life-testing experiment and divide them into n independent groups with k items in each group. When the first failure is observed, discard this corresponding group and R_1 groups from the remaining $(n-1)$ groups randomly. When the second failure occurs, discard the group in which the second failure is observed and R_2 groups from the remaining $(n-R_1-2)$ groups randomly, and so on. This procedure continues until we observe the m-th failure and discard all the remaining groups finally. The number of the groups to be discarded each time $R=(R_1,R_2,\ldots,R_m)$ is fixed, called the censoring scheme. When $k=1$ in each group, this censoring becomes the progressive type-II censoring and it becomes the first-failure censoring when $R_1=R_2=\cdots=R_{m-1}=0$ and $R_m=n-m$.

Based on the pivotal quantity, we can derive the inference of the parameters. This method is proved to be a valid approach to simplifying the inference process when it is difficult to obtain the explicit solutions using maximum likelihood estimation method. [19] considered the pivotal inference for the scaled half logistic distribution with progressively Type-II censored data. They constructed pivot quantities which obey the chi-square distribution. Based on the progressively type-II censoring samples, [20] investigated the pivotal inference for the two-parameter half-logistic distribution. Using pivot quantities, they obtained an unbiased estimator of the location parameter and confidence intervals of the location and scale parameters. [9] presented the inverse moment estimation and joint confidence regions of parameters for exponentiated half logistic distribution based on pivot quantities which have chi-square distribution and F distribution. [18] derived some pivot quantities to obtain the estimation and prediction of exact intervals for the two-parameter Rayleigh distribution based on the Upper record values and prove that the performance of the pivotal inference is quite good. [15] derived the estimators for the Pareto distribution using a pivotal quantity. In addition, the study of the interval estimations based on the pivotal quantities was also discussed by [13, 21].

In this study, we investigate the inference of the location and scale parameters for the two-parameter Rayleigh distribution based on pivotal quantities with progressive first-failure censored data. The remainder of the paper is structured as follows. In Section 2, we introduce the maximum likelihood estimation briefly. Then we derive the pivotal quantities of parameters μ and σ. And we obtain the point and confidence interval estimations of two parameters based on pivotal quantities. In Section 3, we do the tests by Monte Carlo simulations as well as a real data set to assess the performance of the pivotal inference proposed. In Section 4, we draw the conclusions.

2. **Inference Based on Pivotal Quantities.**

2.1. **The likelihood function.** Suppose we have N units in total. Divide them into n $(n\geq m)$ independent groups and each group has k items. Let $X_{1:m:n:k}$, $X_{2:m:n:k},\ldots,$ $X_{m:m:n:k}$ be a progressive first-failure censored sample from the two-parameter Rayleigh distribution and the censoring scheme is denoted by $(R_1,R_2,\ldots,R_m)$. According to [2, 22], the likelihood function based on the progressive first-failure censored data is given by

$$L(\mu,\sigma;x_{1:m:n:k},\ldots,x_{m:m:n:k})=Ck^m\prod_{i=1}^{m}f(x_{i:m:n:k};\mu,\sigma)\left\{1-F(x_{i:m:n:k};\mu,\sigma)\right\}^{k(R_i+1)-1} \quad (3)$$

where $C = n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-m+1-R_1-\cdots-R_{m-1})$.

According to the functions in (1), (2) and (3), the likelihood function for the two-parameter Rayleigh distribution based on progressive first-failure censoring is shown as follows (using x_i instead of $x_{i:m:n:k}$)

$$L(\mu,\sigma) = Ck^m \prod_{i=1}^{m} \frac{x_i-\mu}{\sigma^2} \exp\left\{-\frac{k(R_i+1)(x_i-\mu)^2}{2\sigma^2}\right\} \tag{4}$$

The log-likelihood function is

$$l = \log L(\mu,\sigma) \propto \sum_{i=1}^{m} \log(x_i-\mu) - 2m\log\sigma - \frac{k}{2\sigma^2}\sum_{i=1}^{m}(R_i+1)(x_i-\mu)^2 \tag{5}$$

Normally, to obtain the estimators of μ and σ, we usually derive the MLEs $\hat{\mu}$ and $\hat{\sigma}$. In this method, we should solve the likelihood equations as follows

$$\frac{\partial l}{\partial \mu} = -\sum_{i=1}^{m}\frac{1}{x_i-\mu} + \frac{k}{\sigma^2}\sum_{i=1}^{m}(R_i+1)(x_i-\mu) = 0 \tag{6}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{2m}{\sigma} + \frac{k}{\sigma^3}\sum_{i=1}^{m}(R_i+1)(x_i-\mu)^2 = 0 \tag{7}$$

However, it is not easy to get explicit solution in this case. Alternatively, we could derive the estimators by pivotal quantities, a much more efficient and easier method.

2.2. **Point estimations based on pivotal quantities.** Suppose $X_{1:m:n:k},\ldots,X_{m:m:n:k}$ (using $X_1, X_2, \ldots, X_m$ instead) is the progressive first-failure censored sample with the censoring scheme $(R_1, R_2, \ldots, R_m)$ for the two-parameter Rayleigh distribution.

Let $Y_i = -\log(1-F(X_i)) = \frac{(X_i-\mu)^2}{2\sigma^2}$, so $Y_1 < Y_2 < \cdots < Y_m$ is a progressive first-failure censored sample from a standard exponential distribution and according to Equation (3), the joint probability density function is

$$\begin{aligned} f(y_1,y_2,\ldots,y_m) &= Ck^m \prod_{i=1}^{m} \exp\{-y_i\}\exp\{-y_i[k(R_i+1)-1]\} \\ &= Ck^m \exp\left\{-\sum_{i=1}^{m}[k(R_i+1)]y_i\right\} \end{aligned} \tag{8}$$

where $C = n(n-1-R_1)\cdots(n-m+1-R_1-\cdots-R_{m-1})$.

Then, we do the transformation as follows

$$T_1 = 2knY_1 \tag{9}$$

and

$$\begin{aligned} T_i &= 2k\left[n-\sum_{j=1}^{i-1}(R_j+1)\right](Y_i-Y_{i-1}) \\ &= 2k\left[m-i+1+\sum_{j=i}^{m}R_j\right](Y_i-Y_{i-1}) \qquad (i=2,\ldots,m) \end{aligned} \tag{10}$$

So, the progressive first-failure censored sample $Y_1, Y_2, \ldots, Y_m$ can be expressed as linear combinations of $T_1, T_2, \ldots, T_m$, shown as

$$\begin{aligned} Y_1 &= \frac{T_1}{2kn} \\ Y_2 &= \frac{T_1}{2kn} + \frac{T_2}{2k(n-1-R_1)} \\ &\vdots \\ Y_m &= \frac{T_1}{2kn} + \cdots + \frac{T_m}{2k(n-m+1-R_1-\cdots-R_{m-1})} \end{aligned}$$

Note that the determinant of the Jacobian obtained from the transformation is

$$J = \frac{1}{2^m k^m C} \tag{11}$$

We can derive the joint probability density function for $T_1, T_2, \ldots, T_m$ as

$$f(t_1, \ldots, t_m) = f(y_1, \ldots, y_m) \cdot |J| = \prod_{i=1}^{m} \left\{ \frac{1}{2} e^{-\frac{t_i}{2}} \right\} \tag{12}$$

$T_1, T_2, \ldots, T_m$ are independent and identically from a $\chi^2(2)$ distribution, so we have

$$T_1 = \frac{kn(X_1-\mu)^2}{\sigma^2} \sim \chi^2(2) \tag{13}$$

Let

$$U(\mu, \sigma) = \sum_{i=1}^{m} T_i = \frac{k \sum_{i=1}^{m} (1+R_i)(X_i-\mu)^2}{\sigma^2} \tag{14}$$

and

$$S(\mu, \sigma) = \sum_{i=2}^{m} T_i = U - T_1 = \frac{k \sum_{i=1}^{m} (1+R_i)(X_i-\mu)^2 - kn(X_1-\mu)^2}{\sigma^2} \tag{15}$$

So, on the basis of the property of the chi-square distribution, we know that $U \sim \chi^2(2m)$ and $S \sim \chi^2(2(m-1))$. Moreover, the pivotal quantities S and T_1 are independent.

Then, we obtain the pivotal quantity Q as follows

$$Q(\mu) = \frac{T_1/2}{S/\left[2(m-1)\right]} = \frac{(m-1)n(X_1-\mu)^2}{\sum_{i=1}^{m} (1+R_i)(X_i-\mu)^2 - n(X_1-\mu)^2} \sim F(2, 2(m-1)) \tag{16}$$

As we know, the expectations and variances of pivotal quantities U and Q from chi-square distribution are shown as follows

$$E(U(\mu, \sigma)) = 2m, \quad Var(U(\mu, \sigma)) = 4m \tag{17}$$

and

$$E(Q(\mu)) = \frac{2(m-1)}{2(m-1)-2} = \frac{m-1}{m-2}, \quad Var(Q(\mu)) = \frac{(m-1)^3}{(m-2)^2(m-3)} \tag{18}$$

Lemma 2.1. *The pivotal quantity $U(\mu, \sigma)/2m$ converges to 1 in probability as $m \to \infty$.*

Proof: Since $U(\mu, \sigma)$ has a chi-square distribution with $2m$ degrees of freedom, we have

$$E\left(\frac{U}{2m}\right) = \frac{E(U)}{2m} = 1 \tag{19}$$

and

$$Var\left(\frac{U}{2m}\right) = \frac{Var(U)}{(2m)^2} = \frac{1}{m} \to 0 \qquad \text{as} \quad m \to \infty \tag{20}$$

Given $\varepsilon > 0$, if m is large enough, using Chebyshev's inequality, we have

$$P\left(\left|\frac{U}{2m} - 1\right| > \varepsilon\right) = P\left(\left|\frac{U}{2m} - E\left(\frac{U}{2m}\right)\right| > \varepsilon\right) \leq \frac{Var\left(\frac{U}{2m}\right)}{\varepsilon^2} \to 0$$

as $m \to \infty$.

Lemma 2.2. *The pivotal quantity $Q(\mu)/(m-1)$ converges to $\frac{1}{m-2}$ in probability as $m \to \infty$.*

Proof: Since $Q(\mu)$ has an F distribution with 2 and $2(m-1)$ degrees of freedom,

$$E\left(\frac{Q}{m-1}\right) = \frac{E(Q)}{m-1} = \frac{1}{m-2} \tag{21}$$

and

$$Var\left(\frac{Q}{m-1}\right) = \frac{Var(Q)}{(m-1)^2} = \frac{m-1}{(m-2)^2(m-3)} \to 0 \tag{22}$$

as $m \to \infty$.

Given $\varepsilon > 0$, if m is large enough, using Chebyshev's inequality, we have

$$P\left(\left|\frac{Q}{m-1} - \frac{1}{m-2}\right| > \varepsilon\right) = P\left(\left|\frac{Q}{m-1} - E\left(\frac{Q}{m-1}\right)\right| > \varepsilon\right) \leq \frac{Var\left(\frac{Q}{m-1}\right)}{\varepsilon^2} \to 0$$

as $m \to \infty$.

By Lemma 2.1 and Lemma 2.2, we propose that

$$U(\mu, \sigma) = 2m \tag{23}$$

and

$$Q(\mu) = \frac{T_1/2}{S/\left[2(m-1)\right]} = \frac{m-1}{m-2} \tag{24}$$

Using Equations (24) and (23), we can derive the estimators of μ and σ by solving two equations

$$\frac{(m-1)n(X_1-\mu)^2}{\sum_{i=1}^m (1+R_i)(X_i-\mu)^2 - n(X_1-\mu)^2} = \frac{m-1}{m-2} \tag{25}$$

and

$$\frac{k\sum_{i=1}^m (1+R_i)(X_i-\mu)^2}{\sigma^2} = 2m \tag{26}$$

We obtain the estimators $\hat{\mu}$ and $\hat{\sigma}$ as

$$\hat{\mu} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{27}$$

where

$$A = (m-2)n,\ B = 2\left(\sum_{i=1}^m (1+R_i)X_i - nX_1(m-1)\right),\ C = (m-1)nX_1^2 - \sum_{i=1}^m (1+R_i)X_i^2$$

and

$$\hat{\sigma} = \sqrt{\frac{k\sum_{i=1}^m (1+R_i)(X_i-\mu)^2}{2m}} \tag{28}$$

Note that $\mu < X_1$, so we derive the estimator of μ as

$$\hat{\mu} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{29}$$

2.3. **Exact interval inference based on pivotal quantities.** In this subsection, we consider the exact confidence intervals of the location parameter μ and the scale parameter σ based on the pivotal quantities. According to Equations (14) and (16), we know that $U(\mu, \sigma) \sim \chi^2(2m)$ and $Q(\mu) \sim F(2, 2(m-1))$. We construct the exact $100(1-\alpha)\%$ two-sided confidence intervals for the parameters μ and σ based on the pivotal quantities as follows respectively

$$\left[Q^{-1}\left(F_{\frac{\alpha}{2}}(2, 2(m-1))\right), Q^{-1}\left(F_{1-\frac{\alpha}{2}}(2, 2(m-1))\right)\right] \tag{30}$$

and

$$\left[U^{-1}\left(\chi^2_{\frac{\alpha}{2}}(2m)\right), U^{-1}\left(\chi^2_{1-\frac{\alpha}{2}}(2m)\right)\right] \tag{31}$$

More clearly, the $100(1-\alpha)\%$ confidence interval for μ can be expressed as

$$\begin{aligned} 1 - \alpha &= P\left\{F_{\frac{\alpha}{2}}(2, 2(m-1)) < Q < F_{1-\frac{\alpha}{2}}(2, 2(m-1))\right\} \\ &= P\left\{F_{\frac{\alpha}{2}} < \frac{(m-1)n(X_1 - \mu)^2}{\sum_{i=1}^m (1+R_i)(X_i - \mu)^2 - n(X_1 - \mu)^2} < F_{1-\frac{\alpha}{2}}\right\} \\ &= P\left\{Lower < \mu < Upper\right\} \end{aligned} \tag{32}$$

where $F_{\frac{\alpha}{2}}$ and $F_{1-\frac{\alpha}{2}}$ are the short versions of $F_{\frac{\alpha}{2}}(2, 2(m-1))$ and $F_{1-\frac{\alpha}{2}}(2, 2(m-1))$ respectively and

$$\begin{aligned} Lower &= \max\left\{\frac{-b_1 - \sqrt{b_1^2 - 4a_1c_1}}{2a_1}, \frac{-b_2 + \sqrt{b_2^2 - 4a_2c_2}}{2a_2}\right\} \\ Upper &= \min\left\{\frac{-b_1 + \sqrt{b_1^2 - 4a_1c_1}}{2a_1}, \frac{-b_2 - \sqrt{b_2^2 - 4a_2c_2}}{2a_2}, X_1\right\} \\ a_1 &= a_2 = (m-1)n \\ b_1 &= 2F_{1-\frac{\alpha}{2}} \sum_{i=1}^m (1+R_i)X_i - 2\left(m - 1 + F_{1-\frac{\alpha}{2}}\right) nX_1 \\ c_1 &= \left(m - 1 + F_{1-\frac{\alpha}{2}}\right) nX_1^2 - F_{1-\frac{\alpha}{2}} \sum_{i=1}^m (1+R_i)X_i^2 \\ b_2 &= 2F_{\frac{\alpha}{2}} \sum_{i=1}^m (1+R_i)X_i - 2\left(m - 1 + F_{\frac{\alpha}{2}}\right) nX_1 \\ c_2 &= \left(m - 1 + F_{\frac{\alpha}{2}}\right) nX_1^2 - F_{\frac{\alpha}{2}} \sum_{i=1}^m (1+R_i)X_i^2 \end{aligned}$$

The $100(1-\alpha)\%$ confidence interval for σ according to Equation (31) is shown as

$$\begin{aligned} 1 - \alpha &= P\left\{\chi^2_{\frac{\alpha}{2}}(2m) < U < \chi^2_{1-\frac{\alpha}{2}}(2m)\right\} \\ &= P\left\{\chi^2_{\frac{\alpha}{2}}(2m) < \frac{k\sum_{i=1}^m (1+R_i)(X_i - \mu)^2}{\sigma^2} < \chi^2_{1-\frac{\alpha}{2}}(2m)\right\} \\ &= P\left\{\sqrt{\frac{k\sum_{i=1}^m (1+R_i)(X_i - \mu)^2}{\chi^2_{1-\frac{\alpha}{2}}(2m)}} < \sigma < \sqrt{\frac{k\sum_{i=1}^m (1+R_i)(X_i - \mu)^2}{\chi^2_{\frac{\alpha}{2}}(2m)}}\right\} \end{aligned} \tag{33}$$

So for any $0 < \alpha < 1$, an exact $100(1-\alpha)\%$ confidence interval for μ based on the pivotal quantity is

$$\left[\max\left\{\frac{-b_1-\sqrt{b_1^2-4a_1c_1}}{2a_1}, \frac{-b_2+\sqrt{b_2^2-4a_2c_2}}{2a_2}\right\},\right.$$
$$\left.\min\left\{\frac{-b_1+\sqrt{b_1^2-4a_1c_1}}{2a_1}, \frac{-b_2-\sqrt{b_2^2-4a_2c_2}}{2a_2}, X_1\right\}\right] \tag{34}$$

and the exact $100(1-\alpha)\%$ confidence interval for σ can be constructed as

$$\left[\sqrt{\frac{k\sum_{i=1}^{m}(1+R_i)(X_i-\mu)^2}{\chi^2_{1-\frac{\alpha}{2}}(2m)}}, \sqrt{\frac{k\sum_{i=1}^{m}(1+R_i)(X_i-\mu)^2}{\chi^2_{\frac{\alpha}{2}}(2m)}}\right] \tag{35}$$

Note that the exact confidence interval of the scale parameter σ depends on the location parameter μ, a nuisance parameter. So we use a method based on the generalized pivotal quantity to solve this issue. See the example in [18].

Let μ^* be the unique solution of $Q(\mu) = Q$, where the random variable Q has an F distribution with 2 and $2(m-1)$ degrees of freedom. So we address the nuisance parameter μ based on the generalized pivotal quantity as

$$\mu^* = \frac{-B^* - \sqrt{B^{*2}-4A^*C^*}}{2A^*} \tag{36}$$

where

$$A^* = (m-1)n,\ B^* = 2Q\sum_{i=1}^{m}(1+R_i)X_i - 2(m-1+Q)nX_1,$$
$$C^* = (m-1+Q)nX_1^2 - Q\sum_{i=1}^{m}(1+R_i)X_i^2$$

Then, let U be the random variable from the chi-square distribution with $2m$ degrees of freedom. The generalized pivotal quantity from the pivotal quantity $U(\sigma)$ is shown as

$$V(\mu^*) = \sqrt{\frac{k\sum_{i=1}^{m}(1+R_i)(X_i-\mu^*)^2}{U}} \tag{37}$$

So we generate N (≥ 1000) random samples $Q_1, Q_2, \ldots, Q_N$ and $U_1, U_2, \ldots, U_N$ from distributions $F(2, 2(m-1))$ and $\chi^2(2m)$ respectively. Then we compute $V(\mu^*)_1, V(\mu^*)_2, \ldots, V(\mu^*)_N$ and sort them from small to large. Thus, the exact $100(1-\alpha)\%$ confidence interval for the scale parameter σ based on the generalized pivotal quantity $V(\mu^*)$ can be constructed as:

$$\left[V(\mu^*)_{[(N/100)\times\alpha/2]}, V(\mu^*)_{[(N/100)\times(1-\alpha/2)]}\right] \tag{38}$$

where $[t]$ denotes the largest integer less than or equal to t.

3. **Application.** In this section, we give numerical examples to illustrate the proposed inference methods. First, we do the Monte Carlo simulation study. We compute the estimates and exact confidence intervals (CIs) of the parameters μ and σ. To verify the validity of the inference based on the pivotal quantities derived in this paper, we report the biases and mean square errors (MSEs) of the point estimates while the coverage percentages (CPs) and average lengths (ALs) of the exact confidence intervals. Then we use a real data set from another article for more discussions on the application of the pivotal inference. Also, we compute the maximum likelihood estimates (MLEs) to prove

the accuracy of the pivotal inference results with real data. All calculations are performed using the statistical software R.

3.1. **Simulation study.** Here, we assess the proposed methods by a Monte Carlo simulation study for different combinations of (k, n, m, R) and different true values of location and scale parameters (μ, σ). According to Figure 1, the plots of the two-parameter Rayleigh distribution with different parameters have a large variation. The simulation study should take different combinations of two parameters. So we generate progressive first-failure censored samples from the Rayleigh distribution with $\mu = 1$, $\sigma = 1$ and $\mu = 0.5$, $\sigma = 1.5$ respectively according to the algorithm given by [3]. Obviously, the plots with two parameter combinations show the different forms in Figure 1. To explore the validity of the proposed pivotal inference under different cases, we set different sample sizes n $(n = 60, 100, 200)$, different m $(m = 20, 30, 40, 50, 100)$, different k $(k = 1, 2, 5)$ and different censoring schemes. Note that the censoring schemes in this article have been shown by short notations such as $(1 * 3)$ denoted $(1, 1, 1)$.

We estimate the parameters μ and σ using Equations (29) and (28) and obtain the biases and mean square errors (MSEs) respectively over 1000 replications. Also, the exact 95% confidence intervals (CIs) of μ and σ are computed based on pivotal quantities using Equations (34) and (38), and we report the coverage percentages (CPs) and average lengths (ALs) over 1000 replications. All the results are shown in Table 1 and Table 2. Meanwhile, we select six combinations of n, m and k under $\mu = 1$, $\sigma = 1$ to check the performance of the progressive first-failure censoring. According to Table 1, we plot the corresponding MSEs $(\times 10^{-2})$ of parameters and ALs of the confidence intervals with four different censoring schemes for each combination form, shown from Figure 2 to Figure 13.

From Figure 2 to Figure 13, we find that the MSEs of two parameters become lower when the sample sizes n, m and k become larger, and the ALs of confidence intervals are shorter with larger sample sizes. In practice, we prefer lower errors and shorter average lengths of confidence intervals, so we are supposed to use sample observations as many as possible to obtain better results. However, it is impossible to use the complete samples sometimes because the number of data is quite large. Instead, we can use progressive first-failure censoring method by dividing the large samples into groups randomly and delete the groups and data based on the algorithm proposed. On the other side, the values of MSEs and ALs based on the scheme 1 and scheme 4 are smaller than the values based on scheme 2 and scheme 3, showing that it is better to discard groups at earlier stages. Depending on the biases listed in Table 1 and Table 2, we know that compared with the true values of μ and σ, the estimates of μ are slightly lower while the estimates of σ are slightly larger. In terms of the MSEs, we find that MSEs of $\hat{\mu}$ are lower than $\hat{\sigma}$. However, generally, even for small sample sizes, the performance of point estimates for both μ and σ based on the proposed pivotal quantities is quite good, because the biases and MSEs of two parameters are very small, indicating that the inference method by using pivotal quantities (16) and (14) is applicable. The CPs of exact confidence intervals are approximately equal to 95% and ALs are narrow, which shows that the exact confidence interval inference based on the pivotal quantities proposed is satisfactory.

In general, the proposed pivotal inference based on the progressive first-failure censored data is effective and precise.

3.2. **Real example.** Now, we consider a real example for illustration purpose.

[4] provided a real data set showing the survival times (in years) of 46 patients who received chemotherapy treatment, and indicated that the Rayleigh distribution is a good fitted model for this data set. This data set is shown ascendingly in Table 3.

TABLE 1. Simulation results of the location parameter μ and scale parameter σ based on the pivotal quantities when $\mu = 1$ and $\sigma = 1$

k	n	m	censoring scheme R	$Bias(\mu)$	MSE($\hat{\mu}$)	CP(μ)	AL(μ)	$Bias(\sigma)$	MSE($\hat{\sigma}$)	CP(σ)	AL(σ)
1	60	20	$(40, 0*19)$	−0.02555480	0.01025402	0.948	0.3865644	0.01750024	0.01899363	0.961	0.5613497
			$(0*19, 40)$	−0.02922196	0.01246532	0.951	0.440093	0.02357843	0.02873595	0.956	0.7172582
			$(2*20)$	−0.02625193	0.01190963	0.957	0.4273459	0.02116390	0.02683759	0.952	0.6744886
			$(4*10, 0*10)$	−0.03029477	0.01061416	0.946	0.4130065	0.02388862	0.02306815	0.948	0.6328319
		30	$(30, 0*29)$	−0.02443983	0.01084598	0.957	0.3827941	0.01197175	0.01461458	0.958	0.4691948
			$(0*29, 30)$	−0.02953131	0.01108057	0.949	0.4109853	0.02922284	0.01870743	0.954	0.5507573
			$(1*30)$	−0.03031938	0.01029958	0.955	0.4038235	0.02377045	0.01759375	0.95	0.5276325
			$(3*10, 0*20)$	−0.02756324	0.01012196	0.958	0.3899381	0.01811192	0.01678784	0.955	0.4934792
		40	$(20, 0*39)$	−0.02563377	0.009982662	0.947	0.3748754	0.01649679	0.010521486	0.946	0.4092559
			$(0*39, 20)$	−0.02552188	0.01000724	0.944	0.394475	0.01617708	0.01311114	0.946	0.4587271
			$(1*20, 0*20)$	−0.02403958	0.01005314	0.942	0.3856729	0.01698599	0.01233604	0.954	0.4343874
			$(2*10, 0*30)$	−0.02633723	0.009410344	0.945	0.3801059	0.01506021	0.011116294	0.943	0.4235429
	100	50	$(50, 0*49)$	−0.022086128	0.005381191	0.949	0.2824965	0.009047019	0.008074270	0.941	0.3479576
			$(0*49, 50)$	−0.02222657	0.006043078	0.956	0.2990896	0.01902362	0.011193558	0.949	0.4089273
			$(1*50)$	−0.01976102	0.005734079	0.951	0.2941569	0.01579815	0.009652374	0.948	0.3923275
			$(5*10, 0*40)$	−0.01615296	0.005751541	0.952	0.2863385	0.01102122	0.008317381	0.942	0.3623209
	200	100	$(100, 0*99)$	−0.01114894	0.002529202	0.953	0.1916482	0.00455239	0.003657750	0.939	0.2381855
			$(0*99, 100)$	−0.01155564	0.002879738	0.95	0.1983348	0.01219668	0.005309655	0.952	0.2770386
			$(1*100)$	−0.01530873	0.002783208	0.951	0.196825	0.01106487	0.004903893	0.952	0.2667271
			$(5*20, 0*80)$	−0.014908511	0.002789614	0.946	0.1941354	0.009129241	0.004093604	0.951	0.2499818
2	60	20	$(40, 0*19)$	−0.02245897	0.005020547	0.952	0.2757267	0.01270227	0.018128197	0.952	0.5664879
			$(0*19, 40)$	−0.02672340	0.006069961	0.948	0.3142829	0.03661732	0.029047489	0.947	0.7233965
			$(2*20)$	−0.02033551	0.005816931	0.964	0.3035513	0.02665824	0.028276799	0.949	0.6769999
			$(4*10, 0*10)$	−0.02051114	0.00569644	0.952	0.2903611	0.02427450	0.02402728	0.951	0.6290903
		30	$(30, 0*29)$	−0.02034427	0.005106992	0.947	0.2685868	0.01182119	0.012037739	0.954	0.4641238
			$(0*29, 30)$	−0.01801006	0.005586321	0.944	0.2879806	0.01750749	0.017722836	0.951	0.5469236
			$(1*30)$	−0.01744011	0.005250072	0.946	0.2840598	0.01650828	0.017095409	0.955	0.5264423
			$(3*10, 0*20)$	−0.02411149	0.005120091	0.952	0.2773508	0.01909077	0.016288148	0.945	0.4960408
		40	$(20, 0*39)$	−0.015287205	0.004915872	0.945	0.2674775	0.007169467	0.010468549	0.953	0.4131591
			$(0*39, 20)$	−0.01579758	0.005176539	0.952	0.2775718	0.01681744	0.013474503	0.951	0.4569737
			$(1*20, 0*20)$	−0.01874726	0.005020642	0.949	0.2749155	0.01551905	0.011218572	0.957	0.4376138
			$(2*10, 0*30)$	−0.02122035	0.005011276	0.951	0.2703967	0.01861771	0.010851480	0.961	0.4257562
	100	50	$(50, 0*49)$	−0.01353161	0.00291457	0.951	0.199386	0.01299448	0.00802312	0.941	0.3474466
			$(0*49, 50)$	−0.01386188	0.003014488	0.958	0.2116789	0.01677425	0.011184027	0.961	0.4086002
			$(1*50)$	−0.01366499	0.002789547	0.951	0.2083094	0.01316133	0.010193964	0.952	0.3925486
			$(5*10, 0*40)$	−0.01562199	0.003121823	0.951	0.2028516	0.01432780	0.008793090	0.944	0.3636832
	200	100	$(100, 0*99)$	−0.01112689	0.001260453	0.962	0.1356127	0.01014019	0.003672714	0.954	0.2385124
			$(0*99, 100)$	−0.01115767	0.001336971	0.948	0.1410147	0.01327680	0.005057728	0.949	0.2780136
			$(1*100)$	−0.009132374	0.001434808	0.959	0.1403462	0.011566374	0.004715952	0.95	0.2693214
			$(5*20, 0*80)$	−0.010697643	0.001292663	0.948	0.1369405	0.008086919	0.003959498	0.943	0.2494089
5	60	20	$(40, 0*19)$	−0.01308379	0.001959534	0.95	0.1729934	0.01562159	0.018889497	0.946	0.5626438
			$(0*19, 40)$	−0.01516849	0.002646048	0.951	0.1977435	0.03795912	0.030563225	0.944	0.7179346
			$(2*20)$	−0.01348718	0.002355904	0.951	0.1906584	0.02891513	0.026882569	0.949	0.6736661
			$(4*10, 0*10)$	−0.01509883	0.00227469	0.95	0.1854678	0.03548745	0.02410912	0.946	0.634636
		30	$(30, 0*29)$	−0.01224510	0.002106594	0.949	0.1712656	0.01495641	0.013888650	0.942	0.4682802
			$(0*29, 30)$	−0.01236113	0.002171158	0.953	0.1824322	0.02597863	0.018525526	0.951	0.5471531
			$(1*30)$	−0.01579440	0.002114156	0.947	0.1787314	0.02675868	0.016923373	0.954	0.5220961
			$(3*10, 0*20)$	−0.01294744	0.002027678	0.952	0.1746953	0.02076489	0.014747580	0.95	0.492998
		40	$(20, 0*39)$	−0.01181546	0.002007477	0.941	0.1682397	0.01408755	0.011743743	0.944	0.4102144
			$(0*39, 20)$	−0.01388052	0.002118843	0.949	0.1759298	0.02563276	0.013446681	0.951	0.4579793
			$(1*20, 0*20)$	−0.007968204	0.002010036	0.956	0.1717421	0.011207648	0.011488030	0.948	0.4313804
			$(2*10, 0*30)$	−0.008825889	0.002044704	0.95	0.1707401	0.015218632	0.011920983	0.957	0.4248425
	100	50	$(50, 0*49)$	−0.009991559	0.001060785	0.946	0.1261594	0.012236802	0.007354959	0.948	0.3479323
			$(0*49, 50)$	−0.009185852	0.001214258	0.948	0.132495	0.020905413	0.011728961	0.962	0.4046604
			$(1*50)$	−0.009325096	0.001139876	0.956	0.1312662	0.019262682	0.010108615	0.953	0.3894054
			$(5*10, 0*40)$	−0.00944169	0.001111386	0.946	0.1280554	0.01363894	0.009168792	0.951	0.3636775
	200	100	$(100, 0*99)$	−0.005272802	0.0005422386	0.952	0.08593828	0.005246201	0.0039890148	0.958	0.2389974
			$(0*99, 100)$	−0.006201037	0.0005567293	0.953	0.08910946	0.009561686	0.0049555002	0.968	0.2775163
			$(1*100)$	−0.005920168	0.0005590636	0.958	0.08795682	0.010934677	0.0047461390	0.951	0.2670563
			$(5*20, 0*80)$	−0.005556307	0.0005554492	0.948	0.08681844	0.012706641	0.0043132964	0.957	0.2497196

TABLE 2. Simulation results of the location parameter μ and scale parameter σ based on the pivotal quantities when $\mu = 0.5$ and $\sigma = 1.5$

k	n	m	censoring scheme R	$Bias(\mu)$	MSE($\hat{\mu}$)	CP(μ)	AL(μ)	$Bias(\sigma)$	MSE($\hat{\sigma}$)	CP(σ)	AL(σ)
1	60	20	$(40, 0*19)$	−0.04257607	0.02215878	0.944	0.5844858	0.01631180	0.04216108	0.959	0.8516896
			$(0*19, 40)$	−0.0496086	0.02842874	0.953	0.6649048	0.0486235	0.06691637	0.946	1.079914
			$(2*20)$	−0.04927846	0.02654397	0.951	0.6418047	0.04231967	0.06165199	0.939	1.015625
			$(4*10, 0*10)$	−0.04395464	0.02551892	0.948	0.623464	0.03188296	0.05299528	0.94	0.9542778
		30	$(30, 0*29)$	−0.03837561	0.02133491	0.954	0.5692202	0.01767630	0.02793913	0.948	0.6959507
			$(0*29, 30)$	−0.04165420	0.02478876	0.959	0.6127081	0.03282918	0.04032535	0.949	0.8219353
			$(1*30)$	−0.04739232	0.02300823	0.947	0.6047833	0.03742650	0.03687568	0.944	0.7902139
			$(3*10, 0*20)$	−0.04545496	0.02137672	0.952	0.584126	0.03549118	0.03377651	0.948	0.736571
		40	$(20, 0*39)$	−0.04034188	0.02150454	0.947	0.5650577	0.02453350	0.02286195	0.954	0.6169181
			$(0*39, 20)$	−0.03462147	0.02277376	0.95	0.5889992	0.02390159	0.03048654	0.945	0.6860619
			$(1*20, 0*20)$	−0.04372222	0.02335921	0.944	0.5757702	0.02580068	0.02799871	0.956	0.6482578
			$(2*10, 0*30)$	−0.03829424	0.02265494	0.939	0.5729044	0.01324432	0.02479207	0.953	0.6385658
	100	50	$(50, 0*49)$	−0.030587643	0.01197420	0.954	0.4210589	0.009764954	0.01712393	0.947	0.5190569
			$(0*49, 50)$	−0.02949943	0.01469154	0.952	0.4481184	0.02665890	0.02346939	0.944	0.6115507
			$(1*50)$	−0.02949913	0.01359985	0.958	0.443833	0.02049695	0.02249178	0.949	0.5907913
			$(5*10, 0*40)$	−0.03370589	0.01285907	0.952	0.4307403	0.01967543	0.01777841	0.948	0.5462305
	200	100	$(100, 0*99)$	−0.017418383	0.005746619	0.945	0.2873075	0.008114552	0.008242977	0.959	0.3574692
			$(0*99, 100)$	−0.01676818	0.005973011	0.942	0.2991728	0.01635058	0.010784670	0.95	0.4165813
			$(1*100)$	−0.01994611	0.006309988	0.95	0.2956018	0.01969909	0.010788533	0.948	0.4002543
			$(5*20, 0*80)$	−0.02004627	0.006184999	0.949	0.290464	0.01579616	0.010094916	0.944	0.3739375
2	60	20	$(40, 0*19)$	−0.03031515	0.01138817	0.947	0.4097451	0.02009257	0.04129448	0.953	0.8414394
			$(0*19, 40)$	−0.03956169	0.01452143	0.951	0.4634682	0.05568164	0.07001995	0.958	1.065591
			$(2*20)$	−0.03015643	0.01418671	0.95	0.4539654	0.03523138	0.06333165	0.952	1.015298
			$(4*10, 0*10)$	−0.02609173	0.01284423	0.952	0.4388296	0.02339752	0.05457143	0.956	0.9499343
		30	$(30, 0*29)$	−0.03269347	0.01202904	0.949	0.4031859	0.01135921	0.02984815	0.952	0.6980876
			$(0*29, 30)$	−0.03152345	0.01343595	0.941	0.4341795	0.04013396	0.04432021	0.949	0.8228421
			$(1*30)$	−0.02866373	0.01187176	0.949	0.4304188	0.02591557	0.03841955	0.948	0.7953024
			$(3*10, 0*20)$	−0.02828296	0.01214472	0.952	0.4159078	0.02299327	0.03572808	0.949	0.7440399
		40	$(20, 0*39)$	−0.02413764	0.01107759	0.955	0.3995503	0.02285578	0.02520806	0.945	0.6171057
			$(0*39, 20)$	−0.02311399	0.01227145	0.953	0.4180024	0.02569393	0.02886145	0.944	0.6895156
			$(1*20, 0*20)$	−0.02478395	0.01084932	0.943	0.4086846	0.02151086	0.02805721	0.945	0.6525598
			$(2*10, 0*30)$	−0.02816801	0.01090248	0.951	0.4051043	0.02974152	0.02649455	0.957	0.6384565
	100	50	$(50, 0*49)$	−0.01954276	0.006374518	0.951	0.3003376	0.01735680	0.018275028	0.944	0.5236309
			$(0*49, 50)$	−0.02036315	0.006844358	0.954	0.3155422	0.02755562	0.023214318	0.949	0.6089396
			$(1*50)$	−0.02113483	0.007121828	0.946	0.311577	0.02733789	0.024194837	0.957	0.5864829
			$(5*10, 0*40)$	−0.02144221	0.006580165	0.955	0.3038696	0.02024269	0.019477161	0.947	0.5449425
	200	100	$(100, 0*99)$	−0.01290045	0.002988489	0.955	0.2030536	0.01286536	0.008306989	0.946	0.3573355
			$(0*99, 100)$	−0.01025991	0.003054806	0.956	0.2108915	0.01352083	0.011311125	0.954	0.4154607
			$(1*100)$	−0.01479320	0.003055052	0.955	0.2096771	0.01638298	0.010684340	0.954	0.401827
			$(5*20, 0*80)$	−0.01171053	0.003138003	0.951	0.2053361	0.01392055	0.010189218	0.953	0.3735766
5	60	20	$(40, 0*19)$	−0.01883406	0.004812109	0.952	0.2599334	0.02413902	0.043226342	0.947	0.8448925
			$(0*19, 40)$	−0.01909121	0.005329908	0.952	0.2949481	0.03855028	0.065144997	0.956	1.068722
			$(2*20)$	−0.01699683	0.00502995	0.952	0.2877295	0.02607413	0.05646718	0.948	1.016696
			$(4*10, 0*10)$	−0.02420252	0.004872146	0.951	0.2792151	0.04635826	0.054471881	0.955	0.9541897
		30	$(30, 0*29)$	−0.01931787	0.004361577	0.946	0.25528	0.01505616	0.030643521	0.948	0.6985807
			$(0*29, 30)$	−0.01909652	0.004860025	0.95	0.2756281	0.02826542	0.040688951	0.957	0.8264042
			$(1*30)$	−0.01989897	0.004880958	0.951	0.2687869	0.03368363	0.039410037	0.947	0.7854291
			$(3*10, 0*20)$	−0.01831909	0.004692707	0.954	0.262254	0.02360349	0.036095983	0.941	0.7414038
		40	$(20, 0*39)$	−0.01297138	0.004408639	0.956	0.2520388	0.01713806	0.021914064	0.947	0.6147063
			$(0*39, 20)$	−0.01781543	0.004591847	0.959	0.2629177	0.02379714	0.028720711	0.954	0.6846461
			$(1*20, 0*20)$	−0.01431444	0.004491311	0.959	0.256305	0.02162295	0.026409067	0.951	0.6463899
			$(2*10, 0*30)$	−0.01613159	0.004341664	0.954	0.2551316	0.02324217	0.025662260	0.948	0.6347906
	100	50	$(50, 0*49)$	−0.01488999	0.002293015	0.954	0.1892637	0.01990810	0.018410435	0.95	0.5220009
			$(0*49, 50)$	−0.01446577	0.002684125	0.955	0.2000313	0.03462407	0.025412324	0.945	0.6100712
			$(1*50)$	−0.01292869	0.00247797	0.954	0.1974987	0.01788351	0.02080074	0.944	0.5870104
			$(5*10, 0*40)$	−0.01220493	0.002499114	0.943	0.1932013	0.01580369	0.017328837	0.953	0.5491827
	200	100	$(100, 0*99)$	−0.008281997	0.001182812	0.941	0.1283314	0.011074941	0.008220872	0.955	0.3570408
			$(0*99, 100)$	−0.00870221	0.00126759	0.953	0.1341201	0.01481787	0.01105003	0.954	0.418198
			$(1*100)$	−0.009662421	0.001275237	0.949	0.1323588	0.016183495	0.010835396	0.944	0.4011556
			$(5*20, 0*80)$	−0.007313681	0.001181450	0.946	0.1303568	0.008516027	0.009321047	0.948	0.3753731

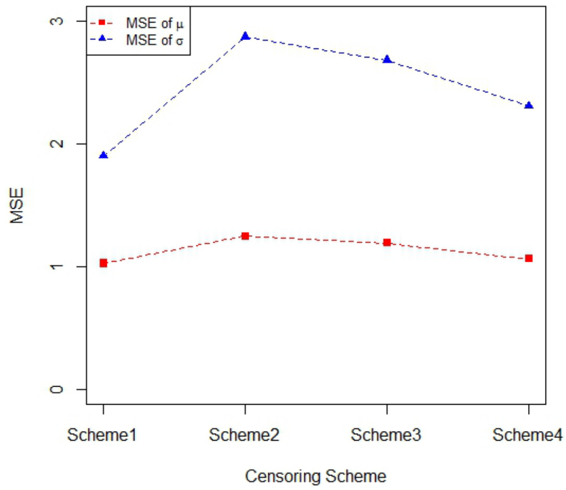

FIGURE 2. MSEs $(\times 10^{-2})$ when $k = 1$, $n = 60$, $m = 20$

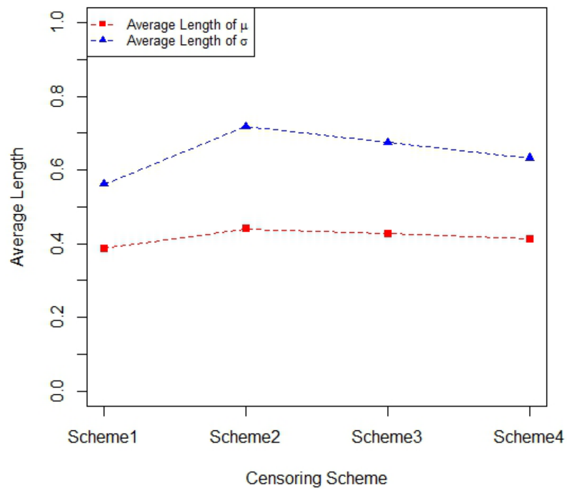

FIGURE 3. ALs when $k = 1$, $n = 60$, $m = 20$

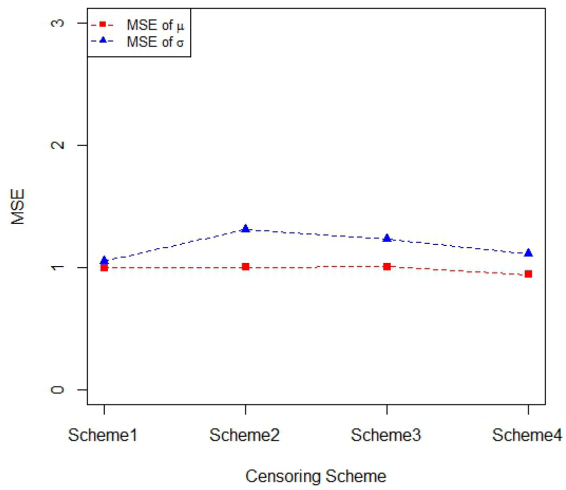

FIGURE 4. MSEs $(\times 10^{-2})$ when $k = 1$, $n = 60$, $m = 40$

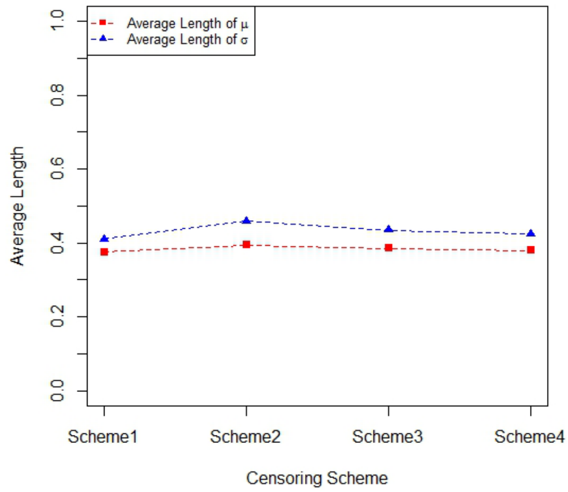

FIGURE 5. ALs when $k = 1$, $n = 60$, $m = 40$

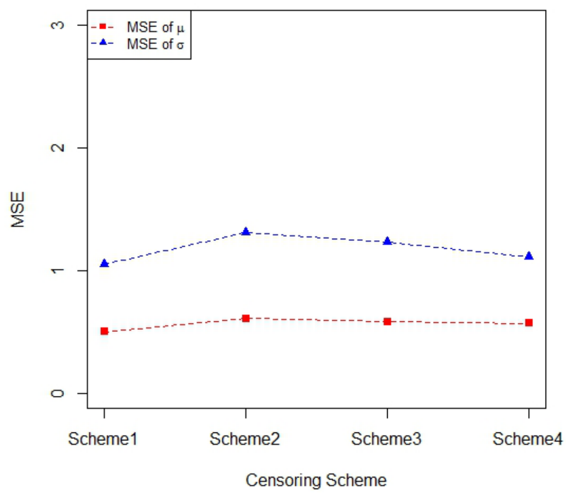

FIGURE 6. MSEs $(\times 10^{-2})$ when $k = 2$, $n = 60$, $m = 20$

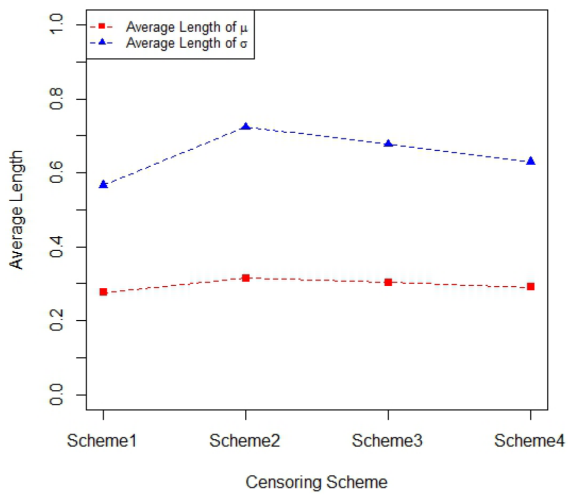

FIGURE 7. ALs when $k = 2$, $n = 60$, $m = 20$

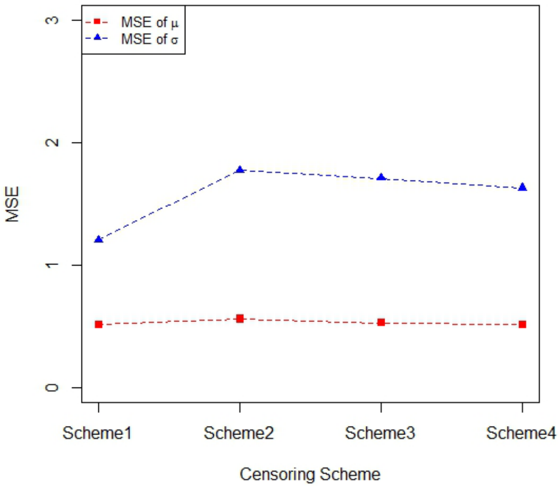

FIGURE 8. MSEs ($\times 10^{-2}$) when $k = 2$, $n = 60$, $m = 30$

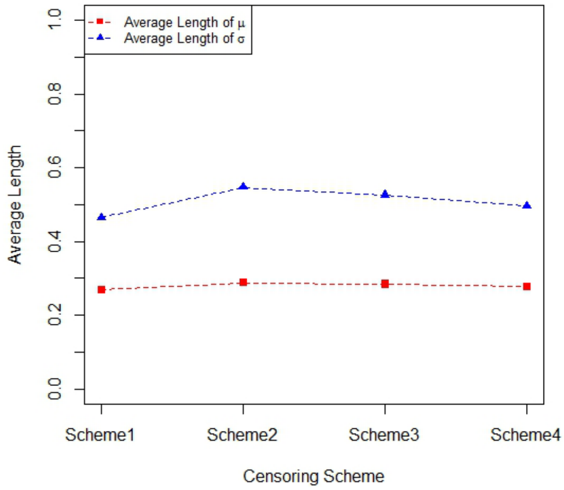

FIGURE 9. ALs when $k = 2$, $n = 60$, $m = 30$

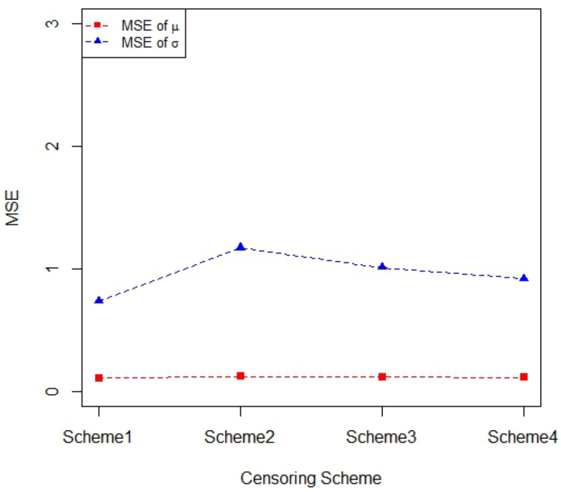

FIGURE 10. MSEs ($\times 10^{-2}$) when $k = 5$, $n = 100$, $m = 50$

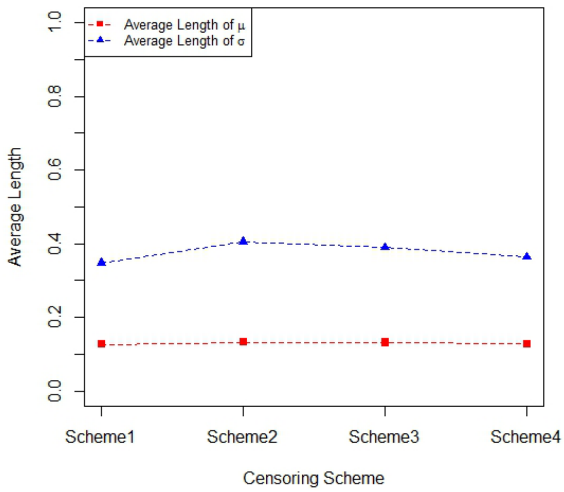

FIGURE 11. ALs when $k = 5$, $n = 100$, $m = 50$

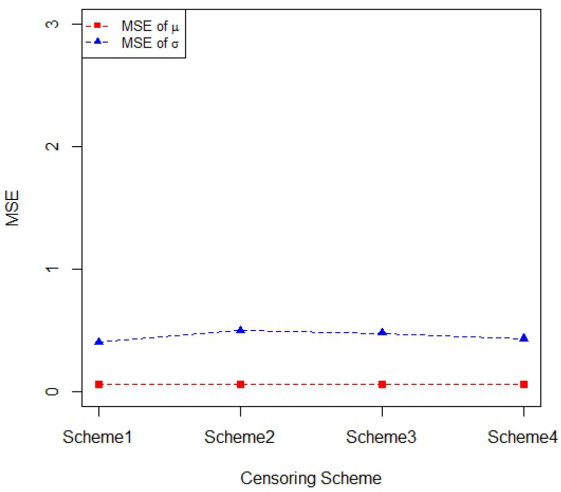

FIGURE 12. MSEs ($\times 10^{-2}$) when $k = 5$, $n = 200$, $m = 100$

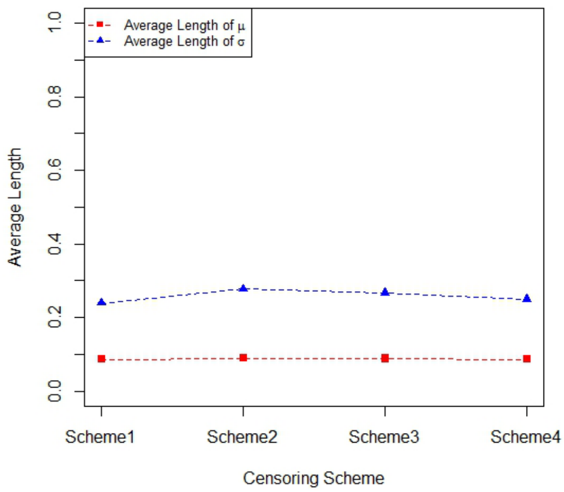

FIGURE 13. ALs when $k = 5$, $n = 200$, $m = 100$

Table 3. Survival times (in years) of 46 patients

0.047	0.115	0.121	0.132	0.164	0.197	0.203	0.260	0.282	0.296	0.334	0.395	0.458	0.466	0.501	0.507	0.529	0.534	0.540	0.570	0.641	0.644	0.696
0.841	0.863	1.099	1.219	1.271	1.326	1.447	1.485	1.553	1.581	1.589	2.178	2.343	2.416	2.444	2.825	2.830	3.578	3.658	3.743	3.978	4.003	4.033

Table 4. Random grouping to the real data set of survival times (in years) of patients

Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7	Set 8	Set 9	Set 10	Set 11	Set 12	Set 13	Set 14	Set 15	Set 16	Set 17	Set 18	Set 19	Set 20	Set 21	Set 22	Set 23
0.047	0.115	0.121	0.164	0.197	0.203	0.282	0.395	0.458	0.466	0.501	0.507	0.529	0.570	0.696	0.841	0.863	1.099	1.219	1.271	1.326	1.447	1.485
2.416	0.132	1.581	0.260	2.444	0.334	0.296	3.743	0.540	0.534	2.178	0.644	3.578	0.641	2.343	1.589	2.825	1.553	3.658	4.033	2.830	4.003	3.978

Table 5. Different progressive first-failure censoring schemes based on survival times (in years) of patients

(k,n,m)	Censoring scheme	Progressive first-failure censored data
$(2,23,20)$	$R_1=(0*17,1*3)$	0.047, 0.115, 0.121, 0.164, 0.197, 0.203, 0.282, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.570, 0.696, 0.841, 0.863, 1.099, 1.219, 1.326
	$R_2=(0*19,3)$	0.047, 0.115, 0.121, 0.164, 0.197, 0.203, 0.282, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.570, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271
	$R_3=(1*3,0*17)$	0.047, 0.115, 0.121, 0.164, 0.197, 0.203, 0.282, 0.458, 0.466, 0.501, 0.529, 0.570, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.447, 1.485
$(2,23,16)$	$R_4=(0*9,1*7)$	0.047, 0.115, 0.121, 0.164, 0.197, 0.203, 0.282, 0.395, 0.458, 0.466, 0.507, 0.529, 0.570, 0.841, 1.099, 1.271
	$R_5=(7,0*15)$	0.047, 0.115, 0.164, 0.197, 0.203, 0.458, 0.466, 0.501, 0.529, 0.570, 0.696, 0.863, 1.219, 1.271, 1.447, 1.485

Table 6. The MLEs and pivotal inference of the parameters for the real data

Censoring scheme	$\hat{\mu}_{MLE}$	$\hat{\mu}_p$	CI(μ)	AL(μ)	$\hat{\sigma}_{MLE}$	$\hat{\sigma}_p$	CI(σ)	AL(σ)
$R_1=(0*17,1*3)$	−0.1477629	−0.154213	(−0.4246923, 0.02044467)	0.4451369	0.9344619	0.9405499	(0.7016315, 1.312071)	0.6104395
$R_2=(0*19,3)$	−0.1420627	−0.1519229	(−0.4196111, 0.02076255)	0.4403737	0.9205130	0.9298448	(0.6934355, 1.297564)	0.6041281
$R_3=(1*3,0*17)$	−0.1318696	−0.1462981	(−0.3999618, 0.02115139)	0.4211132	0.8906413	0.9035523	(0.6803289, 1.249087)	0.5687581
$R_4=(0*9,1*7)$	−0.1420675	−0.1565449	(−0.4447757, 0.02090545)	0.4656812	0.9297448	0.9451699	(0.6787672, 1.390016)	0.7112483
$R_5=(7,0*15)$	−0.1202198	−0.1531807	(−0.4194892, 0.0204299)	0.4399191	0.8986649	0.929548	(0.6837198, 1.331804)	0.6480841

Now, we divide this data set into $n=23$ groups randomly with $k=2$ items in each group, shown in Table 4. And we consider five cases of the progressive first-failure censoring scheme in Table 5.

In Table 6, we find that the estimates based on pivotal quantities (16) and (14) are similar to the MLEs. The average lengths (ALs) of the exact 95% confidence intervals based on pivotal quantities are almost narrow. Figure 14 and Figure 15 show that the pivotal estimate of μ is smaller than MLEs and the pivotal estimate of σ is larger, which is consistent with the results of the simulation study. However, the difference between MLEs and pivotal estimate is quite small. The MLEs of μ and σ are nearly -0.13 and 0.91 respectively while the pivotal estimates of μ and σ are around -0.15 and 0.92. This numerical study indicates that the proposed pivotal inference in this paper can be applied in practice.

4. **Conclusions.** The pivotal inference for the two-parameter Rayleigh distribution based on progressively first-failure censored data is discussed in this article. We derive the pivotal quantities of the location parameter μ and the scale parameter σ. Also, we construct the exact confidence intervals for μ and σ based on pivotal quantities. From the results

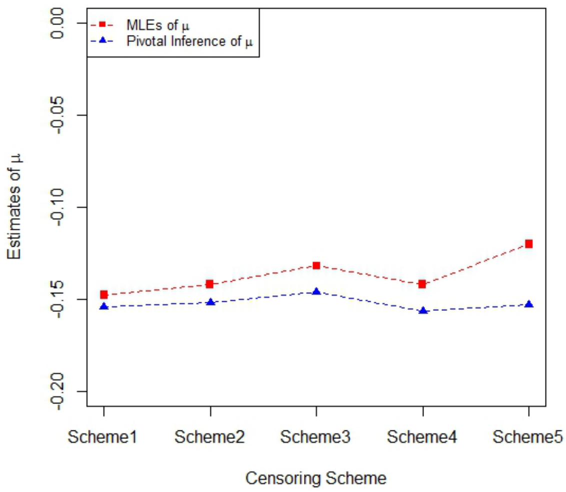

FIGURE 14. MLEs and pivotal inference of the parameter μ under five censoring schemes

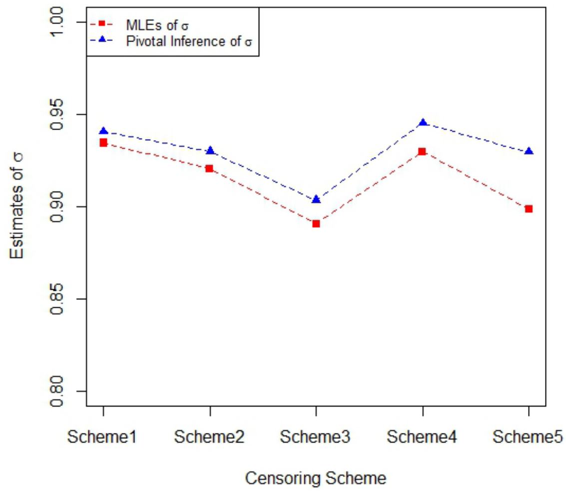

FIGURE 15. MLEs and pivotal inference of the parameter σ under five censoring schemes

of the Monte Carlo simulation study and the real data example, we find that even for small sample sizes, the performance of the proposed pivotal inference in this paper is quite good. Pivotal inference is more convenient than MLE and it can be used widely.

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