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ROBUST GAIN SCHEDULED CONTROLLER DESIGN WITH HARD INPUT CONSTRAINTS. FREQUENCY DOMAIN APPROACH

Vojtech Vesely[1], Jakub Osusky[2] and Danica Rosinova[2]

[1]Department of Robotics and Cybernetics
[2]Department of Automotive Mechatronics
Faculty of Electrical Engineering and Information Technology
Slovak University of Technology in Bratislava
Ilkovicova 3, Bratislava 812 19, Slovakia
{ votech.vesely; jakub.osusky; danica.rosinova }@stuba.sk

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Abstract. *A novel approach to robust gain scheduled controller design in frequency domain with hard input constraints is presented. The proposed design procedure for design of robust gain scheduled controller is based on the integral quadratic constraints (IQC) and one of its corollaries – small gain theory, providing a general robust stability condition. The uncertain multivariable plant input, output and gain scheduled variables are described by a nominal model with additive type uncertainty. We propose a special gain scheduled controller structure, such that hard input constraints are satisfied and the closed loop stability is guaranteed for any rate of corresponding controller gain change. The obtained results and their qualities are illustrated on simulation example.*

Keywords: Gain scheduled robust controller, Small gain theorem, IQC, Hard input constraints

1. **Introduction.** All control systems are subject to actuator magnitude and/or rate limits which lead to a performance degradation and even instability of the closed loop control system. Consideration of hard input constraints belongs to important tasks in controller design procedures. A chronological bibliography on saturating actuators can be found in [1,2]. Necessary and sufficient conditions for controllability of linear systems subject to input/state constraints are given in [3]. Robust stabilization of uncertain systems subject to input constraints is considered in [4], where a piecewise linear control satisfying the input constraints is parameterized by algebraic Riccati equation. Youla parametrization of uncertain plant and stabilizing controllers, which guarantees the prescribed hard constraints on control signal, can be found in [5]. Using the two stage internal model control (IMC) antiwindup design for stable plant and input constraints is described in [6]. The invariant set and an algorithm similar to soft variable structure control approach ensuring soft input constraints for the model predictive control systems, are described in [7]. In [8,9] authors deal with a design of linear systems with saturating actuators where the actuator limitations have to be incorporated into control design. Discrete-time switched system stabilization control under state constraints and quantized control input is proposed via LMI [10]. In the book [11], the methods and algorithms based on state space approach, are designed to overcome the effects of actuator saturation, ensuring the global stability of closed loop systems. The book [12] is devoted to several related topics: anti-windup strategies, model predictive control with constraints, stability analysis and stabilization methods for constrained systems. In [13], the new idea of robust controller design with

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input constraints is given. Fuzzy controller design, which considers the constraints on input and output variables, is given in [14].

The above short survey shows that in the field of robust controller design with input constraints, there exist many different approaches. This observation motivated us to study the following research problem which has not been sufficiently solved yet in the frequency domain. Design the robust gain scheduled controller with hard input constraints, considering the following qualities.

- Guaranteed stability and robustness properties of the closed loop system when the uncertainties both of plant parameters and gain scheduled variables lie in the given uncertainty box. For plant and gain scheduled variable (GSV), the additive type uncertainty is considered.

- Guaranteed closed loop system performance for the considered varying parameters and hard input constraints.

Stability is guaranteed using IQC – small gain theorem based conditions.

Each actuator in practical control systems has constraints. When a controller output achieves maximum value, its further increasing has no effect on controlled system; thus, the closed loop tends to behave as an open loop system. This can impose negative effect on performance when stable system is controlled, and also negative effect on stability when oscillating or unstable system is controlled (e.g., synchronous generators [15]). Avoiding this negative effect is the main motivation for this research.

Developing the controller design method with variable controller output gain, considering the constraints in design stage, which can avoid negative performance and stability effect on control system, is the main aim of this paper.

In this paper, we provide, to the authors' best knowledge, novel alternative approach to the robust gain scheduled controller design with hard input constraints. The proposed uncertainty model, introduced in Section 2, is used to formulate a robust control problem with hard input constraints. The main results are presented in Section 3. The design procedure is based on the new developed robust stability condition for the gain scheduled controller design with hard input constraints. In Section 4, numerical example illustrates the effectiveness of the proposed approach. Finally, some concluding remarks are given in Section 5.

Denotations in this paper are standard in robust controller design in frequency domain.

H_i, $i = 1, 2$ are linear control operators.

$M(s) \in RH_\infty$ is stable transfer function of multivariable systems.

$\sigma_M(\bullet)$ is maximal singular value of the corresponding transfer function.

$R_i(s)$, $i = 0, 1, \dots, p$ are controller transfer functions to be designed.

2. Problem Statement and Preliminaries.

In this section, robust stability conditions based on the IQC approach – small gain theorem are recalled. In the second part, the considered description of multivariable system with additive type uncertainty in a frequency domain and problem formulation are introduced.

2.1. Small gain theorem.

A large number of feedback systems can be put into the form shown in Figure 1 [16] where symbols w_1, w_2 denote the inputs, y_1, y_2 outputs and e_1, e_2 control errors. All these variables are functions of time; usually they are defined for $t \geq 0$. A general problem is: given some assumptions on H_1, H_2 show that if w_1, w_2 belong to some class, then y_1, y_2 and e_1, e_2 also belong to the same class. The feedback equations for the system in Figure 1 are

$$w_1 = e_1 + H_2 e_2; \quad w_2 = e_2 - H_1 e_1 \tag{1}$$

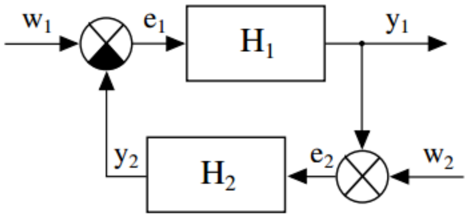

FIGURE 1. Feedback system

Assume that in Figure 1 $H_1 = M(s) \in RH_\infty$ is a linear time-invariant operator with the transfer function $M(s)$ and $H_2 = \Delta_c$ is causal operator with bounded gain. In applications, Δ_c will be used to include the "trouble making" components of a system as nonlinear part, time varying or uncertain parts [17]. Let us recall the robust stability condition of feedback systems using integral quadratic constraints (IQC). Two signals $E_1 \in L_2^k[0,\infty]$, $E_2 \in L_2^l[0,\infty]$ are said to satisfy the IQC defined by a symmetric operator $\Pi(j\omega)$, if

$$\int_{-\infty}^{\infty} \left[\begin{array}{c} E_2(j\omega) \\ E_1(j\omega) \end{array} \right]^* \Pi(j\omega) \left[\begin{array}{c} E_2(j\omega) \\ E_1(j\omega) \end{array} \right] d\omega \geq 0 \tag{2}$$

Robust stability theorem based on IQC for the system in Figure 1 is given in the following lemma [17].

Lemma 2.1. *Let $M(s) \in RH_\infty$ and let Δ_c be a bounded causal operator. Assume that:*
- the interconnection of $M(s)$ and $\tau\Delta_c$ is well posed for $\tau \in <0, 1>$
- the IQC defined by Π is satisfied for $\tau\Delta_c$ where $\tau \in <0, 1>$
- there exists $\varepsilon > 0$ such that

$$\left[\begin{array}{c} M(j\omega) \\ I \end{array} \right]^* \Pi(j\omega) \left[\begin{array}{c} M(j\omega) \\ I \end{array} \right] \leq -\varepsilon I, \quad \forall \omega \in R \tag{3}$$

Then the feedback interconnection of $M(s) - \Delta_c$ is stable.

For the time varying parameters (gain scheduled variables) and uncertainties, IQC matrix Π is given by the constant entries as

$$\Pi = \left[\begin{array}{cc} Q & S \\ S^T & \tilde{R} \end{array} \right] \tag{4}$$

where real Q, $\tilde{R}$, S matrices complying to (2) satisfy

$$Q + S\Delta_c + \Delta_c^T S^T + \Delta_c^T \tilde{R} \Delta_c \geq 0 \tag{5}$$

For the case $Q = I$, $S = 0$, $\tilde{R} = -I$ then (5) implies $I - \Delta_c^T \Delta_c \geq 0 \to \sigma_M(\Delta_c) \leq 1$ and robust stability condition (3) reads as $M(s)^T M(s) - I \leq 0$. Summarizing the above, we get

$$\sigma_M(M(s))\sigma_M(\Delta_c) < 1 \tag{6}$$

The results above are well known as the small gain theorem [18], which guarantees the robust stability condition for $M - \Delta_c$ structure, Figure 1.

Theorem 2.1. [18] *Suppose that in Figure 1, $H_1 = M(s) \in RH_\infty$ and let $\gamma > 0$. Then the interconnected system in Figure 1 is well-posed and internally stable for all $H_2 = \Delta_c \in RH_\infty$ with*
a) $||\Delta_c||_\infty \leq 1/\gamma$ if and only if $||M(s)||_\infty < \gamma$
b) $||\Delta_c||_\infty < 1/\gamma$ if and only if $||M(s)||_\infty \leq \gamma$

2.2. **Problem formulation.** The linear parameter varying (LPV) system is a linear plant whose transfer function matrices are functions of certain time varying parameters, called scheduling parameters, collected in vector $\theta_v(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_p(t)]$. Firstly, we assume that gain scheduled parameters are frozen in the maximal or minimal values. As mentioned above, "trouble making" parts like $\theta(t)$, all uncertainties and time varying parameter $\alpha(t)$ (see later) will be in the next developments gathered to the bounded matrix Δ_c. Consider now an uncertain multivariable LPV system $\overline{G(s)} \in R^{m\times m}$ and LPV controller $R(s) \in R^{m\times m}$ in the form

$$\overline{G(s)} = \overline{G_0(s)} + \sum_{i=1}^{p} \overline{G_i(s)}\theta_i \qquad R(s) = R_0(s) + \sum_{i=1}^{p} R_i(s)\theta_i \tag{7}$$

where (additive type) uncertainties of the plant are given as

$$\overline{G_0(s)} = G_0(s) + w_{0a}(s)\Delta_{0a} \qquad \overline{G_i(s)} = G_i(s) + w_{ia}(s)\Delta_{ia}(s), \quad i = 1, 2, \ldots, p \tag{8}$$

where $G_i(s)$, $i = 0, 1, 2, \ldots, p$ are known plant transfer function matrices, $R_i(s)$ are controller transfer function matrices to be designed, $w_{ia}(s)$ is known stable scalar transfer function, $\Delta_{ia}(s)$ is unknown $m \times m$ transfer function matrix such that $\sigma_M(\Delta_{ia}) \le 1$.

We assume that the exact values of gain scheduled parameters are known in each time and their lower and upper bounds are available, that is $\theta_i \in \Omega$ for $i = 1, 2, \ldots, p$, where

$$\Omega = \left\{\theta_v \in R^p, \underline{\theta_i} \le \theta_i \le \overline{\theta_i}, i = 1, 2, \ldots, p\right\} \tag{9}$$

3. **Main Results.** This section presents the main results of this paper: robust stability condition for LPV system with a gain scheduled controller including hard input constraints. The corresponding procedure to design the gain scheduled controller is provided and simplified version for a decentralized controller is developed as well. Tuneable filters are proposed to satisfy hard input constraints.

Consider an LPV system with m inputs, m outputs and p scheduled parameters. To simplify next developments, the following denotation is introduced.

Consider that matrix $G(s)$ of dimensions $m \times mp$ is composed of transfer function matrices $G_i(s)$ with additive uncertainties

$$G(s) = [G_1(s) \ldots G_p(s)] + [w_{1a}(s)I_m \ldots w_{pa}(s)I_m]\Delta_a \tag{10}$$

where $\Delta_a = blockdiag\{\Delta_{ia}\}$, $i = 1, 2, \ldots, p \in R^{pm\times pm}$. Let us denote

$$W_a(s) = [w_{ia}(s)I_m \ldots w_{pa}(s)I_m] \in R^{m\times mp}$$

$$G_N = [G_1(s) \ldots G_p(s)] \in R^{m\times mp}.$$

Matrix $G(s)$ can be then rewritten as

$$G(s) = G_N(s) + W_a(s)\Delta_a \tag{11}$$

$$\sigma_M(\Delta_a) \le 1$$

where we assume that matrix G_N is stable.

The controller transfer function matrix (7) is considered as

$$R_N(s) = [R_1(s) \ldots R_p(s)] \in R^{m\times mp} \tag{12}$$

assuming that all entries of (12) are stable. Denote the matrix of gain scheduled parameters as

$$\theta_{id} = diag\{\theta_i\}_{m\times m}, \quad \theta = [\theta_{1d} \ldots \theta_{pd}] \in R^{m\times mp}.$$

Using the above notations, we now develop our main result.

3.1. **New gain scheduled control structure for input constraints.** We propose a new gain scheduled control structure considering hard input constraints. The whole scheme, shown in Figure 2, includes three basic parts: gain scheduling controller (to be designed) – on the left hand side, the proposed filter (to be designed) to keep the control action within a prescribed range – in the middle, the controlled gain scheduled system – on the right hand side.

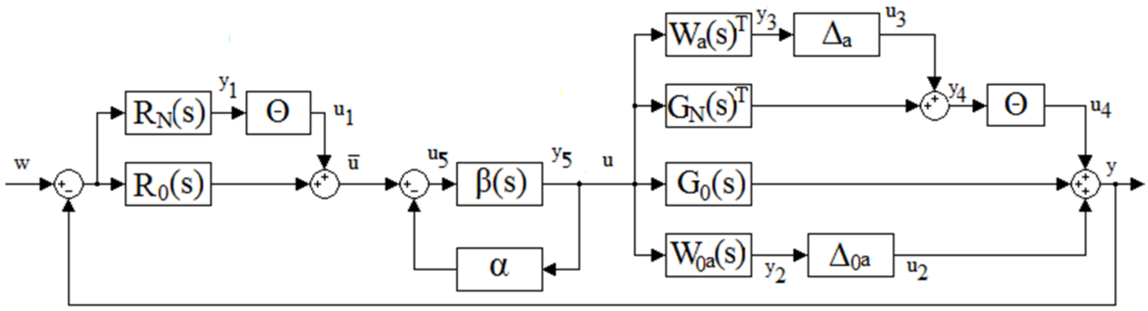

FIGURE 2. The new proposed robust gain scheduled control structure considering hard input constraints

In Figure 2 we consider $\alpha = diag\{\alpha_j\}_{m\times m}$, $\alpha_j \in < \underline{\alpha_j}, \overline{\alpha_j} >$, $\underline{\alpha_j} = -\overline{\alpha_j}$, $j = 1, 2, \ldots, m$, these gains serve to satisfy plant hard input constraints, $\beta(s) = diag\{\beta_j(s)\}_{m\times m}$ is a chosen diagonal transfer function matrix (filter). In Figure 2, the "trouble making" entries for robust stability condition are α, θ, Δ_a and Δ_{0a}, all of them are to be gathered to the block Δ_c and other linear transfer function matrices are included in the block $M(s)$. For outputs and inputs, one obtains:

$$\begin{aligned} y_c &= M(s)u_c \\ u_c^T &= \left[u_1^T\ u_2^T\ u_3^T\ u_4^T\ u_5^T\right] \\ y_c^T &= \left[y_1^T\ y_2^T\ y_3^T\ y_4^T\ y_5^T\right] \end{aligned} \tag{13}$$

where y_i, u_i, $i = 1, 2, 3, 4, 5$ are auxiliary outputs and inputs of the system depicted in Figure 2.

Matrix $M(s) \in R^{m(3p+2)\times 5m}$ can be rewritten as

$$M(s) = M_a(s)M_b(s) \tag{14}$$

where

$$M_a(s) = blockdiag\left\{\ -R_N^T \quad w_{0a}I_m \quad W_a^T \quad I_{mp} \quad I_m\ \right\}_{5\times 5} \tag{15}$$

$$M_a(s) \in R^{m(3p+2)\times m(4+p)}$$

$$N_0(s) = (I + G_0\beta R_0)^{-1}$$

Entries of matrix $M_b(s) = \{m_{ij}\}_{5\times 5} \in R^{m(4+p)\times 5m}$ are as follows

$$\begin{aligned} &m_{11} = G_0N_0\beta,\ m_{12} = N_0,\ m_{13} = N_0,\ m_{14} = N_0,\ m_{15} = -G_0N_0\beta,\ m_{21} = N_0\beta, \\ &m_{22} = -N_0\beta R_0,\ m_{23} = -N_0\beta R_0,\ m_{24} = -N_0\beta R_0,\ m_{25} = -N_0\beta,\ m_{31} = -N_0\beta, \\ &m_{32} = -N_0\beta R_0,\ m_{33} = -N_0\beta R_0,\ m_{34} = -N_0\beta R_0,\ m_{35} = -N_0\beta,\ m_{41} = G_N^TN_0\beta, \\ &m_{42} = -G_N^TN_0\beta R_0,\ m_{43} = -G_N^TN_0\beta R_0,\ m_{44} = -G_N^TN_0\beta R_0,\ m_{45} = -G_N^TN_0\beta, \\ &m_{51} = N_0\beta,\ m_{52} = -N_0\beta R_0,\ m_{53} = -N_0\beta R_0,\ m_{54} = -N_0\beta R_0,\ m_{55} = -N_0\beta \end{aligned} \tag{16}$$

From matrix inversion lemma, [19], it holds $I - G_0N_0\beta R_0 = (I + G_0\beta R_0)^{-1}$ under the assumption that $G_0\beta R_0 = \beta R_0G_0$. This assumption is fulfilled, e.g., for a decentralized controller, where β, R_0 are diagonal matrices. Uncertain matrix Δ_c where we gather all "trouble making" entries is given in the following form

$$\Delta_c = blockdiag\left\{ \theta \quad \Delta_{0a}I \quad \Delta_a\theta \quad \theta \quad \alpha \right\}_{5\times 5} \in R^{m(p+4)\times m(3p+2)} \tag{17}$$

Note that robust stability condition for $M - \Delta_c$ structure is given by (6) or

$$\sigma_M(M)\sigma_M(\Delta_c) < 1 \tag{18}$$

For $\sigma_M(\Delta_c)$ one obtains

$$\begin{aligned}\sigma_M(\Delta_c) &= \max\left(\sigma_M(\theta), |\Delta_{0a}|, \sigma_M(\Delta_a\theta), \sigma_M(\theta), \sigma_M(\alpha)\right)\\ &= \max\left(\max\left(\sum_{i=1}^{p}\theta_i^2\right)^{1/2}, 1, 1, \max_j|\alpha_j|\right)\end{aligned}$$

Without loss of generality and to simplify calculation of $\sigma_M(\Delta_c)$, we can assume that $\max|\alpha_j| = 1$. To find the upper bound on gain scheduled parameters matrix comprised in vector θ even under uncertainties, we assume knowledge of their maximal values as stated in the following remark.

Remark 3.1. *Consider that values of vector entries $\theta_i \in R$ are affected by uncertainties [20], that is gain scheduling parameter $\theta_{\max}$ (19) consists from the exact part $\overline{\theta_i}$ and some error (uncertainty) δ_i, $i = 1, 2, \dots, p$. Uncertain parameter δ_i is defined as*

$$\delta_i = \theta_{\max_i} - \overline{\theta_i} \tag{19}$$

The vector $\delta^T = [\delta_1 \dots \delta_p]$ represents the uncertainties on scheduled parameters. It is supposed that the uncertainties δ_i are independent from each other as well as from actual exact value of gain scheduled parameters θ_i. We assume that for uncertainties both lower and upper bounds are available. Each parameter δ_i lies in hyperrectangle Ω_δ which is known

$$\Omega_\delta = \left\{\delta_i \in <\underline{\delta_i}, \overline{\delta_i}>, i = 1, 2, \dots, p\right\} \tag{20}$$

Then, the upper bound on uncertain gain scheduled variables can be stated as

$$\overline{\theta_i} + \delta_i = \theta_{\max_i} \leq \theta_{\max}.$$

In this case, the norm of uncertainty matrix Δ_c can be bounded by

$$\sigma_M(\Delta_c) = \sqrt{(p)}\theta_{\max} \tag{21}$$

Equation (16) implies that matrix M_b is stable if and only if all entries of (16) are stable, that is if there exists such controller $R_0(s)$ and $\beta(s)$ which will guarantee the stability condition based on Nyquist plot applied for $\det\{I + R_0(s)G_0(s)\beta(s)\}$. Robust stability condition for the closed loop system with gain scheduled controller is then given by the next inequality

$$\sigma_M(M) < \frac{1}{\sigma_M(\Delta_c)} = \frac{1}{\sqrt{(p)}\theta_{\max}} \tag{22}$$

The main result of this paper can be formulated in the next theorem as a sufficient robust stability condition for an LPV system with gain scheduled controller subject to hard control input constraints.

Theorem 3.1. *Consider a matrix $M(s) \in RH_\infty$, given by (14), then the interconnected gain scheduled system in Figure 2 is well-posed and internally stable for all $||\Delta_c|| < 1$ defined by (17) when inequality (22) holds, assuming that $G_N(s)$ and $R_N(s)$ comprise stable matrices.*

The proof of Theorem 3.1 follows from the above observations.

Theorem 3.1 can be used also for a robust controller design under hard control input constraints. Matrices $M(s)$ and Δ_c are in this case simplified to $G_0(s)$ and $R_0(s)$. The details are summarized in Remark 3.2.

Remark 3.2. *Assume the next controller design problem formulation: design the robust controller for the plant with additive uncertainty (8) such that robust stability for the closed loop system is guaranteed for all plant regimes subject to the hard input constraints. In this case we substitute $R_N = 0$, $G_N = 0$ and $W_a(s) = 0$ to (15) and (17); after using the above inversion lemma, we obtain $M_r - \Delta_{cr}$ structure with matrices*

$$M_r(s) = \begin{bmatrix} -w_{0a}I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} R_0N_0 & (I - R_0N_0G_0\beta) \\ R_0N_0 & (I - R_0N_0G_0\beta) \end{bmatrix} \tag{23}$$

and $\Delta_{cr} = blockdiag\{\ \Delta_{0a}I \quad \alpha\ \}_{2\times 2}$.

Robust stability condition (18) is then in the form

$$\sigma_M(M_r(s))\sigma_M(\Delta_{cr}) = \max\{\sigma_M(w_{0a}(s)), 1\} \times \sigma_M(\beta(s))\sigma_M(M_{br}(s)) < 1 \tag{24}$$

where $M_{br}(s)$ is the last matrix of (23). Note that matrix $M_{br}(s) \in RH_\infty$ (stable) if and only if there exists a controller $R_0(s)$ and transfer function matrix $\beta(s)$ such that Nyquist plot of $\det[I + R_0G_0\beta]$ indicates stable system.

Theorem 3.2. *The structure $M_r - \Delta_{cr}$ is well-posed and internally stable for all $||\Delta_{cr}|| < 1$ and $M_r \in RH_\infty$ defined by (23) when inequality (24) holds.*

3.2. **Gain scheduled robust controller design.** Due to essentially simple way from robust controller design to robust gain scheduled controller design, we start with the robust controller design. Robust controller deign procedure consists of three steps.

1st step. Parameters α_j, $j = 1, 2, \ldots, m$ in the control loop in Figure 2 serve to satisfy the hard input constraints. From Figure 2, we obtain for control input u

$$u = (I - \beta(s)\alpha)^{-1}\overline{u} = K_u(s)\overline{u} \tag{25}$$

From (25) one can observe that for $\alpha_j \in <-1, 1>$ and $\beta(s) = diag\left\{\frac{\beta_j}{T_js+1}\right\}_{m\times m}$, where we choose $\beta_j = 0.5$, the gains on the diagonal of transfer function matrix $\beta(s)$ change within $k_{uj} \in <0.333, 1>$ and time constant $T_{jr} \in <0.66, 2> T_j$. These parameter changes can be used to satisfy the hard input constraints or to modify the dynamics of the corresponding closed loop system. For guaranteeing the hard input constraints $|u_j| < u_{Mj}$, $j = 1, 2, \ldots, m$, the required value of k_{uj} is calculated as:

(1) $k_{uj} = 1$ if $|u_j| \le u_{Mj}$

(2) $k_{uj} = \frac{u_{Mj}}{|u_j|}$ if $|u_j| > u_{Mj}$; $j = 1, 2, \ldots, m$

2nd step. When the controller structure is given and $\beta(s)$ is known, the robust controller parameters are to be designed so that robust stability condition (24) holds. One of the suitable frequency method to robust controller parameters design is the method of e-quivalent subsystems [21], including tuning parameter to ensure the robustness properties of the closed loop system. This step results in the design of robust controller parameters.

3rd step. Check the robust stability condition (24). If it is not satisfied, return to the second step and repeat the robust controller design procedure changing the tuning parameter to increase robustness.

For robust gain scheduled controller design, the simplest way is to apply the above-proposed method to the nominal system model G_0 and p models corresponding to the considered operating points. That means $p + 1$ times robust controller design and finally, recalculate the obtained robust controllers to robust gain scheduled controller. The corresponding design procedure is now illustrated on robust gain scheduled controller design for a nonlinear system linearized in p operating points. Below, the complete robust gain scheduling controller design procedure is described and illustrated on the case when LPV system model with GS controller is used to control a nonlinear system, linearized in the determined operating points.

3.2.1. *Gain scheduled LPV model.* Consider the nonlinear systems

$$\dot{x} = f(x, u, w) \qquad y = h(x) \tag{26}$$

where $w \in R^k$ is the vector of system input variables which captures parametric dependency of the plant on the variable w. Assume that for every $w \in \Omega_w$, $f(.,.,.)$ is locally Lipschitz and there exist equilibrium points (operating points) of the system given by triple (x_{ew}, u_{ew}, w_{ew}).

In the set Ω_w define $i = 1, \dots, p+1$ values w_{ei} (equilibrium points) for w_{ew}, where the nonlinear system (26) is replaced by $p+1$ linearized plant models. To obtain the model uncertainty for LPV plant model near each operating point, other families of linearized models need to be obtained, corresponding to vertices of uncertainty domain around operating point. Finally, for each equilibrium point the family of linearized models is obtained

$$\dot{x} = A_{ij}x + B_{ij}u \quad y = C(x) \quad i = 1, 2, \dots, p+1, \quad j = 1, 2, \dots, k \tag{27}$$

where k is the number of linearized models at the i-th operating point. Transforming the above models to frequency domain, one obtains transfer function matrices $F_{ij}(s)$ for each pair i, j. Finally, for each i and additive uncertainty type [22] we have

$$F_i(s) = F_{0i}(s) + v_{0ai}(s)\Delta_{0ai}, \quad i = 1, 2, \dots, p+1 \tag{28}$$

3.2.2. *Gain scheduled controller.* Using Theorem 3.2, a robust controller $K_i(s)$ which guarantees the robust stability around the i-th equilibrium point is designed for $i = 1, 2, \dots, p+1$.

Assume a gain scheduled controller in the form (7)

$$R_g(s) = R_0(s) + \sum_{i=1}^{p} R_i(s)\theta_i \tag{29}$$

For p gain scheduling variables define the values of their lower and upper bounds conforming to (9). To simplify calculation, without loss of generality we recommend to set $\underline{\theta_i} = -1$, $\overline{\theta_i} = 1$, $i = 1, 2, \dots, p$. Assume $p = 2$ and gain scheduled parameter changes as given in (9) then in this case we can write the following linear equations for calculation of gain scheduled controller transfer function matrices

$$\begin{aligned} K_1(s) &= R_0(s) - R_1(s) - R_2(s) \\ K_2(s) &= R_0(s) + R_1(s) - R_2(s) \\ K_3(s) &= R_0(s) + R_1(s) + R_2(s) \end{aligned} \tag{30}$$

Equation (30) can be rewritten in a simple matrix form below which can be used to design a robust GS controller.

$$\begin{bmatrix} I & -I & -I \\ I & I & -I \\ I & I & I \end{bmatrix} \begin{bmatrix} R_0(s) \\ R_1(s) \\ R_2(s) \end{bmatrix} = \begin{bmatrix} K_1(s) \\ K_2(s) \\ K_3(s) \end{bmatrix}$$

3.2.3. *Robustness of gain scheduled controller.* The obtained gain scheduled controller guarantees robust stability condition only around the equilibrium points. To ensure the global robust stability, condition (22) must be satisfied for control loop with the designed gain scheduled controller. If the condition (22) is not satisfied but the obtained controller $R_0(s)$ provides stability to matrix $M(s)$ given by (14), we introduce the tuning scalar parameter $k_t > 0$ for the first step.

$$R_{gt}(s) = k_t[R_1(s) \ldots R_p(s)], \quad k_t \in (0, 1) \tag{31}$$

If there exists such value of tuning parameter $k_t > 0$, that inequalities (22) hold, then the closed loop system with gain scheduled controller

$$R_g(s) = R_0(s) + k_t \sum_{i=1}^{p} R_i(s)\theta_i \tag{32}$$

guarantees robust stability for all plant regimes defined by the set Ω_w. If there does not exist such value of $k_t > 0$ which satisfies (22), design procedure returns to gain scheduled controller design (Section 3.2.2) and design new controllers $K_i(s)$ where the demanded performance indices are increased (increased values of phase margin...); other possibility to satisfy (22) is to redesign the second tuning parameter – the gains of transfer function matrix $\beta(s)$.

4. **Case Study.**

4.1. **Gain scheduled model and uncertainty.**

Consider a multivariable system with two inputs and two outputs, with unstable zero in subsystems and stable interactions identified in three operating points $\overline{G}_1(s)$, $\overline{G}_2(s)$, $\overline{G}_3(s)$. This example serves as a benchmark.

$$\overline{G}_1(s) = \begin{bmatrix} \dfrac{-0.063s + 2.43}{s^2 + 2.15s + 0.83} & \dfrac{-0.05s + 0.081}{s^2 + 0.89s + 0.23} \\ \dfrac{-0.12s + 0.15}{s^2 + 0.87s + 0.21} & \dfrac{-0.48s + 6.53}{s^2 + 6s + 2.38} \end{bmatrix}$$

$$\overline{G}_2(s) = \begin{bmatrix} \dfrac{-0.39s + 3.24}{s^2 + 3s + 1.16} & \dfrac{-0.067s + 0.1}{s^2 + s + 0.27} \\ \dfrac{-0.11s + 0.29}{s^2 + 2.54s + 0.33} & \dfrac{-1.95s + 25.28}{s^2 + 9.15s + 4.16} \end{bmatrix}$$

$$\overline{G}_3(s) = \begin{bmatrix} \dfrac{-0.29s + 1.9}{s^2 + 2.4s + 0.72} & \dfrac{-0.07s + 0.09}{s^2 + 1.03s + 0.24} \\ \dfrac{-0.027s + 0.57}{s^2 + 5.32s + 0.64} & \dfrac{-1.1s + 12.68}{s^2 + 7.41s + 3.91} \end{bmatrix} \tag{33}$$

In each plant regime (operating point), 5% uncertainty is considered and controllers $K_1(s)$, $K_2(s)$, $K_3(s)$ are designed as robust controllers able to cope with this uncertainty.

In the first step, a nominal model $G_0(s)$ was calculated as a model with average parameters considering all three operating points, or using the approach of (30) one obtains $G_0(s)$, $G_1(s)$, $G_2(s)$.

$$G_0 = \begin{bmatrix} \dfrac{-0.34s + 2.56}{s^2 + 3s + 1.16} & \dfrac{-0.07s + 0.095}{s^2 + 1.01s + 0.27} \\ \dfrac{-0.02s + 0.43}{s^2 + 5.33s + 0.71} & \dfrac{-1.03s + 10.89}{s^2 + 9.15s + 4.16} \end{bmatrix} \tag{34}$$

For the additive type uncertainty, weighting function $w_{0a}(s)$ or $W_a(s)$ includes $w_{1a}(s)$, $w_{2a}(s)$ and ensures that the uncertainty norms satisfy $||\Delta_{0a}||_\infty \le 1$ and $||\Delta_{1a}||_\infty \le 1$, $||\Delta_{2a}||_\infty \le 1$. For defined uncertainty and $G_0(s)$, $G_1(s)$, $G_2(s)$ using (35) $l_{0a}(\omega)$ and

$l_{ia}(\omega)$ are calculated. Firstly, functions $l_{0a}(\omega)$ and $(l_{1a}(\omega), l_{2a}(\omega))$ are calculated from (8) as

$$l_{0a}(\omega) = \max_k \sigma_M(G_k(j\omega) - G_0(j\omega)) \tag{35}$$

where k is number of operating points and

$$l_{ia}(\omega) = \max_l \sigma_M(G_l(j\omega) - G_i(j\omega)), \quad i = 1, 2, \dots, p$$

where l is number of uncertain transfer function matrices in each regime. Then the following inequalities hold for weighting functions

$$|w_a(j\omega)| \geq l_a(\omega), \ \forall\omega, \quad |w_{ia}(j\omega)| \geq l_{ia}(\omega), \ \forall\omega \tag{36}$$

According to (36), we below use $l_{0a}(\omega)$ resp. $(l_{1a}(\omega), l_{2a}(\omega))$ instead of $w_a(s)$ or $(w_{1a}(s), w_{2a}(s))$ in robust stability verification. For our system, these frequency depending weighting functions are depicted in Figure 3. These weighting functions can be calculated before controller design.

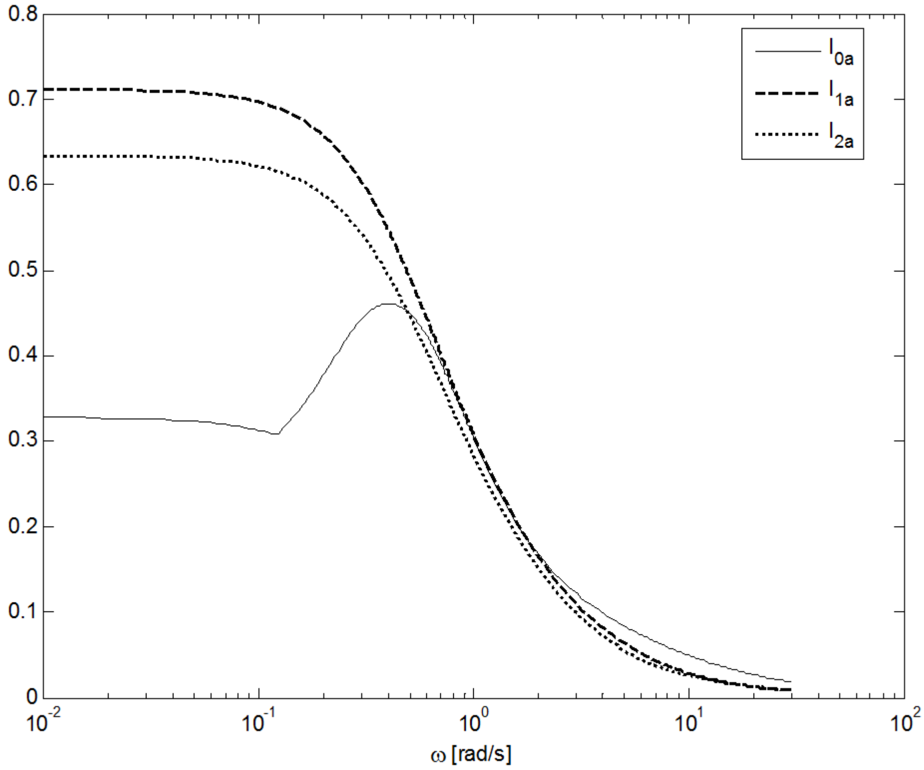

FIGURE 3. Frequency depending weighting functions $l_{0a}(\omega)$ and $(l_{1a}(\omega), l_{2a}(\omega))$

In the next step, it is necessary to choose parameters α_i and β_i to satisfy hard control input constraints. In this case, we consider matrix β as a gain matrix with diagonal elements $\beta_{1,2} = 0.5$. Both α_1 and α_2 can change within interval $\alpha_{1,2} \in < -1, 1 >$, which ensures $||\Delta_c||_\infty = 1$. Thus, the robust stability condition considering also controller output constraint can be calculated according to (24) and the controller output can be possibly constrained from 100% ($\alpha_{1,2} = 1$) to 33.3% ($\alpha_{1,2} = -1$) by corresponding changes of parameter alpha.

4.2. **Robust gain scheduled decentralized controller design.** Decentralized controllers can be designed using any method appropriate for frequency domain. In this paper, we use the method of equivalent subsystems (MES) proposed in [21]. For the

MES, the whole controller design procedure is performed on the subsystem level. We designed controller $K_i(s)$, $i = 1, 2, \ldots, p$ for each equilibrium point. Based on parameters of these controllers, the corresponding gain scheduled controller $R_g(s)$ was calculated. Better results can be obtained if $R_0(s)$ as a part of $R_g(s)$ is not calculated from (30) but is directly designed for $G_0(s)$. The resulting controller $R_0(s)$ designed for $G_0(s)$ is:

$$R_0(s) = \begin{bmatrix} \dfrac{0.522s + 0.165}{s} & 0 \\ 0 & \dfrac{0.51s + 0.28}{s} \end{bmatrix} \tag{37}$$

When $R_0(s)$ is known, controllers $R_1(s)$, $R_2(s)$ are calculated from $K_i(s)$, $i = 1, 2, \ldots, p+1$ using (30) to obtain

$$R_1(s) = \begin{bmatrix} \dfrac{0.01s - 0.112}{s} & 0 \\ 0 & \dfrac{0.023s - 0.02}{s} \end{bmatrix}$$

$$R_2(s) = \begin{bmatrix} \dfrac{0.012s - 0.002}{s} & 0 \\ 0 & \dfrac{0.0035s - 0.0415}{s} \end{bmatrix} \tag{38}$$

Robust stability condition considering hard input constraints and gain scheduled uncertainty 5% was verified graphically using (22) and depicted in Figure 4 for two different values of tuning parameter in (31): $k_t = 1$ and $k_t = 0.8$.

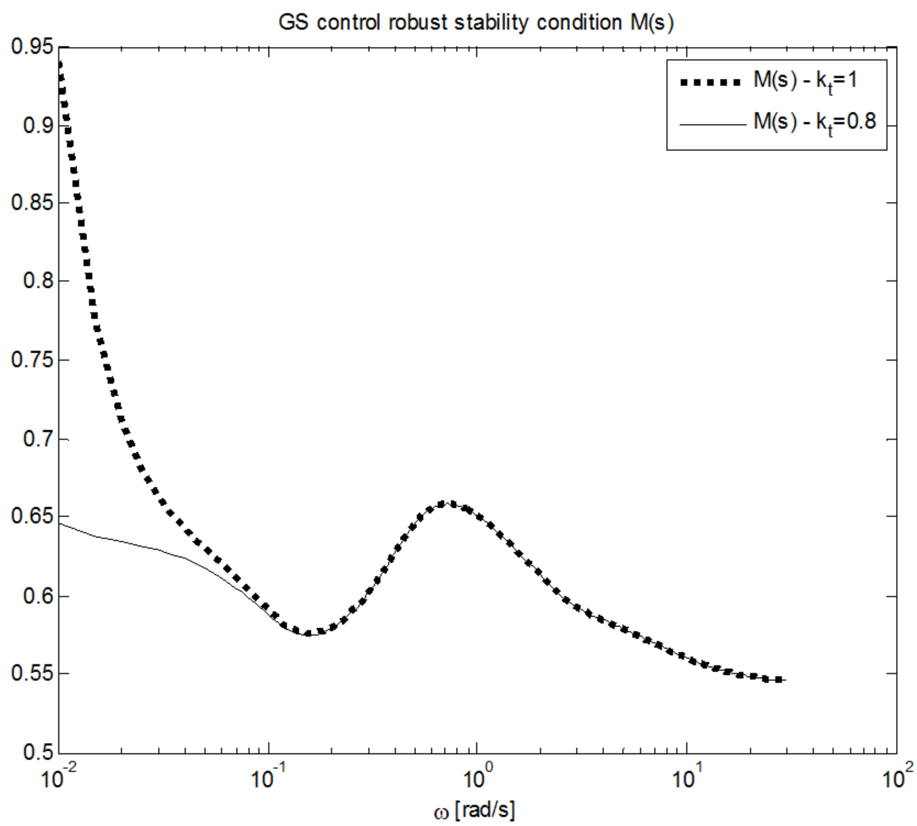

FIGURE 4. Robust stability condition considering hard input constraints

The robust stability condition depicted in Figure 4 shows that the condition is satisfied for the tuning parameter $k_t = 0.8$ and the corresponding system with the designed controller is robustly stable for the case, when the controller output is constrained in interval $< 33\%, 100\% >$ of its nominal value. In this case, when $p = 2$, the robust stability condition has following form, see (21)

$$\sigma_M(M(s)) < 1/\sqrt{2} \tag{39}$$

4.3. **Simulation results.** Simulation results in Figure 5 show performance of the closed loop system with gain scheduled controller where θ_1, θ_2 vary from minimal to maximal value.

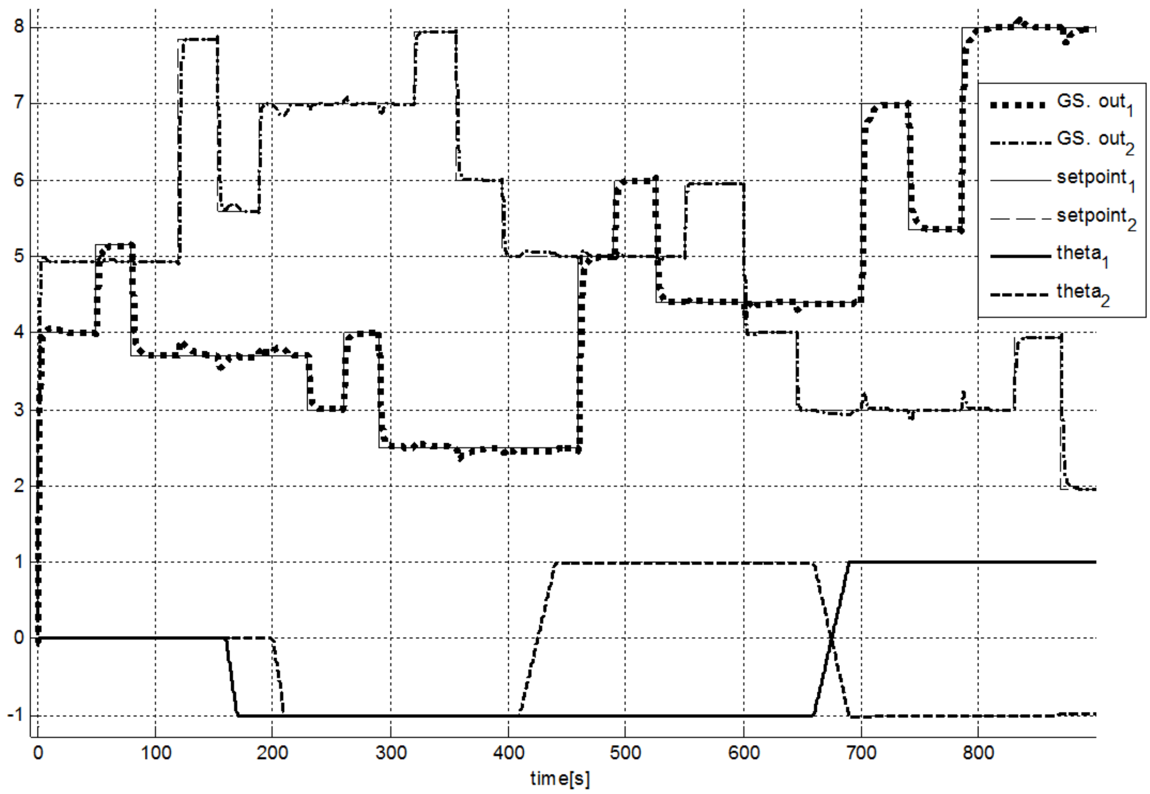

FIGURE 5. Step responses for both outputs in all operating points

TABLE 1. Comparison of gain scheduled control with robust control

ITAE	Loop 1	Loop 2
Gain Scheduled	66.2	38.12
Robust	67.01	43.15

Performance of the designed robust gain scheduled controller was compared with robust controller which has $R_{gt} = 0$, see (31), using ITAE criterion, Table 1 and Figure 6.

Presented control design method ensures the robustness properties of the closed loop system with hard input constraints. Note that to avoid steady state offset, the controller output can be constrained only to a value which enables that the output variable reaches the setpoint. Therefore, in our case controller output in the first loop will be constrained from value of 3.14 to 2.85 and in the second loop from 3.33 to 2.85, which corresponds to about 90.7% in the first loop and 85.5% in the second loop, Figure 7.

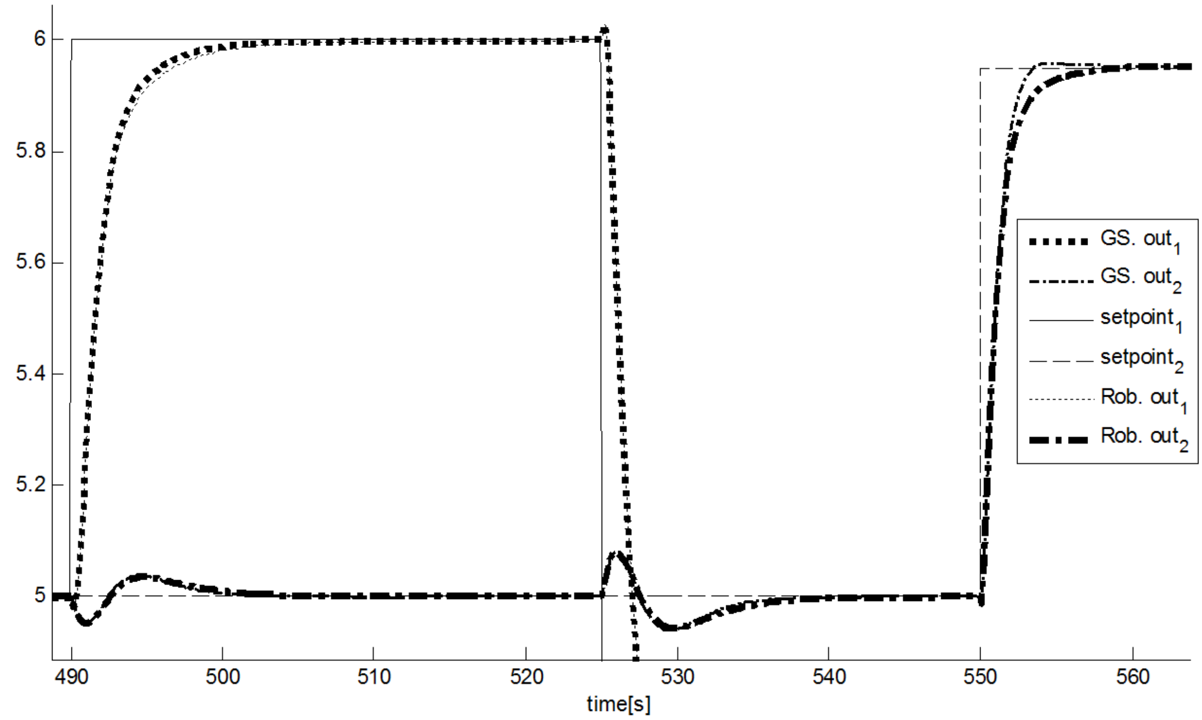

FIGURE 6. Graphical comparison of gain scheduled and robust control

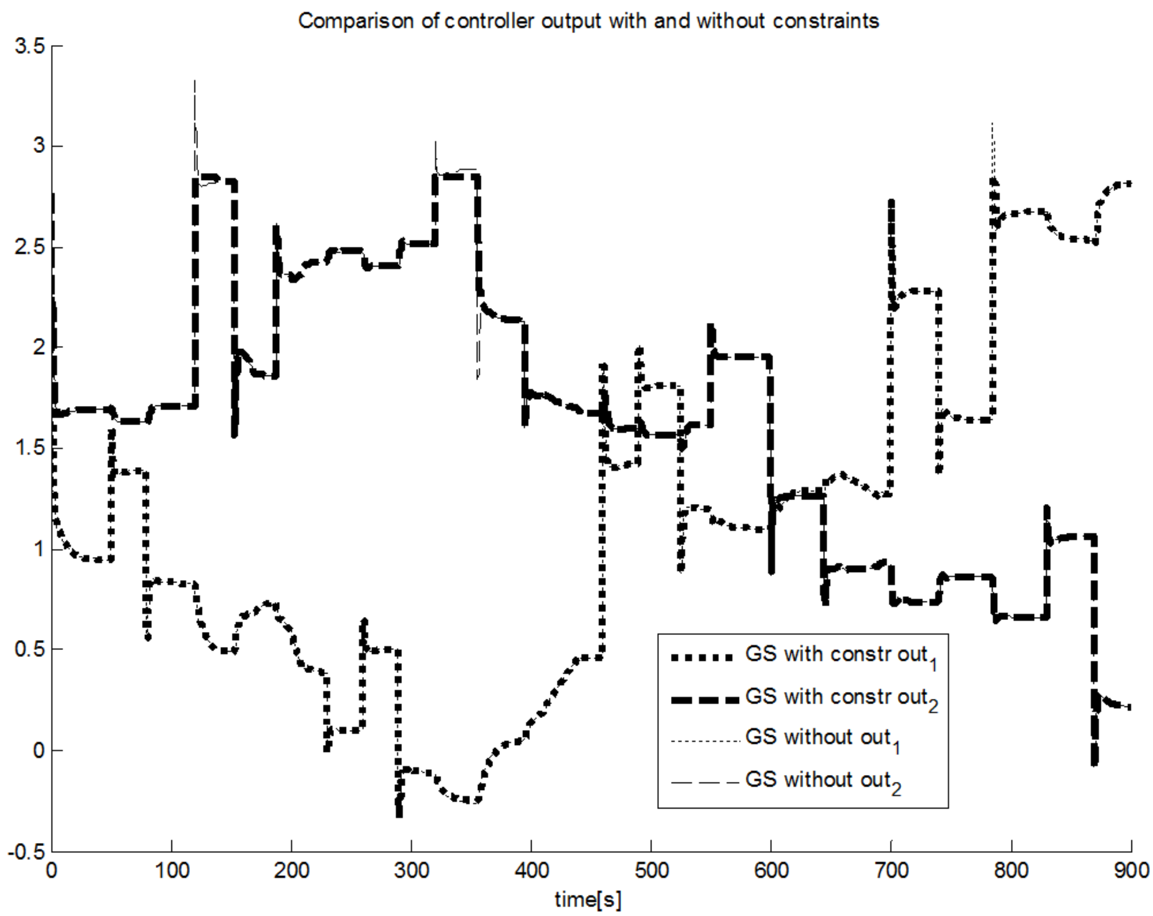

FIGURE 7. Controller outputs with and without constraints

Table 2. Comparison of system with and without constrains of controller output

ITAE	Loop 1	Loop 2
GS without constrains	66.2	38.12
GS with constrains	68.52	51.06

5. **Conclusions.** The main contribution of this paper is in the development of robust stability condition and the corresponding robust gain scheduling controller design procedure for uncertain LPV system with a gain scheduled variable, considering the hard input constraints. All control variables are subject to actuator magnitude and/or rate limits. The presence of constraints often leads to a performance degradation and even instability of the closed loop system; therefore, consideration of control input constraints belongs to very important tasks for control design practice. We developed the new gain scheduled controller design procedure with input constraints without violation of performance and stability of the corresponding closed loop system. The proposed gain scheduled robust controller design procedure is based on integral quadratic constraints and special controller structure to satisfy the hard input constraints such that for the robust decentralized controller design, any multivariable frequency method can be used. The obtained results, illustrated on example, show the applicability of the proposed design procedure to robust gain scheduled controller design which ensures the robustness properties of the corresponding closed loop system with hard input constraints, which is verified by simulations. The proposed approach contributes to the design tools for robust gain scheduled controller design with hard input constraints.

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