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ADAPTIVE SYNCHRONIZATION OF FRACTIONAL-ORDER CHAOTIC NEURAL NETWORKS WITH UNKNOWN PARAMETERS AND TIME-VARYING DELAYS

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Abstract. *The main concern of this paper is to address the synchronization problem of chaotic fractional-order neural networks through designing the novel adaptive control scheme. The objective of the study is to explore the importance of considering parameters uncertainty and time-varying delays. By combining the adaptive control and linear feedback with update law, a simple, analytical, and rigorous adaptive feedback scheme is derived to achieve synchronization of two coupled neural networks with time-varying delay based on the invariant principle of functional differential equations and parameter identification. Besides, the system parameters in the uncertain network can be identified in the process of synchronization. The simulation results are given to demonstrate the rationality of the theoretical analysis.*
Keywords: Fractional-order chaotic neural networks, Adaptive synchronization, Parameter identification, Lyapunov direct method

1. **Introduction.** Fractional calculus is a generalization of integer-order differentiation and integration to arbitrary order [1]. There are many kinds of definitions for fractional integration and fractional differentiation such as Caputo derivative, and Riemann-Liouville derivative. Fractional differential equation has been deemed to be a powerful tool for the modeling of practical problems in biology, chemistry, physics, medicine, economics and other sciences [2]. Contrasting with classical integer-order systems, the reality can be better described by fractional-order systems for the reason that fractional-order differentiation takes into account the present state and all the history of its previous states [3,4]. In other words, fractional-order systems have memory and heredity. Therefore, many scholars have applied fractional operators to neural networks to build fractional models.

Synchronization means two or more systems which are either chaotic or periodic share a common dynamical behavior. It has been shown that this common behavior can be induced by coupling or by external forcing. Because chaotic systems exhibit sensitive dependence on initial conditions, chaotic systems are difficult to be synchronized. Recently, synchronization of fractional-order neural networks has attracted the attention of many researchers. For example, in [5], synchronizing chaotic systems using control based on a special matrix structure were discussed. Synchronization of two identical fractional-order chaotic systems was investigated by using linear feedback control in [6]. In [7], the synchronization and anti-synchronization of uncertain fractional-order chaotic

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systems were proposed by designing active pinning controller. In [8], a nonlinear observer to synchronize a class of identical fractional-order chaotic systems was proposed. In [9], synchronization of fractional-order memristor-based complex-valued neural networks was investigated. It is unavoidable that the coefficient matrices and the activation functions of the drive-response systems are nonidentical. In [10], the global projection synchronization of nonidentical fractional-order neural networks was proposed. In [11], the finite-time global projection synchronization of nonidentical fractional-order neural networks was investigated. In [12], the global asymptotic synchronization problem for nonidentical Riemann-Liouville fractional-order neural networks was proposed.

On the other hand, time-varying delay exists in many network areas unavoidably. A realistic neural network should involve delays because of the network parameter fluctuations of the hardware operation, as well as the finite speed of switching of the transmission and amplifiers of signal in the network [13]. Time-varying delay can usually cause instability and oscillation behavior of systems. So far, a lot of results on fractional-order neural networks with time delay have been obtained. In [14], the quasi-uniform synchronization of fractional-order memristor-based neural networks with delay was studied. In [15], the global Mittag-Leffler projection synchronization of nonidentical fractional-order neural networks with time delays was studied. Based on the nonsmooth analysis and the Razumikhin-type stability theorem, the drive-response synchronization for a class of fractional-order delayed neural networks with discontinuous activations was studied in [16]. Stability and synchronization for Riemann-Liouville fractional-order time-delayed inertial neural networks are investigated in [17].

Note that the aforementioned papers investigated the systems with known parameters. However, there exist many unknown parameters in most of practical situations. Therefore, parameter estimation has become a necessary and significant issue which has attracted increasing concerns. So far, the main method used in the field of parameter identification in unknown network is the so-called synchronization-based method. This concept was firstly proposed by Parlitz [18] and then it has been extensively studied in many other papers concerning the parameter identification of uncertain chaotic systems [19-23]. The synchronization-based method was firstly employed to estimate the topology of a complex network with unknown connections [24]. Adaptive synchronization of a class of chaotic neural networks with unknown parameters was considered in [25], in which parameter identification and synchronization can be achieved simultaneously with this method. Synchronization-based parameter estimation of fractional-order neural networks was investigated in [26]. The identification of parameters and synchronization of fractional-order neural networks with time delays were studied in [27].

Even though numerous works have been reported in the literature regarding synchronization problem of chaotic fractional-order neural networks, there has been a lack of research on involving parameters uncertainty and time-varying delays, which motivated our research. In fact, a time delay will occur in the activation between the neurons in the electronic implementation of dynamical systems, which will affect the dynamical behaviors of the neuron system. Also a chaotic system is extremely sensitive to tiny variations of parameters and in practical situation, some systems parameters cannot be exactly known in prior, the effect of uncertainties will destroy the synchronization and even break it. Therefore, it is important and necessary to study the synchronization in such systems with delays and unknown parameters. To the best of our knowledge, there are few results on the synchronization of fractional-order chaotic neural networks with unknown parameters and time-varying delays.

In this paper, based on fractional adaptive synchronization, an identification method is applied to parameter identification of fractional-order neural networks with time-varying delays. The main contributions of this paper mainly include three aspects. (i) A suitable Lyapunov functional including fractional integral term is constructed, which can avoid computing the fractional-order derivative of Lyapunov such functional to derive the synchronization stability conditions. (ii) Our model of fractional neural networks with time-varying delays and unknown parameters is more general. (iii) An adaptive controller is first proposed, which is more general than the nonlinear controller. Furthermore, the adaptive controller can be used to identify the unknown parameters of nonlinear part, but the nonlinear one may not.

The rest of this paper is outlined as follows. Some preliminaries and lemmas are presented in Section 2. In Section 3, synchronization of fractional-order chaotic neural networks with known parameters is provided. In Section 4, synchronization of fractional-order chaotic neural networks with unknown parameters is obtained by the control strategy. In Section 5, numerical examples are given to demonstrate the effectiveness of theoretical analysis. Finally, conclusions are drawn in Section 6.

2. **Problem Formulation and Preliminaries.**

Several definitions exist regarding the fractional derivative of order $\alpha > 0$, but the Caputo definition is used in most of the engineering applications, since this definition incorporates initial conditions for $f(t)$ and its integer order derivatives, i.e., initial conditions that are physically appealing in the traditional way.

Definition 2.1. *The fractional integral of order $\alpha \in R^+$ on the half axis R^+ is defined as follows*

$$D^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau, \tag{1}$$

where $t > t_0$ and $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *The Caputo fractional derivative of order $\alpha \in R^+$ on the half axis R^+ is defined as follows*

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, \tag{2}$$

where $n-1 \leq \alpha < n$.

In the paper, we consider the following drive and response fractional neural networks with unknown parameters and time-varying delays

$$D^{\alpha}x_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij}f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}g_j(x_j(t-\tau(t))) + I_i \tag{3}$$

which can be rewritten as a compact form

$$D^{\alpha}x(t) = -Cx(t) + Af(x(t)) + Bg(x(t-\tau(t))) + I \tag{4}$$

where $0 < \alpha \leq 1$ is the fractional order, $x(t) = [x_1(t), \ldots, x_n(t)]^T \in R^n$ denotes the state variable associated with the neurons, n denotes the number of neurons in the network, $f(x(t)) = [f_1(x_1(t)), \ldots, f_n(x_n(t))]^T$ and $g(x(t)) = [g_1(x_1(t)), \ldots, g_n(x_n(t))]^T$ are the activation functions of the neurons, $\tau(t)$ is the transmission delay, $C = \mathrm{diag}(c_1, \ldots, c_n)$ is appropriately diagonal matrix with $c_i > 0$, $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are the connection weight matrix and the delayed connection matrix respectively, $I = [I_1, \ldots, I_n]^T$ is a constant external input vector.

Assumption 2.1. *The transmission delay $\tau(t)$ is a differentiable function with $0 \leq \dot{\tau}(t) \leq \kappa < 1$ for all t.*

Assumption 2.2. *Each neuron activation function $f_i(\cdot)$, $g_i(\cdot)$ satisfies the following condition*

$$0 \leq \frac{f_i(x) - f_i(y)}{x - y} \leq \mu_i, \; 0 \leq \frac{g_i(x) - g_i(y)}{x - y} \leq \nu_i, \; \forall x \neq y \in R, \; i = 1, \ldots, n \tag{5}$$

where μ_i, ν_i, $i = 1, \ldots, n$ are positive constants.

Remark 2.1. *Assumption 2.1 and Assumption 2.2 are not restrictive. There are many fractional-order neural networks, such as neural networks model in* [28] *and the reference therein, which satisfy Assumption 2.1 and Assumption 2.2.*

Next, let us give the following lemmas which will be used in the proof of our main results.

Lemma 2.1. [29] *If $x(t)$ is continuous and derivable, then*

$$\frac{1}{2} D^{\alpha} x^T(t) P x(t) \leq x^T(t) P D^{\alpha} x(t) \tag{6}$$

where P is an $n \times n$ positive definite constant matrix.

Lemma 2.2. [30] *If $x(t) \in C^1[0, T]$ for some $T > 0$, then*

$$D^{\alpha_1} D^{\alpha_2} x(t) = D^{\alpha_1 + \alpha_2} x(t) \tag{7}$$

where $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 \leq 1$.

Lemma 2.3. [25] *Let Σ_1, Σ_2, Σ_3 be real matrices of appropriate dimensions with $\Sigma_3 > 0$. Then for any vectors x and y with appropriate dimensions, we have*

$$2x^T \Sigma_1^T \Sigma_2 y \leq x^T \Sigma_1^T \Sigma_3 \Sigma_1 x + y^T \Sigma_2^T \Sigma_3^{-1} \Sigma_2 y \tag{8}$$

Based on the model description and preliminaries, we will study the adaptive synchronization of two coupled fractional-order neural networks with known parameters and time-varying delays in the following section.

3. **Synchronization with Known Parameters.** In this section, we study the adaptive synchronization of two coupled fractional-order neural networks with known parameters. By the adaptive control theory, a simple controller for synchronization is designed and parameters identification is realized.

We study the following drive and response fractional neural networks with time delays:

$$D^{\alpha} x(t) = -Cx(t) + Af(x(t)) + Bg(x(t - \tau(t))) + I \tag{9}$$

and

$$D^{\alpha} y(t) = -Cy(t) + Af(y(t)) + Bg(y(t - \tau(t))) + I + u(t) \tag{10}$$

Define the error between the drive and the response systems as

$$e(t) = y(t) - x(t) \tag{11}$$

Our goal is to design controller $u(t)$, such that the trajectory of the response system (10) with initial condition y_0 can asymptotically approach that of the drive system (9) with initial condition x_0 and, finally, implement synchronization, in the sense that

$$\lim_{t \to \infty} \|e(t)\| = \lim_{t \to \infty} \|y(t) - x(t)\| = 0 \tag{12}$$

where $\|\cdot\|$ is the Euclidean norm.

Subtracting (9) from (10) yields the error dynamical system as follows

$$D^{\alpha} e(t) = -Ce(t) + A\tilde{f}(e(t)) + B\tilde{g}(e(t - \tau(t))) + u(t) \tag{13}$$

where

$$\tilde{f}(e(t)) = f(e(t) + x(t)) - f(x(t))$$

and

$$\tilde{g}(e(t-\tau(t))) = g(e(t-\tau(t)) + x(t-\tau(t))) - g(e(t-\tau(t)))$$

Theorem 3.1. *The drive and response complex networks (9) and (10) can achieve synchronization, if the controller is taken as*

$$u_i(t) = -k_i(t)e_i(t) \tag{14}$$

and the feedback strength $k_i(t)$ is adapted according to the following update law

$$\dot{k}_i(t) = \lambda_i e_i^2(t) \tag{15}$$

where λ_i, $i = 1, \ldots, n$ are arbitrary positive constants.

Proof: Based on the error system (13), we construct the following Lyapunov functional:

$$V(t) = \frac{1}{2}D^{\alpha-1}e^T(t)e(t) + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\lambda_i}(k_i(t)-\rho)^2 + \frac{1}{2(1-\kappa)}\int_{t-\tau(t)}^{t}\widetilde{g}^T(e(s))\widetilde{g}(e(s)) \tag{16}$$

where ρ is a constant to be determined.

By using Lemmas 2.1-2.2, and differentiating $V(t)$ with respect to time along the solution of (13) yields

$$\begin{aligned}
\dot{V}(t) \le\ & e^T(t)D^{\alpha}e(t) + \sum_{i=1}^{n}(k_i(t)-\rho)e_i^2(t) \\
& + \frac{1}{2(1-\kappa)}\widetilde{g}^T(e(t))\widetilde{g}(e(t)) - \frac{1-\dot{\tau}(t)}{2(1-\kappa)}\widetilde{g}^T(e(t-\tau(t)))\widetilde{g}(e(t-\tau(t))) \\
=\ & e^T(t)\Big(-Ce(t) + A\tilde{f}(e(t)) + B\tilde{g}(e(t-\tau(t))) - k(t)e(t)\Big) + \sum_{i=1}^{n}(k_i(t)-\rho)e_i^2(t) \\
& + \frac{1}{2(1-\kappa)}\widetilde{g}^T(e(t))\widetilde{g}(e(t)) - \frac{1-\dot{\tau}(t)}{2(1-\kappa)}\widetilde{g}^T(e(t-\tau(t)))\widetilde{g}(e(t-\tau(t))) \\
=\ & -e^T(t)Ce(t) + e^T(t)A\tilde{f}(e(t)) + e^T(t)B\tilde{g}(e(t-\tau(t))) - e^T(t)k(t)e(t) \\
& + \sum_{i=1}^{n}(k_i(t)-\rho)e_i^2(t) + \frac{1}{2(1-\kappa)}\widetilde{g}^T(e(t))\widetilde{g}(e(t)) \\
& - \frac{1-\dot{\tau}(t)}{2(1-\kappa)}\widetilde{g}^T(e(t-\tau(t)))\widetilde{g}(e(t-\tau(t)))
\end{aligned} \tag{17}$$

Using Assumption 2.1, we can get that

$$\frac{1-\dot{\tau}(t)}{2(1-\kappa)} \le -\frac{1}{2} \tag{18}$$

By using Lemma 2.3, we can obtain that

$$\begin{aligned}
\dot{V}(t) \le\ & -e^T(t)Ce(t) + \frac{1}{2}e^T(t)AA^Te(t) + \frac{1}{2}\tilde{f}^T(e(t))\tilde{f}(e(t)) \\
& + \frac{1}{2}e^T(t)BB^Te(t) + \frac{1}{2}\widetilde{g}^T(e(t-\tau(t)))\widetilde{g}(e(t-\tau(t))) - \rho e^T(t)e(t) \\
& + \frac{1}{2(1-\kappa)}\widetilde{g}^T(e(t))\widetilde{g}(e(t)) - \frac{1}{2}\widetilde{g}^T(e(t-\tau(t)))\widetilde{g}(e(t-\tau(t))) \\
=\ & -e^T(t)Ce(t) + \frac{1}{2}e^T(t)AA^Te(t) + \frac{1}{2}\tilde{f}^T(e(t))\tilde{f}(e(t))
\end{aligned}$$

$$+\frac{1}{2}e^T(t)BB^Te(t) + \frac{1}{2(1-\kappa)}\widetilde{g}^T(e(t))\widetilde{g}(e(t)) - \rho e^T(t)e(t) \tag{19}$$

Recalling Assumption 2.2, we can obtain

$$\begin{aligned} \tilde{f}^T(e(t))\tilde{f}(e(t)) &\le \mu e^T(t)e(t) \\ \tilde{g}^T(e(t))\tilde{g}(e(t)) &\le \nu e^T(t)e(t) \end{aligned} \tag{20}$$

where $\mu = \max\{\mu_i^2 \mid i = 1, \dots, n\}$ and $\nu = \max\{\nu_i^2 \mid i = 1, \dots, n\}$.

Substituting Inequalities (20) into the right side of Inequality (19) yields

$$\begin{aligned} \dot{V}(t) \le& - e^T(t)Ce(t) + \frac{1}{2}e^T(t)AA^Te(t) + \frac{1}{2}e^T(t)BB^Te(t) \\ &+ \left(\frac{1}{2}\mu + \frac{\nu}{2(1-\kappa)} - \rho\right) e^T(t)e(t) \\ \le& \, e^T(t)\left[-\min_{0\le i\le n} c_i + \lambda_{\max}\left(\frac{1}{2}AA^T + \frac{1}{2}BB^T\right) + \frac{1}{2}\mu + \frac{\nu}{2(1-\kappa)} - \rho\right]e(t) \end{aligned} \tag{21}$$

where $\lambda_{\max}(\cdot)$ denotes the maximal eigenvalue of symmetric matrix.

Choose ρ as the following

$$\rho = -\min_{0\le i\le n} c_i + \lambda_{\max}\left(\frac{1}{2}AA^T + \frac{1}{2}BB^T\right) + \frac{1}{2}\mu + \frac{\nu}{2(1-\kappa)} + 1 \tag{22}$$

Then we have

$$\dot{V}(t) \le -e^T(t)e(t) \tag{23}$$

It is obvious that trajectories of the error dynamical system (13) converge asymptotically stable, i.e.,

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y(t) - x(t)\| = 0 \tag{24}$$

Then the two delayed fractional-order neural networks (9) and (10) will achieve synchronization asymptotically. This completes the proof.

As we know, it is common that some systems parameters cannot be exactly known in prior in many applications. We will study the adaptive synchronization fractional-order neural networks with unknown parameters in the following section.

4. **Synchronization with Unknown Parameters.** It is common that some systems parameters cannot be exactly known in prior in many applications. In this section, we deal with the synchronization with unknown parameters. By employing the invariant principle of functional differential equations and adaptive control based on parameter identification, a simple scheme is proposed for synchronization. With this method, one can rapidly achieve global synchronization of such networks while identifying all the unknown parameters dynamically.

The drive system is the same as (9), but the parameters $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are unknown which need be estimated.

The response system is given by the following equation

$$D^\alpha y(t) = -Cy(t) + \hat{A}f(y(t)) + \hat{B}g(y(t-\tau(t))) + I + u(t) \tag{25}$$

where the parameters $\hat{A} = (\hat{a}_{ij})_{n\times n}$ and $\hat{B} = (\hat{b}_{ij})_{n\times n}$ are completely unknown or uncertain.

The goal is to design the updated laws of the coupling strength $k_i(t)$ and parameter adaptive estimation laws of $\hat{A}$ and $\hat{B}$ so that the response system could be synchronized with the drive system and all the parameters $\hat{A} \to A$ and $\hat{B} \to B$ as $t \to \infty$.

Theorem 4.1. *The drive and response complex networks (9) and (25) can achieve synchronization, if the controller and the adaptive laws of parameters are taken as*

$$\begin{cases} u_i(t) = -k_i(t)e_i(t) \\ \dot{\hat{a}}_{ij} = -\xi_{ij}e_i(t)f_j(y_j(t)) \\ \dot{\hat{b}}_{ij} = -\eta_{ij}e_i(t)g_j(y_j(t-\tau(t))) \end{cases} \tag{26}$$

and the feedback strength $k_i(t)$ is adapted according to the following update law

$$\dot{k}_i(t) = \lambda_i e_i^2(t) \tag{27}$$

where ξ_{ij}, η_{ij}, λ_i are arbitrary positive constants.

Proof: From (9) and (25), we can obtain the following error dynamical system

$$\begin{aligned} D^\alpha e(t) = & -Ce(t) + A\tilde{f}(e(t)) + B\tilde{g}(e(t-\tau(t))) + \left(\hat{A} - A\right) f(y(t)) \\ & + \left(\hat{B} - B\right) f(y(t-\tau(t))) + u(t) \end{aligned} \tag{28}$$

Consider the following Lyapunov functional

$$\begin{aligned} V(t) = & \frac{1}{2}D^{\alpha-1}e^T(t)e(t) + \frac{1}{2(1-\kappa)}\int_{t-\tau(t)}^{t} \tilde{g}^T(e(s))\tilde{g}(e(s)) \\ & + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{\lambda_i}(k_i(t)-\rho)^2 + \sum_{j=1}^{n}\frac{1}{\xi_{ij}}\left(\hat{a}_{ij} - a_{ij}\right)^2 + \sum_{j=1}^{n}\frac{1}{\eta_{ij}}\left(\hat{b}_{ij} - b_{ij}\right)^2\right) \end{aligned} \tag{29}$$

where ρ is a constant to be determined.

By using Lemmas 2.1 and 2.2, and differentiating $V(t)$ with respect to time along the solution of (28) yields

$$\begin{aligned} \dot{V}(t) \le & \, e^T(t)D^\alpha e(t) + \frac{1}{2(1-\kappa)}\tilde{g}^T(e(t))\tilde{g}(e(t)) \\ & - \frac{1-\dot{\tau}(t)}{2(1-\kappa)}\tilde{g}^T(e(t-\tau(t)))\tilde{g}(e(t-\tau(t))) \\ & + \sum_{i=1}^{n}\left((k_i(t)-\rho)e_i^2(t) + \sum_{j=1}^{n}\frac{1}{\xi_{ij}}\left(\hat{a}_{ij} - a_{ij}\right)\dot{\hat{a}}_{ij} + \sum_{j=1}^{n}\frac{1}{\eta_{ij}}\left(\hat{b}_{ij} - b_{ij}\right)\dot{\hat{b}}_{ij}\right) \\ = & \, e^T(t)\Big(- Ce(t) + A\tilde{f}(e(t)) + B\tilde{g}(e(t-\tau(t))) + \left(\hat{A} - A\right) f(y(t)) \\ & + \left(\hat{B} - B\right) f(y(t-\tau(t))) - k(t)e(t)\Big) + \frac{1}{2(1-\kappa)}\tilde{g}^T(e(t))\tilde{g}(e(t)) \\ & - \frac{1-\dot{\tau}(t)}{2(1-\kappa)}\tilde{g}^T(e(t-\tau(t)))\tilde{g}(e(t-\tau(t))) + \sum_{i=1}^{n}\Big((k_i(t)-\rho)e_i^2(t) \\ & - \sum_{j=1}^{n}\frac{1}{\xi_{ij}}(\hat{a}_{ij} - a_{ij})e_i(t)f_j(y_j(t)) - \sum_{j=1}^{n}\frac{1}{\eta_{ij}}\left(\hat{b}_{ij} - b_{ij}\right)e_i(t)g_j(y_j(t-\tau(t)))\Big) \\ = & -e^T(t)Ce(t) + e^T(t)A\tilde{f}(e(t)) + e^T(t)B\tilde{g}(e(t-\tau(t))) - \rho e^T(t)e(t) \\ & + \frac{1}{2(1-\kappa)}\tilde{g}^T(e(t))\tilde{g}(e(t)) - \frac{1-\dot{\tau}(t)}{2(1-\kappa)}\tilde{g}^T(e(t-\tau(t)))\tilde{g}(e(t-\tau(t))) \end{aligned} \tag{30}$$

Similar to Theorem 3.1, we can obtain that

$$\begin{aligned}\dot{V}(t) \leq & -e^T(t)Ce(t) + \frac{1}{2}e^T(t)AA^Te(t) + \frac{1}{2}\tilde{f}^T(e(t))\tilde{f}(e(t)) \\ & + \frac{1}{2}e^T(t)BB^Te(t) + \frac{1}{2}\widetilde{g}^T(e(t-\tau(t)))\widetilde{g}(e(t-\tau(t))) - \rho e^T(t)e(t) \\ & + \frac{1}{2(1-\kappa)}\widetilde{g}^T(e(t))\widetilde{g}(e(t)) - \frac{1}{2}\widetilde{g}^T(e(t-\tau(t)))\widetilde{g}(e(t-\tau(t))) \\ = & -e^T(t)Ce(t) + \frac{1}{2}e^T(t)AA^Te(t) + \frac{1}{2}\tilde{f}^T(e(t))\tilde{f}(e(t)) \\ & + \frac{1}{2}e^T(t)BB^Te(t) + \frac{1}{2(1-\kappa)}\widetilde{g}^T(e(t))\widetilde{g}(e(t)) - \rho e^T(t)e(t) \\ \leq & \, e^T(t)\left[-\min_{0\leq i\leq n} c_i + \lambda_{\max}\left(\frac{1}{2}AA^T + \frac{1}{2}BB^T\right) + \frac{1}{2}\mu + \frac{\nu}{2(1-\kappa)} - \rho\right]e(t) \quad (31)\end{aligned}$$

Choose ρ as the following

$$\rho = -\min_{0\leq i\leq n} c_i + \lambda_{\max}\left(\frac{1}{2}AA^T + \frac{1}{2}BB^T\right) + \frac{1}{2}\mu + \frac{\nu}{2(1-\kappa)} + 1 \quad (32)$$

Then we have

$$\dot{V}(t) \leq -e^T(t)e(t) \quad (33)$$

It is obvious that trajectories of the error dynamical system (28) converge asymptotically stable, i.e.,

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y(t) - x(t)\| = 0 \quad (34)$$

Then the two delayed fractional-order neural networks (9) and (25) will achieve synchronization asymptotically. This completes the proof.

Remark 4.1. *It is worth mentioning that we discuss the synchronization problem of fractional-order neural networks with unknown parameters and time-varying delays in this paper, whereas the the existing works are about either with known parameters or with no time delays or with constant time delays. This is to say that the existing works are a special case of our research results.*

In order to illustrate the effectiveness and applicability of the proposed adaptive fuzzy control approach and to confirm the theoretical results, the simulation results are given in the following section.

5. **Numerical Simulations.** In this section, an illustrative example is presented to illustrate the effectiveness and applicability of the proposed adaptive synchronization control approach and to confirm the theoretical results. Consider the following nonlinear fractional-order chaotic neural networks [27].

$$D^\alpha x(t) = -Cx(t) + Af(x(t)) + Bg(x(t-\tau(t))) \quad (35)$$

where

$$A = \begin{pmatrix} 2 & 0.3 \\ 5 & 3 \end{pmatrix},\ B = \begin{pmatrix} -2 & 0.2 \\ 0.3 & -2.5 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (36)$$

and

$$\alpha = 0.995,\ f(x(t)) = g(x(t)) = [\tanh(x_1(t)), \tanh(x_2(t))]^T \quad (37)$$

We choose time-varying delays $\tau(t)$ as follows

$$\tau(t) = \frac{e^{t-0.5}}{1+e^{t-0.5}} \quad (38)$$

The initial conditions for fractional-order chaotic neural networks (35) are given as follows

$$x_1(s) = -0.4,\ x_2(s) = -2,\ \forall s \in [-1, 0] \tag{39}$$

Obviously, Assumption 2.1 and Assumption 2.2 are satisfied. In the numerical simulations, the initial conditions and parameter values are given in the following

$$\begin{cases} x_1(0) = 0.4,\ x_2(0) = 0.6 \\ y_1(0) = 1,\ y_2(0) = 1 \\ k_1(0) = 1,\ k_2(0) = 1 \\ \hat{a}_{11}(0) = 1,\ \hat{a}_{12}(0) = 1,\ \hat{a}_{21}(0) = 1,\ \hat{a}_{22}(0) = 1 \\ \hat{b}_{11}(0) = 1,\ \hat{b}_{12}(0) = 1,\ \hat{b}_{21}(0) = 1,\ \hat{b}_{22}(0) = 1 \end{cases} \tag{40}$$

The simulation results are shown in Figures 1-6. The fractional-order neural networks model (35) is actually chaotic as shown in Figure 1.

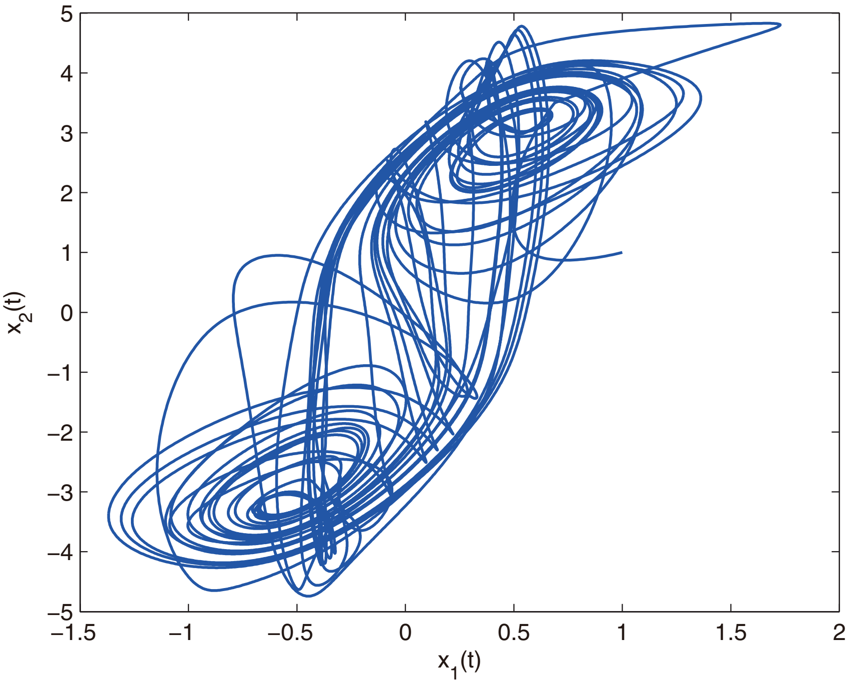

FIGURE 1. Chaotic attractor of system (35)

In the following, we will illustrate that the two couple systems synchronization and uncertain system parameters are achieved by the adaptive control strategy (26) and (27). From Figures 2 and 3, it can be clearly seen that all the unknown system parameters $\hat{a}_{ij}$ and $\hat{b}_{ij}$ are successfully identified, respectively. It is demonstrated that the adaptive synchronization-based method for parameter estimation proposed in the paper is very effective.

From Figure 4, we can see that the adaptive control gains $k_i(t)$, $i = 1, 2$ tend to some positive constants when system (9) and system (25) are synchronized. Furthermore, the controlling strengths are adjusted to fixed values in a very short time from Figure 4. Figure 5 depicts the synchronization error of the state variables between the driver system and the response system. The synchronization error converges to zero asymptotically, which indicates that drive system (9) and response system (25) can achieve global asymptotical synchronization based on the adaptive controller (26) and (27). Figure 6 displays the trajectories of the control inputs. From the simulation results we can say that good control performances have been achieved.

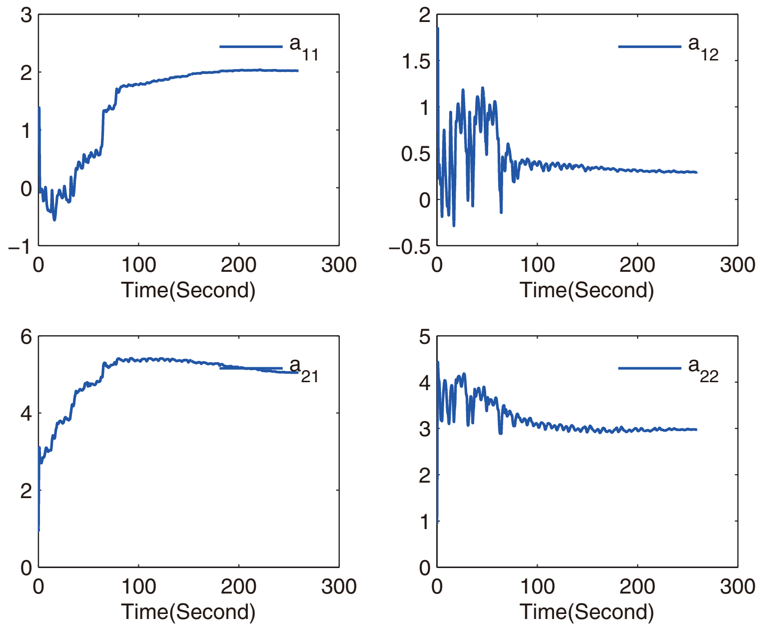

FIGURE 2. Identification of uncertain parameters $\hat{A}$

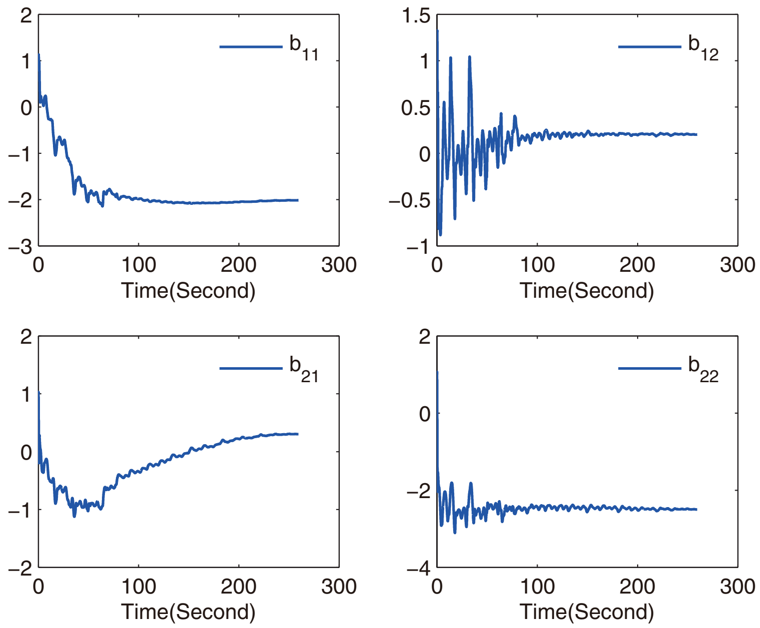

FIGURE 3. Identification of uncertain parameters $\hat{B}$

Remark 5.1. *As an important collective dynamical behavior, synchronization of neural networks has been a hot topic in recent years due to its potential engineering applications in many fields such as secure communications, pseudorandom number generator, image encryption, information science and biological systems [31,32]. The proposed results in this paper can be applied to secure communication [33].*

Remark 5.2. *It should be noticed that the aforementioned papers with respect to synchronization of complex networks always assume that complex networks are known in advance. However, for the real complex networks, there may be much uncertain information, including but not limited to unknown system parameters and uncertain network structures.*

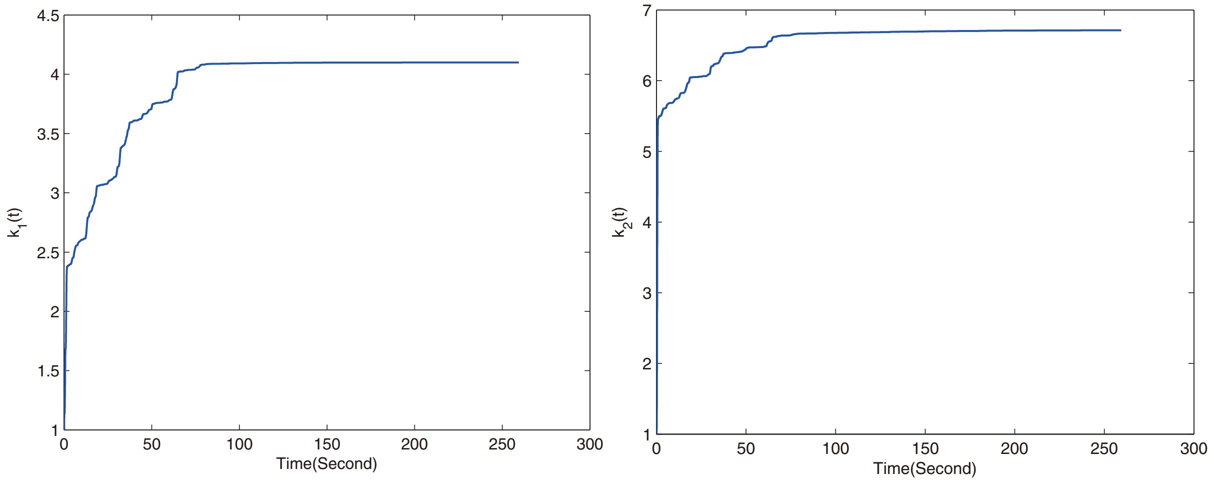

FIGURE 4. Time evolution of the controlling strengths $k_1(t)$ and $k_2(t)$

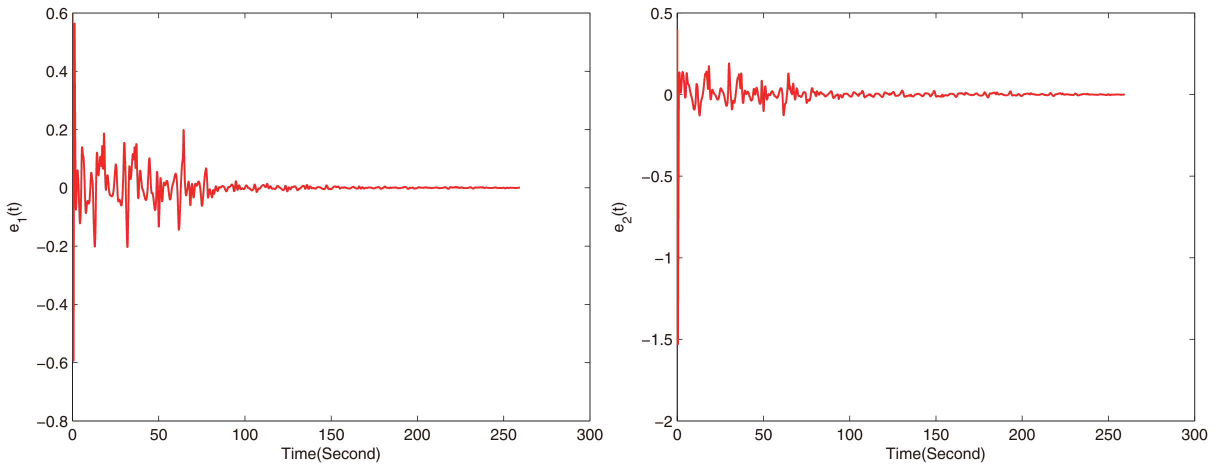

FIGURE 5. Time evolution of synchronization errors $e_1(t)$ and $e_2(t)$

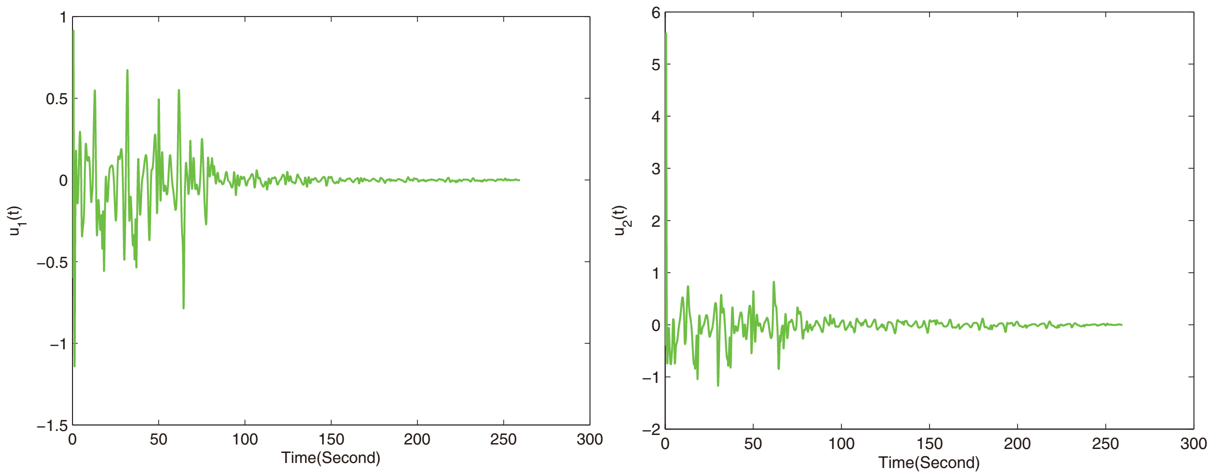

FIGURE 6. Time evolution of the control inputs $u_1(t)$ and $u_2(t)$

6. **Conclusions.** In this paper, the problem of synchronization of a class of fractional-order chaotic neural networks known or unknown parameters and time-varying delays has been considered. The theorems for synchronization are derived. The method introduced for adaptive synchronization and parameters identification of uncertain system is very

effective, and it is simple to implement in practice. Lastly, numerical simulations are given to show the effectiveness and feasibility of the developed method.

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