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STRATEGIC ANALYSIS IN MARKOVIAN QUEUES WITH A SINGLE WORKING VACATION AND MULTIPLE VACATIONS

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Abstract. *This paper considers the customers' equilibrium behavior in Markovian queues with a single working vacation and multiple vacations. That is, if there are customers in the queue upon the end of the working vacation, the new regular busy period will start. Otherwise, the server begins a classical vacation. Arriving customers decide whether to join the system or balk based on the system states and a linear reward-cost structure. Firstly, we study the expected sojourn time of customers in the fully observable and almost observable queues. Secondly, we derive equilibrium strategies for the customers for two cases and analyze the customers' strategic behavior and social welfare under these strategies. Finally, the effect of the information levels as well as system parameters on equilibrium strategies and social welfare are illustrated by numerical examples. We observe that equilibrium thresholds for the almost observable queues are contained within the range between thresholds for the fully observable queues. Moreover, we also find that the state of the server informed to the customers is not necessarily beneficial to social welfare.*

Keywords: Queueing systems, Equilibrium strategies, Multiple vacations, Working vacation, Social welfare

1. **Introduction.** Queuing systems are often studied from the traditional point of view, that is, the customers does not make decisions, even if there are some dynamic controls, the customers are passively accepted after the service system makes changes. However, as the competition of service organizations becomes more and more fierce, customers also have the power to choose among services. Based on the information of the service systems, the customer can use his or her own interests as a starting point to decide whether to enter the system to accept the service or leave. Therefore, in order to further improve the simulation of actual problems, it is necessary to explore the customers' strategy for maximizing their own benefit. For the work of studying customers' strategic behavior, Naor [1] first considered equilibrium and socially optimal strategies in an observable M/M/1 queueing system under a simple linear reward-cost structure. Edelson and Hildebrand [2] reconsidered the above model and studied the unobservable queue in which customers make decisions without information about the system state. From then on, several authors had made further research on the customers' behavior in Markovian queueing systems. Hassin and Haviv [3] summarized main approaches, fundamental results and established some representative models on queueing system.

In recent times, there has been an upsurge of interest towards economic analysis for decision making in queueing systems with vacation. Vacation queues have been used in

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many fields such as computer systems, communication networks, and flexible manufacturing systems and have been extensively studied in many previous studies. As to classical vacation queueing models, Burnetas and Economou [4] first achieved superior results in researching customers' equilibrium balking behavior under Markovian queues with setup times and analyzed the stationary behavior of the system under equilibrium balking strategies. Then Economou and Kanta [5] considered the Markovian single-server queue with breakdowns and repairs. They derived equilibrium threshold strategies in the fully observable and almost observable queues.

Subsequently, Sun et al. [6,7] considered customers' equilibrium and socially optimal balking strategies in Markovian queues with several types of setup/closedown times. Wang and Zhang [8] and Yu et al. [9] discussed equilibrium balking strategies for the observable and unobservable queues with delayed repairs and breakdowns. Economou et al. [10] obtained equilibrium strategies and social optimization in single-server queues with general service time and multiple vacations. In addition, Liu et al. [11] discussed equilibrium threshold strategies in continuous-time and discrete-time queueing systems with a single vacation. The topic of homogenous and heterogeneous customers' equilibrium and socially optimal behavior in the Markovian vacation queue were proposed by Guo and Hassin [12,13], respectively. Following these papers, further research can be referred to Tian et al. [14], Liu and Yu [15], Yu et al. [16] and references therein.

Queueing systems with working vacations have also attracted the attention of many researchers because of their practical applications. Servi and Finn [17] described the detailed description of working vacation policy. Wu and Takagi [18] and Banik et al. [19] extended the results of Servi and Finn [17] to M/G/1 and GI/M/1 queues with multiple working vacations. Liu et al. [20] and Li and Tian [21] considered stochastic decomposition property of the performance measures in continuous-time and discrete-time queueing systems with working vacations. The research of the classic models paves the way for us to further study the strategic behavior in working vacation queues.

As for the work of studying customers' behavior in queues with working vacations, optimal strategies in queues with multiple working vacations were almost simultaneously discussed by Zhang et al. [22] and Sun and Li [23] for different cases with respect to system information. Subsequently, Sun et al. [24] obtained equilibrium strategies and social welfare for M/M/1 queueing systems with two-stage working vacations where the service rate of the second-stage working vacation was lower than that of the first stage. Zhao and Tian [25] considered customers' equilibrium strategies in M/M/1 queuing systems with delayed working vacations. Sun et al. [26] further studied optimal strategies in Markovian queues with double adaptive working vacations and obtained equilibrium and socially optimal strategies for two types of unobservable queues. Guha et al. [27] focused on renewal input batch arrival queues with multiple and single working vacations and investigated equilibrium balking strategy of the customers. Lee [28] obtained equilibrium balking strategies in Markovian queues with a single working vacation and vacation interruption. Tian et al. [29] extended [28] to the queues with multiple working vacations and vacation interruptions and compared equilibrium strategies and socially optimal balking strategies under various levels of information. Ma et al. [30] analyzed strategic joining behavior of customers in a single-server working vacation queue with Bernoulli vacation interruptions. Goswami and Panda [31] studied Markovian queues with Bernoulli-schedule-controlled vacation and vacation interruption, and obtained equilibrium and socially optimal strategies.

Our goal in this paper is to study strategic behavior in Markovian queues with a single working vacation and multiple vacations under a reward-cost structure. When the system becomes empty after the completion of the service, the server begins a working vacation and works with a lower service rate rather than stopping the service completely. After this period, if there are still customers in the system, the server starts a regular busy period. Otherwise, the server starts to take a classic vacation. Meanwhile, when a vacation ends and there are customers in the system, a regular busy period starts. Otherwise, the server continues another vacation. Ye and Liu [32] considered this model, but they only got the steady-state performance measures and stochastic decomposition properties of the system, and did not take into account the customer's behavior. To the best of the authors' knowledge, research on equilibrium strategies in observable Markovian queueing systems with a single working vacation and multiple vacations has not been presented.

The proposed model has potential applications in the design of queueing system. In order to save operation costs, the server waits for the new customers and works with a lower service rate when the system becomes empty. After this period, the server may start a regular busy period if there are still customers in the system. Otherwise, the server may start vacations when there are still no customers in the system. This process includes working vacation and classic vacation, which we call a single working vacation and multiple vacations. For example, in order to keep the service system running efficiently for a long time, it is sometimes necessary to perform some routine maintenance on the service facility. If the system load becomes smaller, the server can choose to run at low speed (working vacation). The maintenance procedure is carried out once the system is vacant, and the time for the implementation of the maintenance is considered as 'a server's vacation'. Customers arriving during this period need to wait until the routine maintenance is completed before they can be received. This type of vacation is also applicable in banking systems and machine failure systems. Another example is provided in a machine system where the pump will remain at a minimum rate after the valve is closed. The pump can be restarted or stopped immediately during the delay stop period.

Motivated by the aforementioned studies and the possible application, we consider customers' equilibrium strategies and social welfare in an M/M/1 queueing model with a single working vacation and multiple vacations. We will consider two cases: 1) the fully observable queues: customers are informed about the number of present customers and the server's state; 2) the almost observable queues: customers are informed only about the number of present customers. For two cases, we will derive equilibrium strategies and social welfare, and discuss the sensitivity of information level to them by numerical examples. The results of this paper have not been reported in the literature on M/M/1 queues with a single working vacation and multiple vacations.

The paper is organized as follows. In Section 2, descriptions of the model are given. In Sections 3 and 4, we study the customers' equilibrium balking strategies and social welfare under the fully and almost observable queues, respectively. In Section 5, several numerical experiments are carried out to illustrate the effect of information levels on equilibrium strategies and social welfare. Finally, we summarize the full paper in Section 6.

2. **Model Description.**

In this paper, we consider an M/M/1 queue with a single working vacation and multiple vacations. The server begins a working vacation as soon as the system becomes empty after the completion of the service. In the working vacation period, the server continues to work at a lower rate. If there are customers in the system at the end of the working vacation, the system switches to the regular busy period. Otherwise, the classic vacation begins. If there are no customers in the system when the vacation ends, another vacation begins. Otherwise, the regular busy period begins. The detailed assumptions of the model are given below.

1) Potential customers arrive according to a Poisson process with rate λ.

2) The service times of the customers in a regular service period are assumed to be exponentially distributed with rate μ. During a working vacation period, arriving customers can be served at a mean rate of μ_1 ($\mu_1 < \mu$), rather than stopping the service completely.

3) The working vacation time is assumed to be exponentially distributed with the rate θ_1 and the classic vacation time is exponentially distributed with rate θ_0.

4) We assume that the inter-arrival times, the service times, the working vacation times and vacation times are mutually independent. In addition, the service order is first come first served (FCFS) discipline.

Let $N(t)$ represent the number of customers at time t, and let $J(t)$ be the server state at time t:

$$J(t) = \begin{cases} 0, & \text{the server is in the vacation period at time } t, \\ 1, & \text{the server is in working vacation period at time } t, \\ 2, & \text{the server is busy at time } t. \end{cases}$$

The process $\{(N(t), J(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\Omega = \{0, 0\} \cup \{0, 1\} \cup \{(n, i) | n \geq 1, i = 0, 1, 2\}$.

Arriving customers are assumed to be identical. Customers' strategic response has practical significance, which is what attracts us most. Customers can receive a reward R after service completion. There is a waiting cost of C units per time unit that the customer remains in the system. The mean sojourn time is denoted by W. So, customer's net welfare is $R - CW$ when the service is completed.

Customers are risk neutral and wish to maximize their expected net welfare. Risk-neutral customers usually do not evade risk or actively pursue risk. The standard for their choice is the size of the expected benefit, regardless of the risk profile. Finally, we assume that there are no retrials of balking customers nor reneging of waiting customers.

In the next sections, we study the fully observable queues and the almost observable queues depending on the information available to the customer at the time of arrival. The fully observable queues mean that arriving customers can observe system information of both the number of present customers $N(t)$ and the state of the server $J(t)$, and the almost observable queues mean they only can observe the number of present customers $N(t)$.

3. **The Fully Observable Queues.**

In this section, we will begin with the fully observable case, a customer can observe both the number of present customers $N(t)$ and the state of the server $J(t)$. We will first analyze the equilibrium thresholds in the fully observable case.

According to the reward-cost structure, the expected net welfare of customers entering the system is expressed as

$$U_{ni} = R - CW_{ni}, \quad i = 0, 1, 2, \tag{1}$$

where W_{ni} represents the customer's expected sojourn time when he enters the system at state (n, i). For each state of the system, there is a corresponding threshold $n(i)$ ($i = 0, 1, 2$). When the number of customers in the system exceeds the threshold, customers refuse to enter the system. So the customers' equilibrium balking threshold at state i can be denoted by $n_e(i)$, and the integrated strategy is expressed as ($n_e(0)$, $n_e(1)$, $n_e(2)$). We first determine the expected sojourn time W_{ni}.

In order to derive the mean sojourn time W_{n1} of the marked customer in case he encounters the system state $(n, 1)$, we make some preliminary work. The random variable S_v of the service time in the working vacation state follows the exponential distribution with the parameter μ_1. Therefore, the convolution $S_v^{(n)}$ of n $\left(n \geq 0, S_v^{(0)} = 0\right)$ independent distributions S_v follows the Erlang distribution with parameter (n, μ_1).

The convolution $S_v^{(n)}$ and random length V of working vacation are mutually independent. If $S_v^{(n)} < V$, the conditional distribution of $S_v^{(n)}$ follows the Erlang distribution with parameters $(n, \theta_1 + \mu_1)$. The probability of the event $\left\{S_v^{(n)} < V\right\}$ is

$$P\left\{S_v^{(n)} < V\right\} = \int_0^\infty P\{u < V\} \frac{\mu_1(\mu_1 u)^{n-1}}{(n-1)!} e^{-\mu_1 u} du = \left(\frac{\mu_1}{\theta_1 + \mu_1}\right)^n. \tag{2}$$

Then the conditional distribution function of $S_v^{(n)}$ follows the Erlang distribution with parameters $(n, \theta_1 + \mu_1)$.

If $S_v^{(n)} < V < S_v^{(n+1)}$, the conditional distribution of V follows the Erlang distribution with parameters $(n+1, \theta_1 + \mu_1)$. The probability of the event $\left\{S_v^{(n)} < V < S_v^{(n+1)}\right\}$ is

$$P\left\{S_v^{(n)} < V < S_v^{(n+1)}\right\} = \left(\frac{\mu_1}{\theta_1 + \mu_1}\right)^n - \left(\frac{\mu_1}{\theta_1 + \mu_1}\right)^{n+1} = \left(\frac{\mu_1}{\theta_1 + \mu_1}\right)^n \frac{\theta_1}{\theta_1 + \mu_1}. \tag{3}$$

The conditional distribution function of V follows the Erlang distribution with parameters $(n+1, \theta_1 + \mu_1)$.

Based on the above results, we obtain the following theorem.

Theorem 3.1. *If an arriving customer finds that the system state is* $(n, 1)$ *and decides to enter, his expected sojourn time is given by*

$$W_{n1} = \frac{n+1}{\mu} + \frac{\mu - \mu_1}{\theta_1 \mu}\left(1 - \left(\frac{\mu_1}{\mu_1 + \theta_1}\right)^{n+1}\right), \quad n = 0, 1, \ldots. \tag{4}$$

Proof: Customer's sojourn time W_{n1} is $n+1$ service times of the lower service rate μ_1 when he finds system is in the state $(n, 1)$, if the residual working vacation time V_R is long enough for service the marked customer. Otherwise, if only j $(0 \leq j \leq n)$ customers have accepted the service during V_R, W_{n1} equals the sum of residual working vacation time V_R and the $n+1-j$ service times at regular service rate μ. Hence, we get the Laplace-Stieltjes transform (LST) of W_{n1}, denoted by $W_{n1}^*(s)$, as follows:

$$\begin{aligned} W_{n1}^*(s) &= P\left\{S_v^{(n+1)} < V_R\right\}\left(\frac{\mu_1 + \theta_1}{\mu_1 + \theta_1 + s}\right)^{n+1} \\ &\quad + \sum_{j=0}^{n} P\left\{S_v^{(n+1)} < V_R < S_v^{(j+1)}\right\}\left(\frac{\mu_1 + \theta_1}{\mu_1 + \theta_1 + s}\right)^{j+1}\left(\frac{\mu}{\mu + s}\right)^{n+1-j} \\ &= \left(\frac{\mu_1 + \theta_1}{\mu_1 + \theta_1 + s}\right)^{n+1} + \left(\frac{\mu}{\mu + s}\right)^{n+1} \frac{\mu\theta_1}{\mu\theta_1 + (\mu - \mu_1)s}\left(1 - \left(\frac{\mu_1(\mu + s)}{\mu(\mu_1 + \theta_1 + s)}\right)^{n+1}\right). \end{aligned} \tag{5}$$

Then we can easily get $W_{n1} = -W_{n1}^{*'}(0)$.

The customer's sojourn time is W_{n1} when the system state is $(n, 1)$, so his expected net welfare after service completion is $R - CW_{n1}$. We obtained equilibrium strategies $n_e(1)$ by solving the equation $R - CW_{n1} = 0$ for n. So we have the following theorem.

Theorem 3.2. *In the fully observable queue with a single working vacation and multiple vacations, there exist equilibrium thresholds* $(n_e(0), n_e(1), n_e(2))$ *which are given by*

$$n_e(0) = \left\lfloor \frac{R\mu}{C} - \frac{\mu}{\theta_0} \right\rfloor - 1, \tag{6}$$

$$n_e(1) = \lfloor x_e \rfloor, \tag{7}$$

$$n_e(2) = \left\lfloor \frac{R\mu}{C} \right\rfloor - 1, \tag{8}$$

where x_e is the unique root of the following equation

$$\frac{n+1}{\mu} + \frac{\mu - \mu_1}{\theta_1 \mu}\left(1 - \left(\frac{\mu_1}{\mu_1 + \theta_1}\right)^{n+1}\right) = \frac{R}{C}.$$

Proof: According to the previous assumptions, we have the following equations:

$$W_{n0} = \frac{n+1}{\mu} + \frac{1}{\theta_0}, \quad n = 0, 1, 2, \dots, \tag{9}$$

$$W_{n2} = \frac{n+1}{\mu}, \quad n = 1, 2, \dots. \tag{10}$$

If the customer's expected net welfare $U_{ni} = R - CW_{ni} > 0$, he enters the system. On the contrary, the customer prefers to balk if $U_{ni} = R - CW_{ni} < 0$. The customer is indifferent if $U_{ni} = R - CW_{ni} = 0$. We assume throughout the paper that $R > \frac{C}{\mu} + \frac{C}{\theta_0}$, which ensures that the customers will enter the system if the system is empty.

$$W_{n1}^{'} = \frac{1}{\mu} - \frac{\mu - \mu_1}{\theta_1 \mu}\left(\frac{\mu_1}{\mu_1 + \theta_1}\right)^{n+1} \ln \frac{\mu_1}{\mu_1 + \theta_1}, \quad n = 1, 2, \dots. \tag{11}$$

$W_{n1}^{'} > 0$ is obvious, so W_{n1} increases with respect to n. Observing (9) and (10), we get W_{n0} and W_{n2} increase with respect to n. Using (4), (9) and (10) to solve $U_{ni} \geq 0$ for n, we obtain unique thresholds $(n_e(0), n_e(1), n_e(2))$. Arriving customer decides to enter if and only if $n \leq n_e(J(t))$ where $(n_e(0), n_e(1), n_e(2))$ are given by (6), (7) and (8).

Remark 3.1. *In the fully observable M/M/1 queues with a single working vacation and multiple vacations, when the number of customers in the system exceeds the threshold, customers refuse to enter the system. The thresholds $(n_e(0), n_e(1), n_e(2))$ represent equilibrium strategies, such that a customer who observes the system at state $(N(t), J(t))$ upon his arrival enters the queue if $N(t) \leq n_e(J(t))$ and balk otherwise.*

For the stationary analysis, the system follows a Markov chain and the state space is

$$\Omega_{fo} = \{0, 0\} \cup \{0, 1\} \cup \{(n, i) \mid n = 1, 2, \dots, n(i) + 1; i = 0, 1, 2\}.$$

And the transition diagram is shown in Figure 1. Denote the stationary distribution as

$$\pi_{ni} = \lim_{t \to \infty} P\{N(t) = n, J(t) = i\}, \quad (n, i) \in \Omega_{fo}.$$

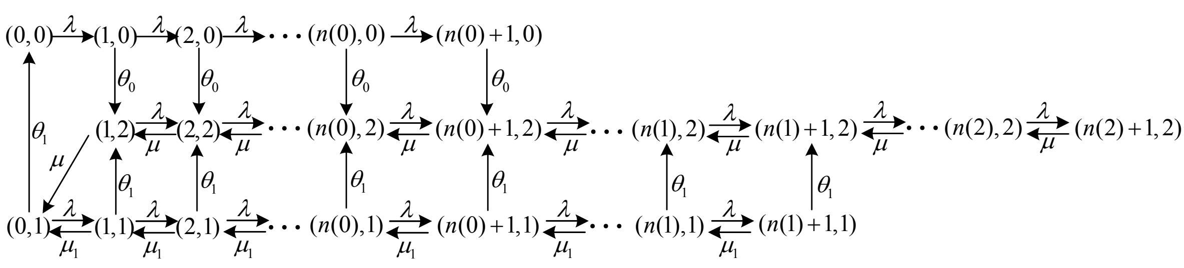

FIGURE 1. Transition rate diagram for the fully observable queues

It is clear that the process $\{N(t), J(t) | t \geq 0\}$ is a continuous-time Markov chain with non-zero transition rates by

$$q_{(ni)(n+1,i)} = \lambda, \quad n = 0, 1, \dots, n(i), \quad i = 0, 1;$$

$$
\begin{aligned}
&q_{(n2)(n+1,2)} = \lambda, \quad n = 1, 2, \ldots, n(2);\\
&q_{(12)(0,1)} = \mu;\\
&q_{(n2)(n-1,2)} = \mu, \quad n = 2, 3, \ldots, n(2) + 1;\\
&q_{(n1)(n-1,1)} = \mu, \quad n = 1, 2, \ldots, n(1) + 1;\\
&q_{(n0)(n2)} = \theta_0, \quad n = 1, 2, \ldots, n(0) + 1;\\
&q_{(01)(00)} = \theta_1;\\
&q_{(n1)(n,2)} = \theta_1, \quad n = 1, 2, \ldots, n(1) + 1.
\end{aligned}
$$

The corresponding stationary distribution $\{\pi_{ni}|(n,i) \in \Omega_{fo}\}$ is obtained as the unique positive normalized solution of the following system of balance equations.

$$\lambda\pi_{00} = \theta_1\pi_{01}, \tag{12}$$

$$(\lambda + \theta_0)\pi_{n0} = \lambda\pi_{n-1,0}, \quad n = 1, 2, \ldots, n(0), \tag{13}$$

$$\theta_0\pi_{n(0)+1,0} = \lambda\pi_{n(0),0}, \tag{14}$$

$$(\lambda + \theta_1)\pi_{01} = \mu_1\pi_{11} + \mu\pi_{12}, \tag{15}$$

$$(\lambda + \mu_1 + \theta_1)\pi_{n1} = \lambda\pi_{n-1,1} + \mu_1\pi_{n+1,1}, \quad n = 1, 2, \ldots, n(1), \tag{16}$$

$$(\mu_1 + \theta_1)\pi_{n(1)+1,1} = \lambda\pi_{n(1),1}, \tag{17}$$

$$(\lambda + \mu)\pi_{12} = \theta_0\pi_{10} + \theta_1\pi_{11} + \mu\pi_{22}, \tag{18}$$

$$(\lambda + \mu)\pi_{n2} = \lambda\pi_{n-1,2} + \mu\pi_{n+1,2} + \theta_0\pi_{n0} + \theta_1\,\pi_{n1}, \quad n = 2, 3, \ldots, n(0) + 1, \tag{19}$$

$$(\lambda + \mu)\pi_{n2} = \lambda\pi_{n-1,2} + \mu\pi_{n+1,2} + \theta_1\pi_{n1}, \quad n = n(0) + 2, \ldots, n(1) + 1, \tag{20}$$

$$(\lambda + \mu)\pi_{n2} = \lambda\pi_{n-1,2} + \mu\pi_{n+1,2}, \quad n = n(1) + 2, \ldots, n(2), \tag{21}$$

$$\mu\pi_{n(2)+1,2} = \lambda\pi_{n(2),2}. \tag{22}$$

By iterating (13), talking into account (12) and (14), we obtain

$$\pi_{n0} = \left(\frac{\theta_1}{\lambda}\right)^2 \left(\frac{\lambda}{\theta_0 + \lambda}\right)^n \pi_{01}, \quad n = 0, 1, \ldots, n(0), \tag{23}$$

$$\pi_{n(0)+1,0} = \frac{\theta_1^2}{\lambda\theta_0} \left(\frac{\lambda}{\theta_0 + \lambda}\right)^{n(0)} \pi_{01}. \tag{24}$$

From (15)-(17), we observe that $\{\pi_{n1}|0 \le n \le n(1) + 1\}$ are the solution of the homogeneous linear difference equation with constant coefficients.

$$\mu_1 x_{n+1} - (\mu_1 + \lambda + \theta_1)x_n + \lambda x_{n-1} = 0, \quad n = 1, 2, \ldots, n(1). \tag{25}$$

So its corresponding characteristic equation is

$$\mu_1 x^2 - (\mu_1 + \lambda + \theta_1)x + \lambda = 0,$$

which has two roots

$$x_{1,2} = \frac{(\mu_1 + \lambda + \theta_1) \pm \sqrt{(\mu_1 + \lambda + \theta_1)^2 - 4\mu_1\lambda}}{2\mu_1}. \tag{26}$$

The general solution of (25) is denoted by $x_n^{hom} = A_1 x_1^n + B_1 x_2^n$ (obviously $x_1 \neq x_2$). Solving the system of (15) and (17) we get the equations about A_1 and B_1 as

$$\begin{cases} A_1(\theta_1 + \lambda - \mu_1 x_1) + B_1(\theta_1 + \lambda - \mu_1 x_2) = \pi_{12}\mu, \\ A_1\left(x_1^{n(1)+1}(\mu_1 + \theta_1) - x_1^{n(1)}\lambda\right) + B_1\left(x_2^{n(1)+1}(\mu_1 + \theta_1) - x_2^{n(1)}\lambda\right) = 0. \end{cases} \tag{27}$$

Solving (27), we obtain A_1 and B_1.

And thus,

$$\pi_{n1} = A_1 x_1^n + B_1 x_2^n, \quad n = 0, 1, \dots, n(1) + 1, \tag{28}$$

where x_1 and x_2 are given by (26).

For the busy period, we first consider the probabilities $\{\pi_{n2} | 1 \le n \le n(0) + 1\}$. From (19), solving following nonhomogeneous linear difference equation we get $\{\pi_{n2} | 2 \le n \le n(0) + 1\}$.

$$\begin{aligned} &\mu x_{n+1} - (\lambda + \mu) x_n + \lambda x_{n-1} \\ &= -\theta_0 \pi_{n0} - \theta_1 \pi_{n1}, \\ &= -\frac{\theta_0 \theta_1}{\lambda} \left(\frac{\lambda}{\theta_0 + \lambda} \right)^n (A_1 + B_1) - \theta_1 (A_1 x_1^n + B_1 x_2^n), \quad n = 2, 3, \dots, n(0). \end{aligned} \tag{29}$$

The general solution of the homogeneous version of (29) is $x_n^{hom} = A_2 1^n + B_2 \rho^n$ (we assume that $\rho \neq 1$), where $\rho = \lambda / \mu$. So the general solution of (29) is denoted by $x_n^{gen} = x_n^{hom} + x_n^{spec}$, where x_n^{spec} is a specific solution. The nonhomogeneous part of the above equation has a geometric part with parameter x_1, x_2 and $\frac{\lambda}{\theta_0 + \lambda}$. So, $x_n^{spec} = C_1 x_1^n + D_1 x_2^n + E_1 \left(\frac{\lambda}{\theta_0 + \lambda} \right)^n$ is a specific solution. Substituting $x_n^{spec} = C_1 x_1^n + D_1 x_2^n + E_1 \left(\frac{\lambda}{\theta_0 + \lambda} \right)^n$ into (29), we get

$$\begin{cases} C_1 = \dfrac{\theta_0 A_1 x_1}{(\mu_1 x_1 - \lambda)(1 - x_1)}, \\ D_1 = \dfrac{\theta_0 B_1 x_2}{(\mu_1 x_2 - \lambda)(1 - x_2)}, \\ E_1 = \dfrac{\theta_0 (\theta_1 + \lambda)(A_1 + B_1)}{\lambda (\mu_1 - \theta_1 - \lambda)}. \end{cases} \tag{30}$$

Hence, the general solution of (29) is given as

$$x_n^{gen} = A_2 1^n + B_2 \rho^n + C_1 x_1^n + D_1 x_2^n + E_1 \left(\frac{\lambda}{\theta_0 + \lambda} \right)^n, \quad n = 1, 2, \dots, n(0) + 1, \tag{31}$$

C_1, D_1 and E_1 are given by (30), and A_2, B_2 are to be obtained. Substituting (31) in (18), we then obtain

$$\begin{cases} A_2 + \rho B_2 = \pi_{12} - \left(C_1 x_1 + D_1 x_2 + E_1 \dfrac{\lambda}{\theta_0 + \lambda} \right), \\ (A_2 + \rho^2 B_2) \mu_1 = (\lambda + \mu) \pi_{12} - (A_1 x_1 + B_1 x_2) \theta_1, \\ \qquad - \left(C_1 x_1^2 + D_1 x_2^2 + E_1 \left(\dfrac{\lambda}{\theta_0 + \lambda} \right)^2 \right) \mu - \dfrac{\theta_0 \theta_1}{\theta_0 + \lambda} (A_1 + B_1). \end{cases} \tag{32}$$

After solving (32), we have A_2 and B_2.

And thus, from (31)

$$\pi_{n2} = A_2 + B_2 \rho^n + C_1 x_1^n + D_1 x_2^n + E_1 \left(\frac{\lambda}{\theta_0 + \lambda} \right)^n, \quad n = 1, 2, \dots, n(0) + 1. \tag{33}$$

Let $n = n(0) + 1$ in (19), we find $\pi_{n(0)+2,2}$.

Now, we consider the probabilities $\{\pi_{n2} | n(0) + 1 \le n \le n(1) + 2\}$. From (20), they can be obtained by solving the following nonhomogeneous linear difference equations:

$$\begin{aligned} \mu x_{n+1} - (\lambda + \mu) x_n + \lambda x_{n-1} &= -\theta_1 \pi_{n1} \\ &= -\theta_1 (A_1 x_1^n + B_1 x_2^n), \quad n = n(0) + 2, \dots, n(1) + 1. \end{aligned} \tag{34}$$

By observing the homogeneous version of (34), we can get the general solution as $x_n^{hom} = A_3 1^n + B_3\rho^n$. So the general solution denoted by $x_n^{gen} = x_n^{hom} + x_n^{spec}$, where x_n^{spec} is a specific solution of (34). Because geometric with parameter x_1, x_2 in the nonhomogeneous part of (34), we obtain $x_n^{spec} = C_2 x_1^n + D_2 x_2^n$ is a specific solution. Substituting $x_n^{spec} = C_2 x_1^n + D_2 x_2^n$ into (34), we get

$$\begin{cases} C_2 = \dfrac{\theta_1 x_1 A_1}{(x_1 - 1)(\lambda - \mu x_1)}, \\ D_2 = \dfrac{\theta_1 x_2 B_1}{(x_2 - 1)(\lambda - \mu x_2)}. \end{cases} \tag{35}$$

Hence, the general solution of (34) is given as

$$x_n^{gen} = A_3 1^n + B_3 \rho^n + C_2 x_1^n + D_2 x_2^n, \quad n = n(0) + 1, \ldots, n(1) + 2, \tag{36}$$

where C_2, D_2 are given by (35), and A_3, B_3 are to be obtained. Taking into account (19) and (33), we then obtain the equations about A_3 and B_3. However, their expressions are too lengthy and verbose to display detailed results in this page. Solving these two equations, we get A_3 and B_3. And thus, from (36)

$$\pi_{n2} = A_3 + B_3 \rho^n + C_2 x_1^n + D_2 x_2^n, \quad n = n(0) + 1, \ldots, n(1) + 2. \tag{37}$$

Finally, we consider the probabilities $\{\pi_{n2} | n(1) + 1 \leq n \leq n(2) + 1\}$. From (21), we get them by solving for following homogeneous linear equation.

$$\mu x_{n+1} - (\lambda + \mu) x_n + \lambda x_{n-1} = 0, \quad n = n(1) + 2, \ldots, n(2). \tag{38}$$

It can be known from the homogeneous version of (38) that the general solution is $x_n^{hom} = A_4 1^n + B_4 \rho^n$, where A_4 and B_4 are to be determined. Considering the (22) and (37), we obtain

$$\begin{cases} (\mu - \lambda) A_4 + (\mu\rho - \lambda) B_4 \rho^{n(2)} = 0, \\ A_4 + B_4 \rho^{n(1)+2} = A_3 + B_3 \rho^{n(1)+2} + C_2 x_1^{n(1)+2} + D_2 x_2^{n(1)+2}. \end{cases} \tag{39}$$

Solving (39), we get

$$\begin{cases} A_4 = 0, \\ B_4 = \dfrac{A_3 + B_3 \rho^{n(1)+2} + C_2 x_1^{n(1)+2} + D_2 x_2^{n(1)+2}}{\rho^{n(1)+2}}. \end{cases} \tag{40}$$

And thus,

$$\pi_{n2} = B_4 \rho^n, \quad n = n(1) + 1, \ldots, n(2) + 1, \tag{41}$$

where B_4 is given by (40).

In conclusion, we find that the steady-state probabilities $\{\pi_{ni} | (n, i) \in \Omega_{fo}\}$ are related to π_{12}. Using the normalization condition

$$\sum_{(n,i) \in \Omega_{fo}} \pi_{ni} = 1,$$

we get the result of π_{12}.

Based on the stationary probabilities and the PASTA property, the social welfare per time unit, denoted by $S_{fo}(n(0), n(1), n(2))$, equals

$$\begin{aligned} S_{fo}(n(0), n(1), n(2)) = {} & \lambda R (1 - \pi_{n(0)+1,0} - \pi_{n(1)+1,1} - \pi_{n(2)+1,2}) \\ & - C \left(\sum_{n=0}^{n(0)+1} n \pi_{n0} + \sum_{n=0}^{n(1)+1} n \pi_{n1} + \sum_{n=1}^{n(2)+1} n \pi_{n2} \right). \end{aligned} \tag{42}$$

When all customers follow the equilibrium threshold strategy $(n_e(0), n_e(1), n_e(2))$, the social welfare per time unit in equilibrium can be expressed as $S_{fo}(n_e(0), n_e(1), n_e(2))$.

4. **The Almost Observable Queues.** In this section, we consider the almost observable case, where the customers are only informed the number of customers $N(t)$ in the system. Thus, the customers follow the same threshold strategy n_e. Taking $n(0) = n(1) = n(2) = n_e$, we get stationary distribution of the corresponding Markov chain. The transition diagram is shown in Figure 2.

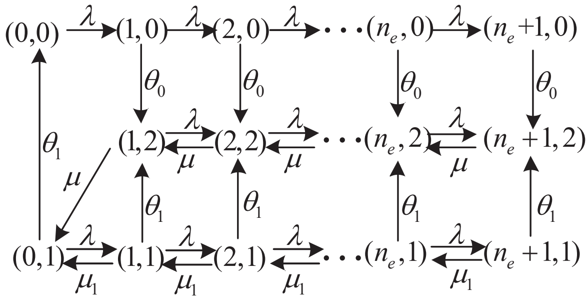

FIGURE 2. Transition rate diagram for the almost observable queues

Our first concern is the expected net welfare of arriving customer if there are n customers in system. We have the following result.

Lemma 4.1. *In the almost observable queues with a single working vacation and multiple vacations, customers use same threshold n_e. Customers enter system if $N(t) \le n_e$, else customers balk. If there are n customers in the system, the net welfare of a customer that decides to enter is given by*

$$U_n = R - C\left(\frac{n+1}{\mu} + \frac{1}{\theta_0}\right)\pi_{I|N}(0|n) - C\left(\frac{n+1}{\mu} + \frac{\mu-\mu_1}{\theta_1\mu}\left(1 - \left(\frac{\mu_1}{\mu_1+\theta_1}\right)^{n+1}\right)\right) \tag{43}$$
$$\times\, \pi_{I|N}(1|n) - C\tfrac{n+1}{\mu}\pi_{I|N}(2|n), \quad n = 0, 1, \dots, n_e + 1,$$

where $\pi_{I|N}(i|n)$ $(i = 0, 1, 2)$ is the probability that an arriving customer finds the server at state i, given that there are n customers.

Proof: The net welfare of a customer that observes n customers decides to enter is given by:

$$U_n = R - CW_n, \tag{44}$$

$W_n = E[S|N^- = n]$ means his mean sojourn time when he finds n customers in the system before he arrives. Conditioning on the state of the server that he finds upon arrival we obtain

$$W_n = W_{n0}\pi_{I|N}(0|n) + W_{n1}\pi_{I|N}(1|n) + W_{n2}\pi_{I|N}(2|n), \tag{45}$$

and

$$\pi_{I|N}(i|n) = \frac{\lambda\pi_{n0}}{\lambda\pi_{n0} + \lambda\pi_{n1} + \lambda\pi_{n2}}, \quad n = 0, 1, \dots, n_e + 1,$$

where $\pi_{02} = 0$.

Using the stationary probabilities, we obtain the probabilities $\pi_{I|N}(0|n)$, $\pi_{I|N}(1|n)$ and $\pi_{I|N}(2|n)$ for $n = 0, 1, \dots, n_e + 1$. So, we obtain W_n $(n = 1, 2, \dots, n_e + 1)$. It is worth noting that customer does not enter the empty system if $U_0 < 0$. Otherwise, he enters the queue.

The sojourn time of arriving customer who finds j $(1 \leq j \leq n_e + 1)$ customers in the system is greater than $j - 1$ customers in the system. In other words, W_n increases with respect to n. Form (44), we obtain that U_n decreases with respect to n.

Making definitions, $f_1(n_e) = U_{n_e}$, $f_2(n_e + 1) = U_{n_e+1}$.

$$
\begin{aligned}
f_1(n) = R - C\left(\frac{n+1}{\mu} + \frac{1}{\theta_0}\right) & \frac{\left(\frac{\theta_1}{\lambda}\right)^2 \left(\frac{\lambda}{\lambda+\theta_0}\right)^n (A_1 + B_1)}{\psi_1(n)} \\
-C\left(\frac{n+1}{\mu} + \frac{\mu - \mu_1}{\theta_1 \mu}\left(1 - \left(\frac{\mu_1}{\mu_1 + \theta_1}\right)^{n+1}\right)\right) & \frac{A_1 x_1^n + B_1 x_2^n}{\psi_1(n)} \qquad (46) \\
-C\frac{n+1}{\mu} & \frac{A_2 + B_2\rho^n + C_1 x_1^n + D_1 x_2^n + E_1\left(\frac{\lambda}{\lambda+\theta_0}\right)^n}{\psi_1(n)}, \quad n = 0, 1, \ldots,
\end{aligned}
$$

$$
\begin{aligned}
f_2(n) = R - C\left(\frac{n+1}{\mu} + \frac{1}{\theta_0}\right) & \frac{\frac{\theta_1^2}{\lambda\theta_0} \left(\frac{\lambda}{\lambda+\theta_0}\right)^n (A_1 + B_1)}{\psi_2(n)} \\
-C\left(\frac{n+1}{\mu} + \frac{\mu - \mu_1}{\theta_1 \mu}\left(1 - \left(\frac{\mu_1}{\mu_1 + \theta_1}\right)^{n+1}\right)\right) & \frac{A_1 x_1^n + B_1 x_2^n}{\psi_2(n)} \qquad (47) \\
-C\frac{n+1}{\mu} & \frac{A_2 + B_2\rho^n + C_1 x_1^n + D_1 x_2^n + E_1\left(\frac{\lambda}{\lambda+\theta_0}\right)^n}{\psi_2(n)}, \quad n = 1, 2, \ldots,
\end{aligned}
$$

where $\psi_1(n) = (\frac{\theta_1}{\lambda})^2 \left(\frac{\lambda}{\lambda+\theta_0}\right)^n (A_1 + B_1) + A_1 x_1^n + B_1 x_2^n + A_2 + B_2\rho^n + C_1 x_1^n + D_1 x_2^n + E_1\left(\frac{\lambda}{\lambda+\theta_0}\right)^n$, $\psi_2(n) = \frac{\theta_1^2}{\lambda\theta_0}\left(\frac{\lambda}{\lambda+\theta_0}\right)^n (A_1 + B_1) + A_1 x_1^n + B_1 x_2^n + A_2 + B_2\rho^n + C_1 x_1^n + D_1 x_2^n + E_1\left(\frac{\lambda}{\lambda+\theta_0}\right)^n$.

Function $f_1(n)$ is the expected net welfare of a customer that observes n customers and decides to enter if given that all other customers follow n threshold strategy, and $f_2(n)$ is the expected net welfare of a customer that observes n customers and decides to enter if given that all other customers follow $n - 1$ threshold strategy. Moreover $f_1(n) > f_2(n)$, $n = 0, 1, \ldots$.

We have that if $W_0 > 0$ and $\lim_{n\to\infty} f_1(n) = -\infty$ then $f_1(0) > 0$. So, we can find finite number $n_U + 1$ in the sequence $(f_1(n))$ that satisfies inequality

$$f_1(0), f_1(1), \ldots, f_1(n_U) > 0, f_1(n_U + 1) \leq 0. \tag{48}$$

Obviously, $f_1(n) > f_2(n)$ $(n = 0, 1, \ldots)$, so $f_2(n_U + 1) < f_1(n_U + 1) \leq 0$. In the range from 0 to $n_U + 1$, we can find that n_L satisfies the inequality

$$f_2(n_L) > 0, f_2(n_L + 1), f_2(n_L + 2), \ldots, f_2(n_U + 1) \leq 0, \tag{49}$$

where n_L is the first positive term of the sequence $(f_2(n))$. If the sequence $(f_2(n))$ is non-positive between 0 and $n_U + 1$, then we have

$$f_2(0), f_2(1), \ldots, f_2(n_U), f_2(n_U + 1) \leq 0. \tag{50}$$

Next, we consider equilibrium threshold strategies and have the following theorem.

Theorem 4.1. *In the almost observable queues with a single working vacation and multiple vacations, $n_e = n_L, n_L + 1, \ldots, n_U$ are equilibrium strategies.*

Proof: If an arriving customer finds the number of the customers in the system is n ($n \le n_e$), the customer prefers to enter. It can be known from expected net welfare $f_1(n) > 0$ because of (44), (46) and (48).

If an arriving customer finds the number of customers in the system is $n_e + 1$, the customer prefers to balk. It can be known from expected net welfare $f_2(n_e + 1) \le 0$ because of (44), (47), (49) and (50).

The social welfare per time unit when all customers follow the threshold policies n_e given in Theorem 4.1 equals:

$$\begin{aligned} S_{ao}(n_e) &= \lambda R(1 - \pi_{n_e+1,0} - \pi_{n_e+1,1} - \pi_{n_e+1,2}) \\ &\quad - C\left(\sum_{n=0}^{n_e+1} n\pi_{n0} + \sum_{n=0}^{n_e+1} n\pi_{n1} + \sum_{n=1}^{n_e+1} n\pi_{n2}\right). \end{aligned} \tag{51}$$

5. **Numerical Example.** In this section, the effects of several parameters on customers' behavior in the fully observable and almost observable queues were studied by numerical experiments. Specifically, the equilibrium thresholds and social welfare under equilibrium thresholds have attracted our interest.

We first consider the sensitivity of the system parameters (service reward R, service rate μ, vacation rates θ_0, θ_1 and arrival rate λ) to equilibrium threshold strategies. The results are shown in Figures 3-7. The meaningful facts are found in these figures. Firstly, the difference between thresholds $n_e(0)$ and $n_e(2)$ for the fully observable queues is relatively large. Secondly, the server's state has a significant impact on the thresholds of the fully observable queues. Finally, the range of thresholds for the almost observable queues $\{n_L, n_L+1, \ldots, n_U\}$ is contained within the range between thresholds $n_e(0)$ and $n_e(2)$ for the fully observable queues.

Using the following numerical examples, we observe the sensitivity of the particular parameter. Form Figure 3, we observe that when service reward R increases, the thresholds increase in a linear fashion. Customers prefer to join if they can receive higher values of the benefit upon service completion. In Figure 4, we find that equilibrium thresholds

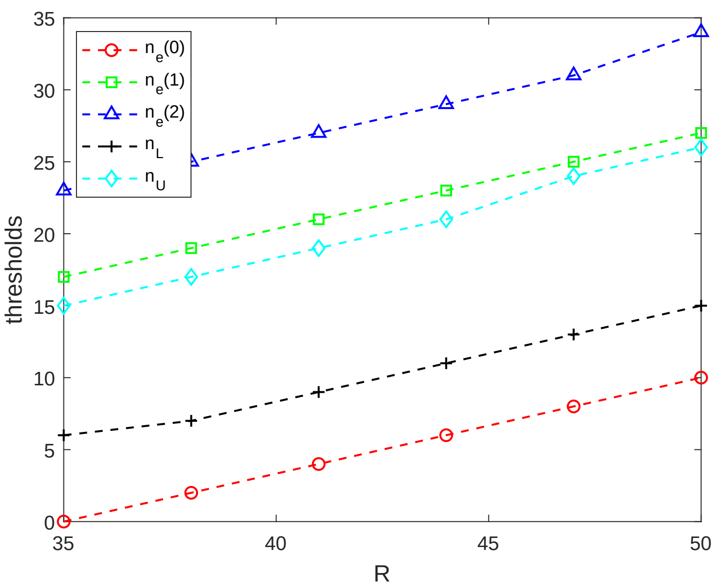

FIGURE 3. Equilibrium thresholds for the fully and almost observable queues when $C = 1$, $\theta_0 = 0.03$, $\theta_1 = 0.06$, $\mu_1 = 0.3$, $\mu = 0.7$, $\lambda = 0.2$

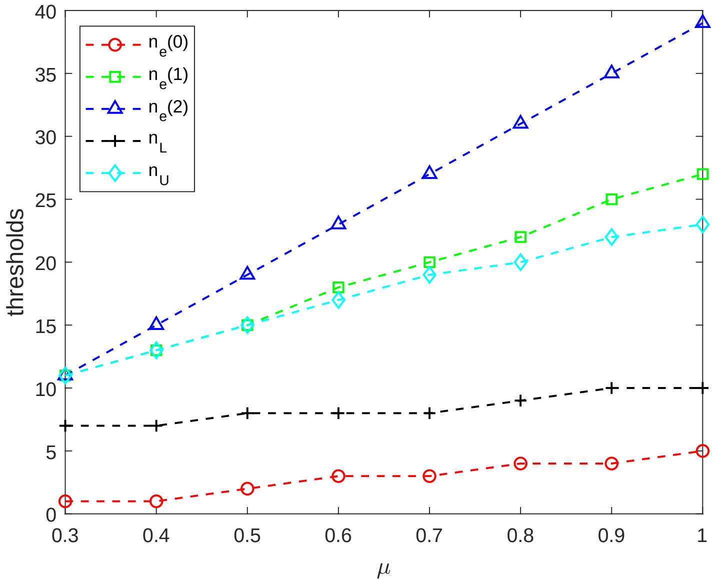

FIGURE 4. Equilibrium thresholds for the fully and almost observable queues when $R = 40$, $C = 1$, $\theta_0 = 0.03$, $\theta_1 = 0.06$, $\mu_1 = 0.3$, $\lambda = 0.2$

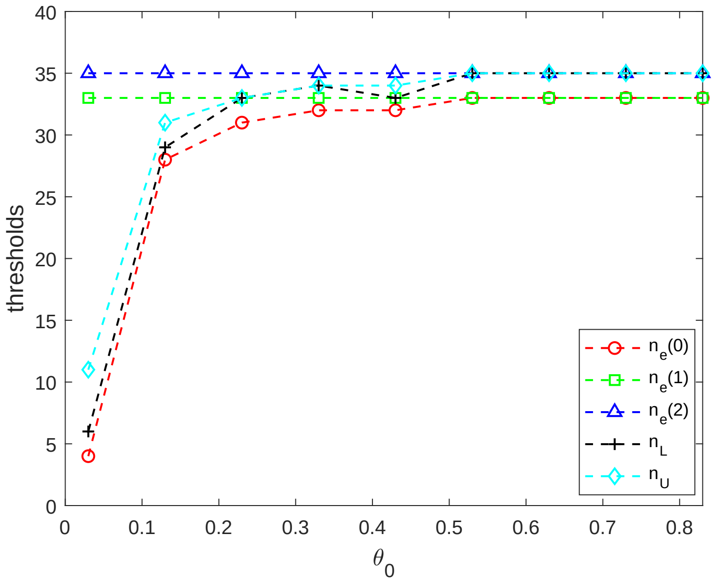

FIGURE 5. Equilibrium thresholds for the fully and almost observable queues when $R = 40$, $C = 1$, $\theta_1 = 0.3$, $\mu_1 = 0.5$, $\mu = 0.9$, $\lambda = 0.2$

are all increasing with respect to the service rate μ. The reason is that more customers the server serves per unit time the less waiting time the customers have. So, customers are more willing to enter the system. Figure 5 and Figure 6 show that the thresholds increase as vacation rates θ_0 and θ_1 increase except $n_e(2)$ which is a constant. It is easy to understand that when the server is activated faster, the customer generally has more motivation to enter the system. The arrival rate λ has no effect on the fully observable queues thresholds as described in Figure 7. This can be derived from Theorem 3.2, the arrival rate λ has nothing to do with the customer's decision. The thresholds for the almost observable queues increase with λ. When customers only know the number of

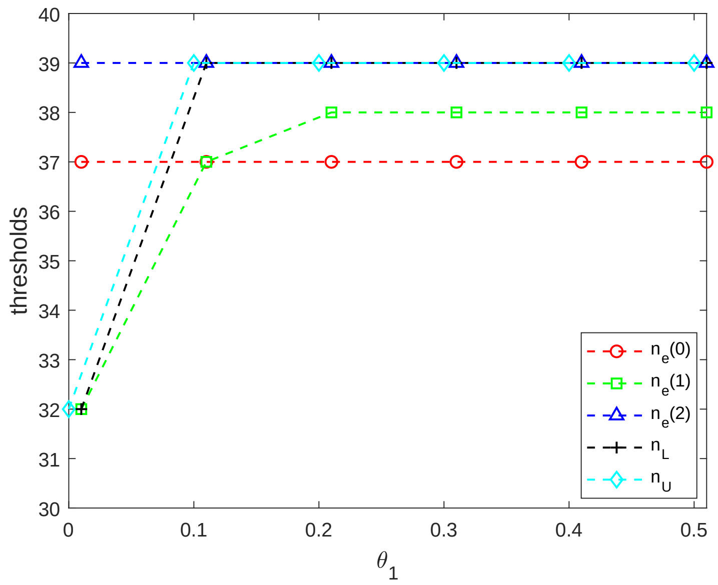

FIGURE 6. Equilibrium thresholds for the fully and almost observable queues when $R = 40$, $C = 1$, $\theta_0 = 0.6$, $\mu_1 = 0.8$, $\mu = 1$, $\lambda = 0.5$

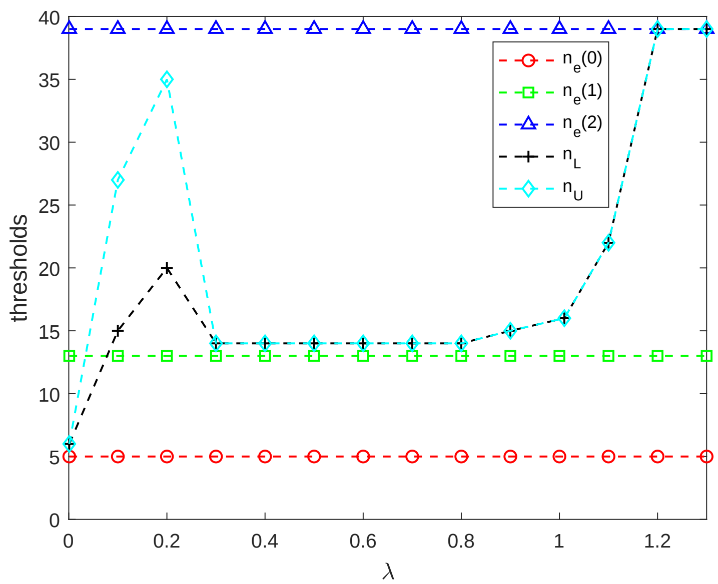

FIGURE 7. Equilibrium thresholds for the fully and almost observable queues when $R = 40$, $C = 1$, $\theta_0 = 0.03$, $\theta_1 = 0.01$, $\mu_1 = 0.3$, $\mu = 1$

customers in the system, they are more willing to enter the system under higher arrival rate.

Subsequently, we focus on the social welfare under the equilibrium thresholds for the different information levels. Figures 8-10 show the sensitivity in parameters service reward R, service rate μ and arrival rate λ to social welfare. There are multiple equilibrium strategies in the almost observable queues, $\{n_L, n_L + 1, \ldots, n_U\}$. In order to observe the range of social welfare, we use extreme threshold values n_L and n_U to calculate. The difference between n_L and n_U is small, which may result in very close values for social welfare $S_{ao}(n_L)$ and $S_{ao}(n_U)$. By comparisons, the fully observable queues do not always

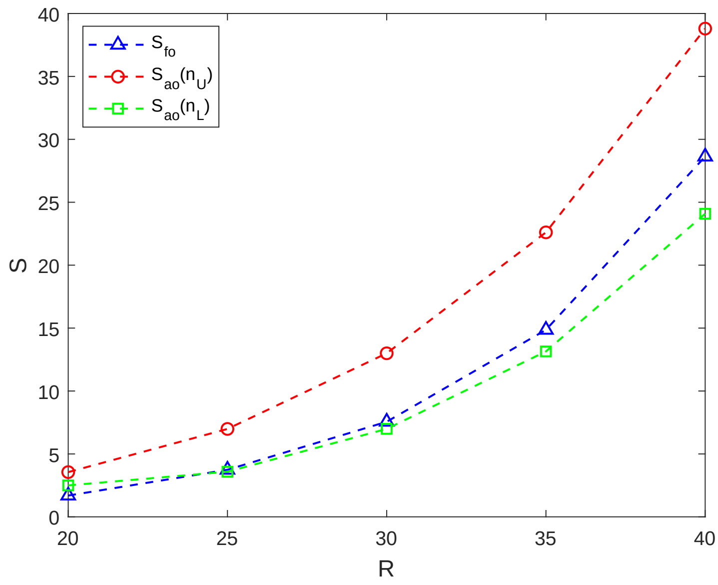

FIGURE 8. Social welfare for the fully and almost observable queues when $C = 1$, $\theta_0 = 0.05$, $\theta_1 = 0.05$, $\mu_1 = 0.3$, $\mu = 1$, $\lambda = 0.2$

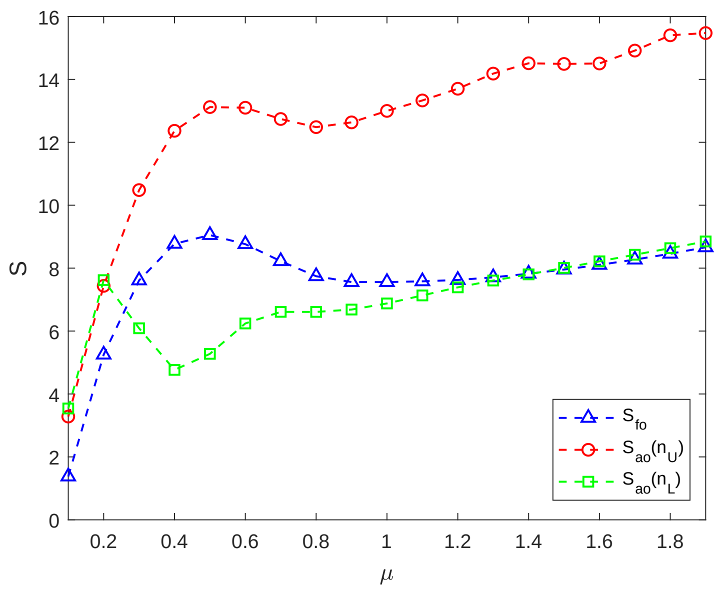

FIGURE 9. Social welfare for the fully and almost observable queues when $R = 30$, $C = 1$, $\theta_0 = 0.05$, $\theta_1 = 0.05$, $\mu_1 = 0.3$, $\lambda = 0.2$

have higher social welfare. This reveals that additional information of server state is not always beneficial to increase social welfare.

What is more, Figures 8-10 show the results. In Figure 8, social welfare increases with respect to service reward R, which is intuitive. Similarly, form Figure 9, we find that social welfare increases as service rate μ increases. The reason for this behavior is that an increasing of normal service rate can reduce customers' expected waiting time. For the arrival rate λ, Figure 10 shows that the social welfare first increases then decreases, and has a maximum. Form Figure 7 we find that n_U and n_L increase rapidly when $\lambda > 0.9$, so $S_{ao}(n_U)$ and $S_{ao}(n_L)$ increase rapidly in the range of $\lambda = 0.9$ to $\lambda = 1.1$. The reason

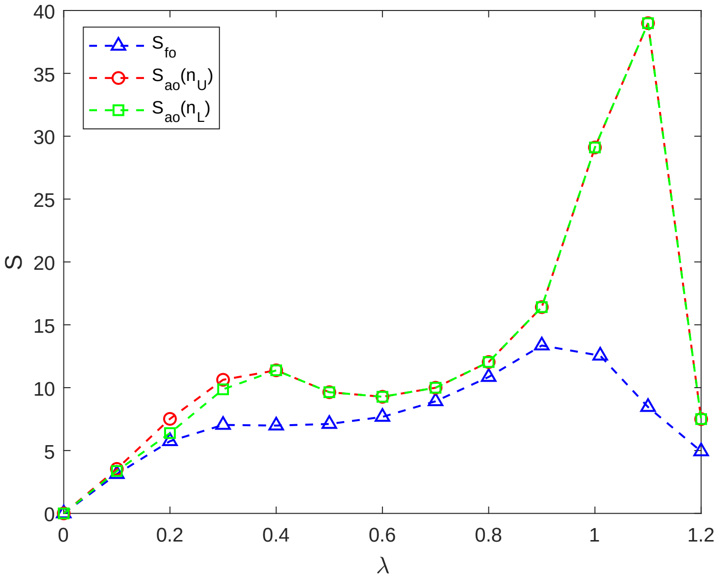

FIGURE 10. Social welfare for the fully and almost observable queues when $R = 40$, $C = 1$, $\theta_0 = 0.03$, $\theta_1 = 0.01$, $\mu_1 = 0.3$, $\mu = 1$

is that when λ is small, customers do not have to wait a long time to get service, which improves social welfare. However, as the arrival rate continues to increase, more and more customers are in the system, which leads customers to face higher and higher waiting costs and social welfare is decreasing. So, system congestion has a harmful effect on the social welfare.

6. **Conclusion.**

In this paper we analyzed the customer strategic behavior in the M/M/1 queueing systems with a single working vacation and multiple vacations. The system information is used by the customer to decide whether to enter or balk. We derived the equilibrium strategies of the fully observable and almost observable queues. We also discussed the sensitivity of information level to equilibrium strategies and social welfare and conjectured that customers know all the information of server state is not always beneficial to increase social welfare. This paper focuses on equilibrium analysis. An interesting extension is to study social optimization issues that maximize the overall welfare of all customers.

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