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CONTROLLABILITY AND OBSERVABILITY FOR A CLASS OF DYNAMIC SYSTEMS WITH INTERNAL CAUSES

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Abstract. *In this paper, controllability and observability are investigated for a dynamic system with an internal cause. The dynamic system is represented as a nonlinear model which is linearized to a linear one by a Taylor expansion method. Moreover, Gramian and rank criteria are developed to analyze the controllability and observability for the dynamic system. Based on the developed criteria, logical relationships of input-state and state-output can be analyzed profoundly for the dynamic system. Finally, an example is provided to verify the controllability and observability conclusions.*

Keywords: Dynamic system, Internal cause, Controllability, Observability

1. **Introduction.** For dynamic systems, states of motion always evolve with time according to deterministic or deterministic statistical laws which are called internal causes in [1]. Internal causes include rule, command, constraint, psychology, emotion and other factors influencing evolving processes of dynamic systems [2]. For example, a price game between buyers and sellers is an internal cause affecting stock prices in economic systems [3]. In [4], a novel dynamic system with an internal cause is proposed to model a cyber-physical system in an insecure network environment. Furthermore, some criteria are investigated to guarantee that the novel dynamic system is globally asymptotically stable and quadratically stable in [5]. It is noted that an internal cause can determine system state information and then affect logical relationships of input-state and state-output for dynamic systems. Therefore, controllability and observability analyses are important in the research on the dynamic system with an internal cause. Although this topic has been investigated preliminarily, controllability and observability have not been considered in [5], which motivated us carry on this research work.

Considering a dynamic system, its controllability should be analyzed and studied firstly, which means that the system input of the dynamic system can affect internal states [6]. Meanwhile, observability of the dynamic system is another important issue to be investigated [7]. The observability of the dynamic system indicates whether the internal states can be reflected by the system output [8]. In [9], some sufficient uncontrollability and unobservability criteria are given for nonlinear dynamic systems on the basis of known uncontrollability and unobservability criteria for linear systems and additional linear conditions. Moreover, the concepts of local controllability and observability are studied for nonlinear discrete-time systems with the Caputo-, Riemann-Liouville- and Grünwald-Letnikov-type h-difference fractional order operators in [10]. However, an internal cause is not considered for nonlinear dynamic systems in above literature where

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system state information cannot be altered by changing an internal cause. It is obvious that the controllability and observability correspond to an internal description of dynamic systems, which prompts us to investigate the controllability and observability for the dynamic systems with an internal cause.

In this paper, controllability and observability are investigated for a dynamic system with an internal cause. Gramian and rank criteria are used to analyze the controllability and observability, and logical relationships of the input-state and state-output are more deeply described and investigated for the dynamic system. Moreover, a numerical result is presented to show effectiveness of the proposed technique in this paper.

2. **Problem Statement and Preliminaries.** Based on [4], a dynamic system with an internal cause is modeled as a nonlinear model in the following

$$\dot{e}(t) = \mathcal{E}\left(t, e(t), g'(t)\right), \tag{1}$$

$$e'(t) = \mathcal{I}\left(t, e(t), p(t)\right), \tag{2}$$

$$y(t) = \bar{\mathcal{E}}\left(t, e'(t)\right), \tag{3}$$

where $g'(t) \in R^S$ denotes the control input, $p(t) \in R$ denotes the internal cause, $e(t) \in R^\Phi$ denotes the state before changing the internal cause, $e'(t) \in R^\Phi$ denotes the state after changing the internal cause, $y(t) \in R$ denotes the output, $\mathcal{E}(\cdot)$, $\mathcal{I}(\cdot)$ and $\bar{\mathcal{E}}(\cdot)$ are a series of nonlinear functions. Suppose that function $\mathcal{I}(t, e(t), p(t))$ is invertible for the time instant t. Then it is easy to find an inverse function such that $e(t) = \mathcal{I}^{-1}\left(t, e'(t), p(t)\right)$ holds. The dynamic system with an internal cause (1)-(3) is rewritten as

$$\dot{e}'(t) = F\left(t, e'(t), g'(t)\right), \tag{4}$$

$$y(t) = \bar{\mathcal{E}}\left(t, e'(t)\right), \tag{5}$$

where

$$F\left(t, e'(t), g'(t)\right) = \frac{\partial \mathcal{I}}{\partial t} + \frac{\partial \mathcal{I}}{\partial e(t)} \cdot \mathcal{E}\left(t, \mathcal{I}^{-1}\left(t, e'(t), p(t)\right), g'(t)\right) + \frac{\partial \mathcal{I}}{\partial p(t)} \cdot \dot{p}(t).$$

To simplify complexity of the nonlinear system (4)-(5), a linear model is obtained by taking Taylor expansion of system (4)-(5) at a neighborhood of (e'_0, g'_0) and ignoring high order terms, which is shown as

$$\dot{e}'(t) = A(t)e'(t) + B(t)g'(t), \tag{6}$$

$$y(t) = C(t)e'(t), \tag{7}$$

where

$$A(t) = \left(\frac{\partial F}{\partial e'^T}\right)_{e'_0, g'_0}, \quad B(t) = \left(\frac{\partial F}{\partial g'^T}\right)_{e'_0, g'_0}, \quad C(t) = \left(\frac{\partial \bar{\mathcal{E}}}{\partial e'^T}\right)_{e'_0, g'_0},$$

and (e'_0, g'_0) is the operating point which is selected by actual control requirements.

Definition 2.1. *For the linear dynamic system (6)-(7) and given initial time instant $t_0 \in J$ where J is the given interval, if there exist a time instant $t_1 \in J$ with $t_1 > t_0$ and an unconstrained admissible control $g'(t)$ with $t \in [t_0, t_1]$ such that the state transfers from $e'(t_0) = e'_0$ to $e'(t_1) = 0$, then a nonzero state e'_0 is controllable at the time instant t_0.*

Definition 2.2. *For the linear dynamic system (6)-(7) and given initial time instant $t_0 \in J$, if all nonzero states are controllable, then the linear dynamic system (6)-(7) is regarded as completely controllable at the time instant t_0.*

Definition 2.3. *For the linear dynamic system (6)-(7), if the initial time instant is $t_0 \in J$ and the observation time instant is $t \in J$, then the state transition matrix is denoted as an $n \times n$ solution matrix $\Phi(t, t_0)$ shown as*

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t, t_0) = \boldsymbol{I}.$$

3. **Controllability of Linear Dynamic Systems.** Controllability of the linear dynamic system (6)-(7) is studied in this section. Here are the main results in this paper.

Theorem 3.1. *The linear dynamic system (6)-(7) is completely controllable at the time instant $t_0 \in J$ if and only if there exists a finite time instant $t_1 \in J$ with $t_1 > t_0$ such that the Gramian matrix*

$$W_c[t_0, t_1] = \int_{t_0}^{t_1} \Phi(t_0, t_1)B(t)B^T(t)\Phi^T(t_0, t_1)dt$$

is nonsingular.

Proof: The proof processes are divided into two steps. Firstly, sufficiency of the theorem will be proven. Since $W_c[t_0, t_1]$ is nonsingular, there always exists $W_c^{-1}[t_0, t_1]$. For any initial state $e_0' \neq 0$, the input of the linear dynamic system (6)-(7) is constructed as

$$g'(t) = -B^T(t)\Phi^T(t_0, t_1)W_c^{-1}[t_0, t_1]e_0'.$$

Under the control of input $g'(t)$, it is easy to obtain that

$$\begin{aligned} e'(t_1) &= \Phi^T(t_0, t_1)e_0' + \Phi(t_0, t_1)\int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)d\tau \\ &= \Phi^T(t_0, t_1)e_0' - \Phi(t_0, t_1)W_c[t_0, t_1]W_c^{-1}[t_0, t_1]e_0' \\ &= \Phi^T(t_0, t_1)e_0' - \Phi^T(t_0, t_1)e_0' = 0. \end{aligned}$$

In view of Definition 2.2, the linear dynamic system (6)-(7) is completely controllable. Therefore, the sufficiency of the theorem is proved.

Secondly, necessity of the theorem will be proven. It is known that the linear dynamic system (6)-(7) is completely controllable. Assume that matrix $W_c[t_0, t_1]$ is singular. Then there exists a nonzero state e_0' such that

$$\begin{aligned} 0 = e_0'^T W_c[t_0, t_1]e_0' &= \int_{t_0}^{t_1} e_0'^T \Phi(t_0, \tau)B(\tau)B^T(\tau)\Phi^T(t_0, \tau)e_0' d\tau \\ &= \int_{t_0}^{t_1} \left\| e_0'^T \Phi(t_0, \tau)B(\tau) \right\|^2 d\tau. \end{aligned}$$

Therefore, it is not difficult to obtain that $e_0'^T \Phi(t_0, \tau)B(\tau) = 0$. And if the linear dynamic system (6)-(7) is completely controllable, one has that

$$e_0'^T e_0' = -\int_{t_0}^{t_1} \left[e_0'^T \Phi(t_0, \tau)B(\tau)\right] u(\tau)d\tau = 0$$

for state e_0'. That is, condition $e_0' = 0$ contradicts with the known condition $e_0' \neq 0$. Thus, matrix $W_c[t_0, t_1]$ is nonsingular. □

Theorem 3.2. *Consider an n-dimensional linear dynamic system (6)-(7) where matrices $A(t)$ and $B(t)$ are $n - 1$ order continuous differentiable for the time instant t. If there exists a finite time instant $t_1 \in J$ with $t_1 > t_0$ satisfying*

$$\operatorname{rank}\begin{bmatrix} M_0(t_1) & M_1(t_1) & \cdots & M_{n-1}(t_1) \end{bmatrix} = n, \tag{8}$$

where

$$M_0(t) = B(t),$$
$$M_1(t) = -A(t)M_0(t) + \frac{d}{dt}M_0(t),$$
$$\vdots$$
$$M_{n-1}(t) = -A(t)M_{n-2}(t) + \frac{d}{dt}M_{n-2}(t),$$

then the n-dimensional linear dynamic system (6)-(7) is completely controllable at the time instant $t_0 \in J$.

Proof: In order to make the proof clear, it is divided into the following three steps. Firstly, considering $\Phi(t_0,t_1)B(t_1) = \Phi(t_0,t_1)M_0(t_1)$, one has that

$$\frac{\partial}{\partial t_1}[\Phi(t_0,t_1)B(t_1)] = \left[\frac{\partial}{\partial t}[\Phi(t_0,t)B(t)]\right]_{t=t_1},$$

which results in

$$\begin{aligned}&\left[\begin{array}{cccc}\Phi(t_0,t_1)B(t_1) & \frac{\partial}{\partial t_1}\Phi(t_0,t_1)B(t_1) & \cdots & \frac{\partial^{n-1}}{\partial t_1^{n-1}}\Phi(t_0,t_1)B(t_1)\end{array}\right]\\ &= \Phi(t_0,t_1)\left[\begin{array}{cccc}M_0(t_1) & M_1(t_1) & \cdots & M_{n-1}(t_1)\end{array}\right].\end{aligned} \tag{9}$$

Based on Equations (8) and (9), it is obtained that

$$\operatorname{rank}\left[\begin{array}{cccc}\Phi(t_0,t_1)B(t_1) & \frac{\partial}{\partial t_1}\Phi(t_0,t_1)B(t_1) & \cdots & \frac{\partial^{n-1}}{\partial t_1^{n-1}}\Phi(t_0,t_1)B(t_1)\end{array}\right] = n. \tag{10}$$

Secondly, this step will prove that matrix $\Phi(t_0,t)B(t)$ is column linear independent for $t \in [t_0,t_1]$, which is proven by reduction to absurdity. Supposing Equation (10) holds and $\Phi(t_0,t)B(t)$ is linear dependent, then there exists a nonzero constant vector α satisfying

$$\alpha\Phi(t_0,t)B(t) = 0. \tag{11}$$

Besides, for $t \in [t_0,t_1]$ and $k = 1,2,\ldots,n-1$, one has that

$$\alpha\frac{\partial^k}{\partial t^k}\Phi(t_0,t)B(t) = 0. \tag{12}$$

By Equations (11) and (12), it is obtained that

$$\alpha\left[\begin{array}{cccc}\Phi(t_0,t)B(t) & \frac{\partial}{\partial t}\Phi(t_0,t)B(t) & \cdots & \frac{\partial^{n-1}}{\partial t^{n-1}}\Phi(t_0,t)B(t)\end{array}\right] = 0,$$

which means that

$$\left[\begin{array}{cccc}\Phi(t_0,t)B(t) & \frac{\partial}{\partial t}\Phi(t_0,t)B(t) & \cdots & \frac{\partial^{n-1}}{\partial t^{n-1}}\Phi(t_0,t)B(t)\end{array}\right] = 0$$

is linear dependent. The fact is contradictory to the known condition, which suggests that $\Phi(t_0,t)B(t)$ is linear independent.

Thirdly, this step will prove that matrix $W_c[t_0,t_1]$ is nonsingular using reduction to absurdity. Supposing $W_c[t_0,t_1]$ is singular, then there exists a nonzero constant vector α satisfying

$$0 = \alpha W_c[t_0,t_1]\alpha^T = \int_{t_0}^{t_1}[\alpha\Phi(t_0,t)B(t)]\left[\alpha\Phi^T(t_0,t)B(t)\right]^T dt = \int_{t_0}^{t_1}\|\alpha\Phi(t_0,t)B(t)\|^2 dt.$$

The integrand in the integral above is a continuous function and is nonnegative for all $t \in [t_0, t_1]$, which leads to $\alpha\Phi(t_0, t)B(t) = 0$ for $t \in [t_0, t_1]$. It is contradictory to the fact that $\Phi(t_0, t)B(t)$ is linear independent. Therefore, matrix $W_c[t_0, t_1]$ is nonsingular.

In conclusion, according to the Gramian matrix criterion, the n-dimensional linear dynamic system (6)-(7) is completely controllable at the time instant t_0. □

Remark 3.1. *Although the form of the Gramian criterion in Theorem 3.1 is simple, its application is limited due to the difficulty in solving the state transition matrix. Moreover, rank criterion in Theorem 3.2 is characterized by a direct use of the coefficient matrix to judge controllability of the linear dynamic system (6)-(7).*

4. **Observability of Linear Dynamic Systems.** Observability of the linear dynamic system (6)-(7) is studied in this section. It is noted that observability is a system property to describe whether state $e'(t)$ can be fully reflected by output $y(t)$. Therefore, observability is uncorrelated with input $g'(t)$. To simplify analysing processes, input $g'(t)$ is selected as $g'(t) = 0$. Then the linear dynamic system (6)-(7) is further rewritten as

$$\dot{e}'(t) = A(t)e'(t), \tag{13}$$

$$y(t) = C(t)e'(t). \tag{14}$$

Then some definitions on observability of the linear dynamic system (13)-(14) are given in the following.

Definition 4.1. *For the linear dynamic system (13)-(14) and given initial time instant $t_0 \in J$, if there exists a time instant $t_1 \in J$ with $t_1 > t_0$ such that output $y(t) \equiv 0$ holds for any $t \in [t_0, t_1]$, then a nonzero state e'_0 is unobservable at the time instant t_0.*

Definition 4.2. *For the linear dynamic system (13)-(14) and given initial time instant $t_0 \in J$, if all nonzero states e'_0 are not unobservable at the time instant t_0, then the linear dynamic system (13)-(14) is completely observable at the time instant t_0.*

Based on given definitions on observability of the linear dynamic system (13)-(14), Gramian and rank criteria are given in the following theorems.

Theorem 4.1. *Let $\Phi(\cdot)$ be a state transition matrix. The linear dynamic system (13)-(14) is completely observable at a time instant $t_0 \in J$ if and only if there exists a time instant $t_1 \in J$ with $t_1 > t_0$ such that the following Gramian matrix*

$$W_o[t_0, t_1] = \int_{t_0}^{t_1} \Phi(t, t_0)C^T(t)C(t)\Phi(t, t_0)dt \tag{15}$$

is a nonsingular matrix.

Proof: The proof processes are divided into two steps. Firstly, sufficiency of the theorem will be proven. If the Gramian matrix (15) is nonsingular, then matrix $W_o^{-1}[t_0, t_1]$ holds. Therefore, for any $y(t) \neq 0$ in the time interval $[t_0, t_1]$, a corresponding unique initial state e'_0 is constructed as

$$W_o^{-1}[t_0, t_1] \int_{t_0}^{t_1} \Phi(t, t_0)C^T(t)y(t)dt = W_o^{-1}[t_0, t_1] \int_{t_0}^{t_1} \Phi(t, t_0)C^T(t)C(t)\Phi(t, t_0)dte'_0$$
$$= W_o^{-1}[t_0, t_1]W_o[t_0, t_1]e'_0 = e'_0,$$

which means that any nonzero state e'_0 is observable. According to Definition 4.2, the linear dynamic system (13)-(14) is completely observable. Therefore, the sufficiency of the theorem is proven.

Secondly, reduction to absurdity is utilized to prove the necessity of the theorem. If matrix $W_o[t_0, t_1]$ is singular, then there exists a nonzero state e'_0 such that

$$0 = e'^T_0 W_o[t_0, t_1] e'_0 = \int_{t_0}^{t_1} e'^T_0 \Phi^T(t, t_0) C^T(t) C(t) \Phi(t, t_0) e'_0 dt = \int_{t_0}^{t_1} \|C(t)\Phi(t, t_0)e'_0\|^2 dt$$

holds. Therefore, one has that

$$y(t) = C(t)\Phi(t, t_0)e'_0 \equiv 0, \ \forall t \in [t_0, t_1]. \tag{16}$$

Equation (16) means that e'_0 is an unobservable state, which contradicts the definition of completely observable. Therefore, $W_o[t_0, t_1]$ is nonsingular. □

Theorem 4.2. *Consider an n-dimensional linear dynamic system (13)-(14) in which $A(t)$ and $C(t)$ are $n-1$ order continuous differentiable at the time instant t. If there exists a time instant $t_1 \in J$ with $t_1 > t_0$ satisfying*

$$\text{rank}\left[\ N_0(t_1), N_1(t_1), \ldots, N_{n-1}(t_1)\ \right]^T = n, \tag{17}$$

where

$$\begin{aligned} &N_0(t) = C(t), \\ &N_1(t) = N_0(t)A(t) + \frac{d}{dt}N_0(t), \\ &\quad\vdots \\ &N_{n-1}(t) = N_{n-2}(t)A(t) + \frac{d}{dt}N_{n-2}(t), \end{aligned}$$

then the n-dimensional linear dynamic system (13)-(14) is completely observable at the time instant $t_0 \in J$.

Proof: The proof processes are divided into three steps to make the proof clear. Firstly, based on $C(t_1)\Phi(t_0, t_1) = N_0(t_1)\Phi(t_0, t_1)$ and

$$\frac{\partial}{\partial t_1}\left[C(t_1)\Phi(t_0, t_1)\right] = \left[\frac{\partial}{\partial t}C(t)\Phi(t_0, t)\right]_{t=t_1},$$

one has that

$$\begin{aligned} &\left[\ C(t_1)\Phi(t_0, t_1), \frac{\partial}{\partial t_1}C(t_1)\Phi(t_0, t_1), \ldots, \frac{\partial^{n-1}}{\partial t_1^{n-1}}C(t_1)\Phi(t_0, t_1)\ \right]^T \\ &= \left[\ N_0(t_1), N_1(t_1), \cdots, N_{n-1}(t_1)\ \right]^T \Phi(t_0, t_1). \end{aligned} \tag{18}$$

Then owing to non-singularity of matrix $\Phi(t_0, t_1)$, it is obtained that

$$\text{rank}\left[\ C(t_1)\Phi(t_0, t_1), \frac{\partial}{\partial t_1}C(t_1)\Phi(t_0, t_1), \ldots, \frac{\partial^{n-1}}{\partial t_1^{n-1}}C(t_1)\Phi(t_0, t_1)\ \right]^T = n. \tag{19}$$

Secondly, this step will prove that matrix $C(t)\Phi(t_0, t)$ is column linear independent for $t \in [t_0, t_1]$, which is proven by reduction to absurdity. Assume that matrix $C(t)\Phi(t_0, t)$ is column linear dependent. There exists an $n \times 1$-dimensional nonzero vector β such that

$$C(t)\Phi(t_0, t)\beta = 0 \tag{20}$$

holds for any $t \in [t_0, t_1]$. Therefore, for any $t \in [t_0, t_1]$ and $k = 1, 2, \ldots, n-1$, one has

$$\frac{\partial^k}{\partial t^k}C(t)\Phi(t_0, t)\beta = 0. \tag{21}$$

According to Equations (20) and (21), one can obtain that the following equation holds for any $t \in [t_0, t_1]$

$$\left[\begin{array}{c} C(t)\Phi(t_0,t), \frac{\partial}{\partial t}C(t)\Phi(t_0,t), \ldots, \frac{\partial^{n-1}}{\partial t^{n-1}}C(t)\Phi(t_0,t) \end{array} \right]^T \beta = 0. \tag{22}$$

Equation (22) holds meaning that matrix

$$\left[\begin{array}{c} C(t)\Phi(t_0,t), \frac{\partial}{\partial t}C(t)\Phi(t_0,t), \ldots, \frac{\partial^{n-1}}{\partial t^{n-1}}C(t)\Phi(t_0,t) \end{array} \right]^T$$

is column linear dependent, which contradicts Equation (19). Therefore, the assumption does not hold, that is, $C(t)\Phi(t_0,t)$ is column linear independent for any $t \in [t_0, t_1]$.

Thirdly, this step will prove that matrix $W_o[t_0, t_1]$ is nonsingular by reduction to absurdity. Assume that matrix $W_o[t_0, t_1]$ is singular. Then there exists an $n \times 1$-dimensional nonzero vector β such that

$$0 = \beta^T \left[\int_{t_0}^{t_1} \Phi(t_0,t)C^T(t)C(t)\Phi(t_0,t)dt \right] \beta = \int_{t_0}^{t_1} \beta \|C(t)\Phi(t_0,t)\|^2 dt$$

holds. The above integral is a continuous function and the function is nonzero for any $t \in [t_0, t_1]$. Therefore, one has that $\beta C(t)\Phi(t_0,t) = 0$ for all $t \in [t_0, t_1]$, which contradicts that matrix $C(t)\Phi(t_0,t)$ is column linear independent. Therefore, matrix $W_o[t_0, t_1]$ is singular. □

5. **Numerical Example.** In this section, a numerical example is given to demonstrate effectiveness of the results proposed in this paper. A dynamic system with an internal cause is given as follows

$$\begin{cases} e_1'(t) = \sqrt[3]{e_1(t)}, \\ e_2'(t) = e_2(t), \\ \dot{e}_1(t) = e_1(t) + tg'(t), \\ \dot{e}_2(t) = te_2(t) + g'(t), \\ y(t) = te_1'(t) + e_2'(t). \end{cases} \tag{23}$$

Note that system (23) is transformed into the following form by algebraic transformation

$$\begin{cases} \dot{e}_1'(t) = \frac{1}{3}e_1'(t) + \frac{1}{3}[e_1'(t)]^{-2}tg'(t), \\ \dot{e}_2'(t) = te_2'(t) + g'(t). \end{cases} \tag{24}$$

According to Theorem 3.1, the operating point of the dynamic system (24) is taken as $(e_0', g_0') = (1, 1)$. Then the dynamic system (24) is further linearized into the following linear dynamic system as

$$\begin{cases} \dot{e}'(t) = A(t)e'(t) + B(t)g'(t), \\ y(t) = C(t)e'(t), \end{cases} \tag{25}$$

where

$$A(t) = \left[\begin{array}{cc} \frac{1}{3} - \frac{2}{3}t & 0 \\ 0 & t \end{array} \right], \ B(t) = \left[\begin{array}{c} \frac{1}{3}t \\ 1 \end{array} \right], \ C(t) = \left[\begin{array}{cc} t & 1 \end{array} \right].$$

From Theorem 3.2, it is obtained that

$$M_0(t) = \begin{bmatrix} \frac{1}{3}t & 1 \end{bmatrix}^T, \; M_1(t) = \begin{bmatrix} \frac{2}{9}t^2 - \frac{1}{9}t + \frac{1}{3} & -t \end{bmatrix}^T.$$

Since there does not exist a solution t with $t \in R^+$ such that

$$\left\| \begin{matrix} \frac{1}{3}t & \frac{2}{9}t^2 - \frac{1}{9}t + \frac{1}{3} \\ 1 & -t \end{matrix} \right\| = 0$$

holds, one can obtain that

$$\text{rank} \begin{bmatrix} \frac{1}{3}t & \frac{2}{9}t^2 - \frac{1}{9}t + \frac{1}{3} \\ 1 & -t \end{bmatrix} = 2.$$

According to the rank criterion in Theorem 3.2, the state of linear dynamic system (25) is completely controllable in a time interval $[0, t]$.

Similarly, based on the rank criterion of observability in Theorem 4.2, one has that

$$N_0(t) = \begin{bmatrix} t & 1 \end{bmatrix}, \; N_1(t) = \begin{bmatrix} \frac{1}{3}t - \frac{2}{3}t^2 + 1 & t \end{bmatrix}.$$

Since there does not find a solution t with $t \in R^+$ such that

$$\left\| \begin{matrix} t & 1 \\ \frac{1}{3}t - \frac{2}{3}t^2 + 1 & t \end{matrix} \right\| = 0$$

holds, one has that

$$\text{rank} \begin{bmatrix} t & 1 \\ \frac{1}{3}t - \frac{2}{3}t^2 + 1 & t \end{bmatrix} = 2.$$

According to the rank criterion of observability in Theorem 4.2, states of the linear dynamic system (25) are completely observable in a time interval $[0, t]$.

6. **Conclusions.** In this paper, controllability and observability have been studied for a dynamic system with an internal cause. Gramin criteria which are simple in form have been provided to judge the controllability and observability of the dynamic system. Furthermore, rank criteria which are characterized by a direct use of the coefficient matrix of the dynamic system have also been given. The criteria are related to the internal cause of the dynamic system and can be used to judge the input-state and state-output relationships. Some researches on finite-time stability will be investigated for the dynamic system with an internal cause in our further work.

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