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# QUANTUM EXPRESSION OF NEURAL NETWORK AND CIRCUITS (ITS LOGIC OF CALCULATION)

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Abstract. *We have mentioned the theory of the neural network using quantum theory (wave function and path integrals) of quantized polarization wave, what we call, polaritons, and made some mathematical forms and an expression of calculations for arbitrary neural networks' circuits. The significantly important difference between the ordinary (classical) neural networks and our quantum networks was in whether our focused systems had quantum theoretical interferences or not. The quantum mechanical system had essentially quantum fluctuation and its interference in those systems. And quantum probability has relation to the probability amplitude, propagators and wave functions, which were ordinary complex numbers and those functions. So, common classical probability never contained any interference because of the real numbers' field. Then concretely we have shown how our quantum system contained much interference, which was applied to the Bayes' theory, the entropy of information theory, and the three-step's neural network of multi channels. And we knew that the quantum neural networks and polariton's model were related to the ordinary quantum information, classical neural networks and some information theories. We would like to propose a kind of complex fuzzy system (or complex neural networks). Both systems (i.e., fuzzy system and quantum system) have something in common on these shapes. We expect that some parts of brain functions and its networks are able to describe with methods of complex fuzzy neural networks or quantum fuzzy processes.*
**Keywords:** Classical neural network, Quantum interference and Feynman path integral, Bayes' theory, Entropy of information theory, Quantum neuron, Schrödinger equation, Complex fuzzy system

## 1. Introduction.

The research of the famous physiological and neural systems is established by Hodgkin and Huxley, whose theory is applied by mathematical cable theory, ionic current ($Na^+$, $K^+$), leak currents, ionic gates and conductions of action potentials [1]. They have had greatest success with this model in branch of biological and physiological neural theory, that model and its neural network. Modern neuro-physiology is variously based on their model and theory, whose theory mentions that each neural impulse and action potential never interfere with each other, because each neural axon has independence of each neuron and it is insulated by its myelin sheath. Arvanitaki, however, proved an existence of ephapse, which is an electrical interference of each neural axon [2-4]. They said that his discovery and experiment are thought to discover a kind of artificial neuron [4,5]. Moreover, it is known that some pathological states and neurons, for such as neuralgia and causalgia, are caused by ephapse's phenomena. So, we have been studying a model of neural electromagnetic interferences, polization waves and the network. And finally, we proposed new engineering (not always biological) neural models based on

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quantized polarization waves, the polariton, which are quantum quasi particles, massive photons with spin 1, running quasi particles, massive photons with spin 1, running along on each axon. In other words, its process is caused by a series of neural activities, whose process is described as polarization, depolarization, and hyper-polarization.

The previous papers have mentioned the nano-mechanism of neural action potential and its activities on the axons [17-20]. On the second stage, we have mentioned the methods and tools of description for neural circuits, amida-lots, multi-step bifurcation, small quantum logical circuits (we named them as q-AND, q-NOT, q-OR), based on propagators, wave mechanics and path integrals [5]. Thus, by applying both path integral and propagator to common logical circuits, we could rewrite quantum mechanical formulae to arbitrary various neural networks and Beyes' form [5,20].

This paper, to refer to the differences between classical information theory and quantum one, we perform concretely to calculate the classical Bayes' probability and quantum one, informational entropy, and output of quantum neural networks. And we would like to give an expression of the cases by applying quantum tools developed in previous papers for classical Beyes' probability. As you know, Bayes' theory is applied to many network theories, predictions, and some control systems. So, many excellent books and reports already have been published in the branch of information science [6,7]. Bayes' statistics are said to be subjective probability, when the Bayes' method is compared with common probability (objective probability) [8-12]. We think that the classical mechanics essentially have an apparent pathway between initial point and end point, what is called, which is deterministic treaties. However, the quantum path is said to essentially probabilistic phenomena since its time development is governed by the complex probability amplitude of Schrödinger equation, Proca equation and their fields. As we have already discussed in previous papers, our polariton's propagation of neural network theory can be described as motion of massive relativistic particles governed by the Proca equation, or non-relativistic quaternary Schrödinger equations [13,14,18]. We recognize that many interesting characteristics of quantum mechanics are contained in the interference of phenomena, a mixing state of pure state vectors, superpositions and tunnel effects. The common Bayes' probability is classical theory, we say, because of not being considered the interferences of phenomena between each event. We know that all events are independent of each other. We think it is interesting to research where the quantum interference comes from and how it effects on establishing Bayesian theorem, the entropy and information.

One of the purposes of this paper is that we attempt to show an expression of quantum Bayes' rule, instead of classical Bayes' theorem, by using a basic set of orthogonal state vectors and simple model. And we clearly describe the differences between classical Bayes' theory and quantum form. The second step, we compared their entropies of both systems, and in the two-step's neural networks of multiple channels, we could approximately obtain a solution by means of perturbation method and path integrals.

Finally, we would like to point out similarities of formal descriptions between soft scientific theories and quantum control systems. The first example refers to neural network control and its neuro-synaptic junction between each neuron and the second is similarities between fuzzy probability and quantum expectation values.

2. **Neural Conduction.** The previous paper [17-20] has said, that the equation of polaritons on neurons, and the polarities (quantized polaization waves), exactly obey to the Proca equation Equation (1), which is the relativistic and massive photon's equation:

$$\left(\partial_\mu \partial^\mu + m^2\right) A^\mu = J^\mu, \quad J^\mu(x) \equiv (\rho(\mathbf{x},t), i(\mathbf{x},t)) \approx j^\mu_{Na} + j^\mu_K. \tag{1}$$

The symbol $m$ is the mass of polariton, and $J^\mu$ means its quaternary vector currents, which is composed of both current. The one is sodium ionic current and the other is potassium ionic current. The classical active neuron's theory like as Hodgkin and Huxley model [1] shows, that the polariton corresponds to quantized polarization wave, whose impulse cusses from each neuron and its action potential. To derive non-relativistic polariton's equation from relativistic equation, Proca equation, instead of the wave function $A^\mu$ with natural unit, we will return to that of MKS unite. So vector potential $A^\mu$ is

$$A^\mu(\mathbf{x},t) = \varphi^\mu(\mathbf{x},t)\cdot\exp\left(-\frac{i}{\hbar}mc^2 t\right). \tag{2}$$

Then, we part vector potential $A^\mu$ into two terms and then the one was having the rest polariton's mass, $m$. The non-relativistic limit, includes the kinetic energy $E_k$, whose value is so small that we can describe that energy term $\phi(A)$ as

$$E_K = E - mc^2, \quad E' << mc^2. \tag{3}$$

The limit of non-relativistic kinetic energy $E_k$ becomes

$$\left|i\frac{\partial\varphi^\mu}{\partial t}\right| \approx E_K\varphi^\mu << mc^2\varphi^\mu, \quad \frac{\partial A^\mu}{\partial t} \approx -i\frac{mc^2}{\hbar}\varphi^\mu\cdot\exp\left(-\frac{i}{\hbar}mc^2 t\right),$$
$$\frac{\partial^2 A^\mu}{\partial t^2} \approx \left[-i\frac{2mc^2}{\hbar}\frac{\partial\varphi^\mu}{\partial t} - i\frac{m^2c^4\varphi^\mu}{\hbar^2}\right]\cdot\exp\left(-\frac{i}{\hbar}mc^2 t\right). \tag{4}$$

Inserting that result into following relativistic relation, we have

$$p^\mu p_\mu A^\nu + m^2c^2A^\nu = j^\nu/c. \tag{5}$$

We finally obtain the 4-component non-relativistic expressions (quaternary form) of vector potential like as Schrödinger equation. We notice the result means non-relativistic polariton's relationship of electromagnetic potential,

$$i\hbar\frac{\partial A^\mu}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]A^\mu, \quad A^\mu = (\phi,\mathbf{A}) \quad j^\nu\hbar^2/(2mc) \Leftrightarrow \hat{V}A^\nu. \tag{6}$$

We find the final polariton's equation with 4-components whose motion is described by above quaternary equations: $A^\mu$ is composed of the scalar potential $\mathbf{A}_0 = \phi$ and vector potential $\mathbf{A}$. If the quaternary vector potential of polaritons is having $\mathbf{A}$ = constant or slowly changing of $\mathbf{A}$ (i.e., stationary magnetic field), then Equation (6) reduces to common Schrödinger equation with only one element (the scalar potential $\phi$).

$$i\hbar\frac{\partial\phi(\mathbf{x},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]\phi(\mathbf{x},t) = \hat{H}\phi(\mathbf{x},t) \quad \because \mathbf{B}(\mathbf{x},t) = rot\mathbf{A}(\mathbf{x},t) \approx 0,$$
$$\mathbf{E}(\mathbf{x},t) = -grad\phi(\mathbf{x},t). \tag{7}$$

The polariton's motion is so slow that we can regard as about zero magnetic field, and the quaternary solution nearly equals $A^\nu = (\phi$, Constant $\mathbf{A})$.

3. **Kernels and Description.** In our previous papers, we showed the equations between kernels and wave functions, and they are based on the path integral of quantum theory [13,14].

1) The solution $\phi$ of Equation (7) is given by kernel $K(B,A)$ of $\phi$, which is free propagation of polariton (where, $\phi(A)$ is an initial condition): quaternary potential

$$i\hbar\frac{\partial\phi}{\partial t} = \hat{H}\phi \Rightarrow \phi(\mathbf{x},t) = \int K(B,A)\phi(A)dA \Rightarrow A^\mu = (\phi(\mathbf{x},t), \text{constant } \mathbf{A}_c)$$
$$\because B \equiv (\mathbf{x},t), \quad A \equiv (\mathbf{x}_0,t_0). \tag{8}$$

The characters $A$ and $B$ mean points of 3-dimensional space and time (space-time). And $A$ is an initial point of space-time and $B$ is an end point. For the case of slow changing or constant magnetic field, the $A^\mu$ reduces to one component's function $\phi(\mathbf{x},t)$, which is called scalar potential equation, and $A^i$ becomes nearly to constant. So, we think Equation (6) reduced to Equation (7), and finally we reach the expression of Equation (8) by kernel $K(B,A)$. The kernel $K(B,A)$ of free polatriton is represented by Equation (9)

$$K(B,A) = \left[\frac{2\pi i\hbar(t-t_0)}{m}\right]^{-1/2} \exp\left[\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right] = {}_t\langle B|A\rangle_0. \tag{9}$$

And the position $B$ means

$$x(t) = x_0 + \frac{x-x_0}{t-t_0}(t-t_0). \tag{10}$$

2) If kernel $K_C(B,A)$ can be separated into two parts at a relay's point $C$, then the $K_C(B,A)$

$$K_C(B,A) \equiv {}_t\langle B|_C A\rangle_0 = \int K(B,C)K(C,A)dC \tag{11a}$$

is represented by path integral. If polariton has diffraction at point $D$, then we reach a similar relation with slit width $\delta$:

$$K_C(B,A) \equiv {}_t\langle B|_C A\rangle_0 = \int_0^\delta K(B,D)K(D,A)dD \tag{11b}$$

Those kernels $K(B,A)$ are governed by Schrödinger equation,

$$i\hbar\frac{\partial K(B,A)}{\partial t} = \hat{H}K(B,A). \tag{12}$$

3) If state vector $|\phi(t)\rangle$ is projected into $x$-axis of coordinate, then wave function $\phi(\mathbf{x},t)$ becomes

$$\phi(\mathbf{x},t) \equiv \langle x|\phi(t)\rangle, \quad \because |\phi(t)\rangle = U(t,t_0)|\phi(t_0)\rangle \tag{13}$$

with time developing unitary operator.

4) We substitute Equation (13) into Equation (8), and then the unitary operator $U(t,t_0)$ is described by the same Schrödinger equation. Thus, the unitary operator

$$U(t,t_0) = \exp\left(-i\hat{H}(t-t_0)\middle/\hbar\right) \tag{14}$$

is finally applied to $K(B,A)$ and we have the expression of time development of kernel,

$$K(B,A) = {}_t\langle B|A\rangle_0 = \left\langle B\left|\hat{U}(t,t_0)\right|A\right\rangle. \tag{15}$$

5) The special type of kernel,

$$K_C(B,A) = {}_t\langle B|_C A\rangle_t = \left\langle B\left|\hat{U}(t,t_C)\hat{U}(t_C,t)\right|A\right\rangle \tag{16}$$

is equal to this delta function at fixed time $t$, and we have

$$\int K(B,C)K(C,A)dC = \int {}_t\langle B|C\rangle_{tc}\langle C|A\rangle_t dC = \langle B|A\rangle_t = \delta(\mathbf{x}-\mathbf{x}_0)$$
$$\because \int dX|X\rangle\langle X| = 1. \tag{17}$$

6) If polariton is scattered by general potentials $V$ (for example, atomic structure or by switch function $S$ of electronic circuit) at point $C$, we have a similar scattering description by Equation (15):

$$K_C(B,A) \equiv {}_t\left\langle B\left|\hat{S}_C\right|A\right\rangle_0 = \int K(B,C)S(C)K(C,A)dC$$
$$= \int \left\langle B\left|\hat{U}(t,t_C)\right|C\right\rangle S(C)\left\langle C\left|\hat{U}(t_C,t_0)\right|A\right\rangle dC. \tag{18}$$

7) The scalar potential $\phi$ of polariton is one of the elements of quaternary Schrödinger equation, and a time development state $|\phi(t)\rangle$ of the formal solution of Equation (8) is obtained as follows:

$$|\phi(t)\rangle = e^{-i\hat{H}t/\hbar}|\phi(0)\rangle. \tag{19}$$

The completeness of the eigen-state vector $|\psi_j(\mathbf{x},t)\rangle$ gives the kernel $K(B,A)$ of proper wave function $\psi_j(\mathbf{x},t)$.

$$K(B,A) = {}_t\langle B|A\rangle_0 = \sum_j \psi_j(\mathbf{x})\psi_j^*(\mathbf{x}_0)\exp\left(-iE_j(t-t_0)\right) \quad \because |\phi(t)\rangle = e^{-i\hat{H}t/\hbar}|\phi(0)\rangle,$$
$$\sum_j |\psi_j(\mathbf{x})\rangle\left\langle\psi_j^*(\mathbf{x}_0)\right| = 1, \quad \langle x|\phi(t)\rangle = \phi(x,t). \tag{20}$$

8) The diffraction at point $D$ by slit of width $\delta$ and the potential scattering at point $C$ are given by kernel $K_{DC}(B,A)$:

$$K_{DC}(B,A) = {}_t\left\langle B\left|\hat{S}_C\right|_D A\right\rangle_0 = \int dDdCdEK(B,E)K(E,C)S(C)K(C,D)K(D,A)$$
$$= \int_0^\delta \int dCK(B,C)S(C)K(C,A). \tag{21}$$

Thus, utilizing these tools, various classical neural networks can be re-written to the quantum ones. And these expressions have dynamics and time developments of quantum neural systems like conductions or motion of polariton.

4. **Bayes' Theory of Quantum Expression.** We would like to show relations between the classical Bayes' theory and our quantum Bayes' style. The Bayesian probability $P_{Cl}$ is defined as the ratio when an event $A_K$ ($k = 1 \sim N$) arises. Then we obtained an ordinary Bayes' theorem:

$$P_{Cl}(A_K|B) = \frac{P(B|A_K)\cdot P(A_K)}{P(B)}, \quad \because P(B) = \sum_K^n P(B|A_K)\cdot P(A_K). \tag{22}$$

The $P(A_K)$, an initial probability, is regarded as a probability of occurrence of event $A$, and the $P(B|A_K)$ is a correspondence probability. And so, its meaning of probability $P(B|A_K)$, represents a condition when an event $A_K$ is propagated to the state $B$. Then the event $A_K$ takes place at an occurrence probability $P(A_K)$. So, the $P(B|A_K)$ is regarded as classical propagator of probability $P(A_K)$. We commonly think that Equation (22) is regarded as classical Bayes' theorem. So we attempted to translate the classical theorem into quantum description, and then we set a rule that the classical Bayes' theorem should be expressed if the expectation value of quantum operators is taken by an eigen state and its value. The expectation value of quantum Maxwell equations (what we call quantum electrodynamics) should be obeyed to classical Maxwell equation. Thus, both $P(A_K)$ and $P(B|A_K)$ should be regarded as operators' description of Maxwell equation, and so those eigen functions have complex probability amplitudes. In order to re-interpret classical relations into quantum operators, we would like to show the simplest case of quantum Bayes' form. Notice that the form of quantum Bayes is given as Equation (23):

$$\left\langle A_K \left| \hat{P}(A|B) \right| B \right\rangle \equiv \frac{\left\langle B \left| \hat{P}(B|A) \cdot \hat{P}(A) \right| A_K \right\rangle}{\sum_j^n \left\langle B \left| \hat{P}(B|A) \cdot \hat{P}(A) \right| A_j \right\rangle}. \tag{23}$$

We know that quantum expression is similar to classical Bayes' ones except using expectation value; then, all probabilities' relations are not $c$-numbers but $q$-numbers, what is called, operators' relationship in quantum Bayes' theorem. Vector $|A_K\rangle$ is one of the initial conditions, and the final state vector means $|B\rangle$. We can reduce Equation (23) to simplified expression by conditions being related to both initial and final state vectors. And quantum neural networks have many axons and synapses with quantum interferences. Calculating Equation (23), we introduce some rules of eigen state vectors, completeness and orthonormality.

5. **Quantum Bayes' without Errors.** The quantum state vector means that initial state vector $|A_K\rangle$ converges at the final state vector $|B\rangle$, and the each character $\beta_K$ is the occurring probability-amplitude of the initial state vectors $|A_K\rangle$. The $\hat{\eta}$ which corresponds to $\hat{P}(B|A)$ is the propagating operator (propagator), and it means a transition of states from state A to state B. The $\hat{\eta}$ determines the propagating conduction-rate of state vectors. For obtaining the simpler expressions to Equation (23), we adopt state vectors concepts of quantum mechanics. Thus, it is completeness for bra & ket vectors,

$$\sum_j^n |A_j\rangle\langle A_j| = 1. \tag{24}$$

Utilizing Equation (24) and substituting it into Equation (23), we are able to rewrite the numerator of Equation (23), and we obtain

$$\left\langle B \left| \hat{P}(B|A) \cdot \hat{P}(A) \right| A_K \right\rangle = \sum_j^n \left\langle B \left| \hat{P}(B|A) \right| A_j \right\rangle \left\langle A_j \left| \hat{P}(A) \right| A_K \right\rangle. \tag{25}$$

We should note that the second term $\left\langle A_j \left| \hat{P}(A) \right| A_K \right\rangle$ of Equation (25) is the expectation value of the occurring operator $\hat{P}(A)$. We are able to name $\hat{P}(A)$ as the occurring operator. Then the state vector $|A_K\rangle$ transits to the other state vector $|A_j\rangle$ by the operator $\hat{P}(A)$. Thus the total occurring amplitude from $|A_K\rangle$ to $|A_j\rangle$ is written as $\left\langle A_j \left| \hat{P}(A) \right| A_K \right\rangle$.

The first term $\left\langle B \left| \hat{P}(B|A) \right| A_j \right\rangle$ of Equation (25) corresponds to the transitional and propagator's amplitude. To simplify those expressions, we set some rules. The initial $N$-numbers' vectors make a complete and orthogonal set $\{|A_K\rangle, k = 1, \dots, N\}$.

$$\langle A_j | A_K \rangle = \delta_{jK}. \tag{26}$$

An arbitrary vector $|M\rangle$ can be mathematically expanded by those initial vectors.

$$|M\rangle = \sum_j^n a_j |A_j\rangle. \tag{27}$$

The initial state vectors $|A_K\rangle$ are in some pure states at start point $t = 0$, and then we assume that those vectors are satisfied with eigen equations

$$\hat{P}(A)|A_j\rangle = \beta_j |A_j\rangle \quad \because \hat{P}(A) \equiv \hat{P}(x, -i\hbar\partial/\partial x). \tag{28}$$

Even if signals or information are propagating their communication channels and those processes are free from mistakes, we cannot escape the attenuation, exhaustion, dissipation at various junctions (neuro-synaptic junction, etc.). So, a mixture of various states occurs at final state $|B\rangle$. If the propagating states are expressed as

$$\hat{P}(B|A)|A_j\rangle \equiv \hat{\eta}(A)|A_j\rangle = \eta_j|A_j\rangle \tag{29}$$

then both Equation (28) and Equation (29) tell us that those two operators $\hat{P}(A)$ and $\hat{\eta}$ are commutative to each other. The operator $\hat{\eta}$ is called the propagator. So we know

$$\left[\hat{P}(A), \hat{\eta}(A)\right] \equiv \hat{P}\hat{\eta} - \hat{\eta}\hat{P} = 0. \tag{30}$$

And we should notice that final state vector $|B\rangle$ is not in pure state but in a mixed state, and then final state is given by superposition of initial pure states $|A_K\rangle$. We give an expansion of the final state by using superposition of the initial state, $|A_K\rangle$. The mixed state $|B\rangle$,

$$|B\rangle = \sum_j^n C_j|A_j\rangle \tag{31}$$

becomes summing up possible state vectors $|A_K\rangle$ by using Equation (27). Applying Equations (28) and (29) to Equation (25), the numerator of Equation (25) can be expressed by a simple relation,

$$\left\langle B \left| \hat{P}(B|A)\hat{P}(A) \right| A_K \right\rangle = \sum_j^n \left\langle B \left| \hat{P}(B|A) \right| A_j \right\rangle \left\langle A_j \left| \hat{P}(A) \right| A_K \right\rangle = C_K^* \beta_K \eta_K \tag{32}$$

and the same procedure is practiced to the denominator of Equation (23), whose result is

$$\sum_K^n \left\langle B \left| \hat{P}(B|A)\hat{P}(A) \right| A_K \right\rangle = \sum_K^n \left\langle B \left| \hat{P}(B|A) \right| A_K \right\rangle \beta_K = \sum_j^n C_j^* \beta_j \eta_j. \tag{33}$$

Finally, we obtain quantum Bayes' form: it is not probability but probability amplitude.

$$\left\langle A_K \left| \hat{P}(A|B) \right| B \right\rangle = \frac{\left\langle B \left| \hat{P}(B|A) \cdot \hat{P}(A) \right| A_K \right\rangle}{\sum_j^n \left\langle B \left| \hat{P}(B|A) \cdot \hat{P}(A) \right| A_j \right\rangle} = \frac{C_K^* \beta_K \eta_K}{\sum_j^n C_j^* \beta_j \eta_j}. \tag{34}$$

The above result, Equation (34) is almost similar to the classical calculation – Equation (22); however, Equation (34) has complex coefficient $C_K$ whose complex number causes an essential difference between the classical Bayes' theorem and the quantum Bayes' one of Equation (34). So, the classical Bayes' probability has real numbers, and on the other hand, quantum Bayes' form becomes complex numbers. Thus, the quantum Bayes' form applied for polaritons on neurons has a lot of interferences among polaritons and neural networks. Equation (34) shows polaritons to possess the phase and complex numbers, which mean to arise quantum effect interferences and probability amplitude. However, the classical Bayes' form has real numbers which directly mean the probability or probability density, and the real values cannot cause the interferences between each neuron. We would like to develop the calculation of Equation (34) by using wave function. The state vector $|A_K\rangle$ obeys Equation (8),

$$i\hbar\frac{\partial}{\partial t}\langle x_K|A_K\rangle = \hat{H}\langle x_K|A_K\rangle \Rightarrow \phi_K(x_K, t_K) \equiv \langle x_K|A_K\rangle. \tag{35}$$

Performing some calculations, finally we reach quantum Bayes' interference:

$$P_Q(A_K|B) = \frac{P(B|A_K)\cdot P(A_K)}{\left\{\sum_j^n P(B|A_j)\cdot P(A_j)\right\}\cdot(1+N+M+MN)}$$

$$\mathrm{Re}(Z) = \sum_{j>K}^{n} \mathrm{Re}\left[\beta_j^*\beta_K\eta_j^*\eta_K\cdot\exp i\left(\theta_j-\theta_K\right)\right]$$

$$N \equiv -\sum_{j\neq K}^{n}|\beta_j\eta_j|\cdot|\beta_K\eta_K| \Big/ \left(\sum_K^n|\beta_K\eta_K|\right)^2, \quad M \equiv \frac{\mathrm{Re}(Z)}{\sum_j^n|\beta_j|^2|\eta_j|^2}$$

$$|\beta_K| \to P(A_K), \quad |\eta_K| \to P(A_K|B)$$

$$\langle x|B\rangle = \sum_j^n \frac{1}{\sqrt{n}}\langle x|A_j\rangle\cdot\exp(i\theta_j), \quad \langle x|A_j\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(-\frac{i}{\hbar}(p_jx-E_jt_j)\right). \tag{36}$$

This is written as similar to classical Bayes' expression except the term of $\mathrm{Re}(Z)$, $N$ and $M$. So the additional terms $N$ and $M$ are effects of the quantum interferences. All pure states are mixing each other, and the new mixed state $|B\rangle$ is generated at the above junction regions. If we apply Equation (36) for quantum neural networks, the mixed state $|B\rangle$ represents the states of an information around neuro-synaptic junctions, or each of axon's interferences (or cables of artificial neurons, ephapse). We would like to obtain the entropy of both occurrence probabilities of the classical case and quantum one, since entropy is one of the most important elements of information theory. Both of the classical occurring probability and the propagating probability have their values of real numbers $P(A_K)$ and of non-negative ones $P(B|A_K)$. On the other hand, quantum case does not mean direct probability, but the quantum form corresponds to the eigen values of operator $\hat{P}(A)$ and $\hat{P}(B|A)$, and their counter probability amplitudes, $\beta_K$ and $\eta_K$. The $\hat{P}(B|A)$ shows concepts of the occurrence probabilities of events, quantum occurring operators, and aspects of the propagation of the probabilities, and its quantum version of network's path (they are really communication paths or axons). When the occurrence probability is $P(A_K)$ at point $A_K$, and the information propagates from $A_K$ to $B$, the entropy is defined as

$$H(B|A_K) = -P(A_K)P(B|A_K)\log_2 P(A_K)P(B|A_K). \tag{37}$$

Thus, total entropy from all of $A$ to $B$ is given as

$$H(B|A) = -\sum_k^n \alpha_k\lambda_k\log_2(\alpha_k\lambda_k) \quad \because \alpha_k \equiv P(A_k), \quad \lambda_k \equiv P(B|A_k). \tag{38}$$

If we pay only attention to occurrence probability, its entropy is calculated by the result:

$$H(A) = -\sum_k^n \alpha_k\log_2\alpha_k. \tag{39}$$

It should be noticed that the $\alpha_K$ is the positive real number. However, the amplitude of entropy of quantum system $\sigma_q(B|A_K)$, which is not always real number, is defined as

$$\begin{aligned}\sigma_q(B|A_K) &\equiv \left\langle B\left|\hat{P}(B|A)\hat{P}(A)\cdot\log\left(\hat{P}(B|A)\hat{P}(A)\right)\right|A_K\right\rangle \qquad (40)\\ &= -\sum_j^n\left\langle B\left|\hat{P}(B|A)\hat{P}(B|A)\right|A_j\right\rangle\cdot\left\langle A_j\left|\log\left\{\hat{P}(B|A)\hat{P}(A)\right\}\right|A_K\right\rangle\\ &= -C_K^*\beta_K\eta_K\cdot\log_2(\beta_K\eta_K),\end{aligned}$$

by using eigen-state vectors, Equations (28) and (29), and taking an expectation's value of operators. Moreover, the amplitude of entropy for $A_K$ is $\sigma_q(A_K)$ which is related to the state $A_K$:

$$\sigma_q(A_K) \equiv -\left\langle A_j \left| \hat{P}(A)\log_2\left\{\hat{P}(A)\right\}\right| A_K \right\rangle \delta_{jK} = -\left|\beta_K\right| e^{i\gamma_K}\left(\log_2\left|\beta_K\right| + \log_2 e^{i\gamma_K}\right)$$
$$\because -\pi \le \mathrm{Im}(\beta_K) \le \pi, \quad \arg(\beta_K) = \gamma_K \tag{41}$$

by using eigen-state vectors, Equations (30) and (31) and the operations of expectation values. Then the entropy of the occurrence of quantum system, $H_q(A_K)$ for $A_K$, is written as

$$\begin{aligned} H_q(A_K) &\equiv -\left[\sigma_q^*(A_K)\sigma_q(A_K)\right]^{1/2} = -|\beta_K| \cdot \left\{(\log_2|\beta_K|)^2 + (\gamma_K/\ln 2)^2\right\}^{1/2} \\ &= -(1+J_K)\cdot P(A_K)\log_2 P(A_K) \quad \because J_K \equiv \frac{1.041\gamma_K^2}{(\log_2|\beta_K|)^2}. \end{aligned} \tag{42}$$

We sum up the entropy of each pure state to obtain the total entropy of probabilities:

$$\begin{aligned} H_{Tq}(A_K) &= -\sum_K^n |\beta_K| \cdot \left\{(\log_2|\beta_K|)^2 + (\gamma_K/\ln 2)^2\right\}^{1/2} \\ &\approx -\sum_K^n (1+J_K)\cdot P(A_K)\log_2 P(A_K). \end{aligned} \tag{43}$$

Comparing the above result of Equation (43) with the classical result of Equation (39), we immediately find the $J_K$-term to be added to $P(A_K)\log_2 P(A_K)$ whose additional term is directly generated by a phase of the wave function of the $A_K$, and the phase affects the occurring probability, and it gives rise to an interference, reflexive interaction and transitional action. However, if both operators, $\hat{P}(A_K)$ and $\hat{P}(B|A_K)$, are Hermitian and the counter states belong to their pure states, then their eigen values become real numbers since their phase $\gamma_K$ reduces to zero. Thus, the quantum results of Equation (43) perfectly coincides with the classical case of Equation (39). Using those relations for Equation (40), the total entropy amplitude of the final state $|B\rangle$ is given as $\sigma_{Tq}(B|A_K)$, which is not pure state but it is clearly the mixed state superposed by many pure states: the $\sigma_{Tq}(B|A_K)$ is

$$\sigma_{Tq}(B|A) = -\sum_K^n \sigma_q(B|A_K) = -\sum_K^n C_K^*\beta_K\eta_K \cdot \log_2(\beta_K\eta_K). \tag{44}$$

Finally, the total entropy is calculated by those equations, and we obtain the final result of the quantum expression corresponding to classical relation (38):

$$\begin{aligned} H_{Tq}(B|A) &= -\left[\sum_K^n \sigma_{Tq}^*(B|A_K)\sigma_{Tq}(B|A_K)\right]^{1/2} \\ &= -\sum_K^n |C_K|\cdot|\beta_K\eta_K|\cdot\log_2|\beta_K\eta_K|\,(1+D)\,(1+B+E) \end{aligned}$$

$$D = -\frac{\sum_{J\ne K}^n |C_J C_K \beta_J \beta_K \eta_J \eta_K| \log_2|\beta_K\eta_K| \cdot \log_2|\beta_J\eta_J|}{\left(\sum_J^n |C_J\beta_J\eta_J| \cdot \log_2|\beta_J\eta_J|\right)^2},$$
$$\because \beta_K = |\beta_K| e^{i\gamma_K}, \quad \eta_K = |\eta_K| e^{i\delta_K}$$

$$B = \frac{\left(\sum_K^n \tau_K \log_2 e\right)^2}{\sum_K^n |C_K^*\beta_K\eta_K|^2 \cdot (\log_2|\beta_K\eta_K|)^2}, \quad E = \frac{\sum_{J>K}^n \mathrm{Re}(W_{J,K})}{\sum_K^n |C_K^*\beta_K\eta_K|^2 (\log_2|\beta_K\eta_K|)^2}. \tag{45}$$

It should be noticed that entropy of the quantum system has a lot of complex additional terms whose effects arise from much interference and a mixture of pure states. Comparing Equation (45) with classical entropy Equation (39), we find the same term, $P(B|A)P(A)\log_2 P(B|A)P(A)$ which means classical effect, and the other residual terms are corresponding to much interference of quantum system. Considering both result Equation (42) and Equation (45), we conclude that generally the entropy of quantum system is greater than that of the classical system since the other residual terms are additional and non-negative. Thus, the quantum interference between pure states makes out an increase of entropy larger than the case of the classical system.

6. **Multi Channels and Steps with Errors.** We reported quantum channel and those steps without noise and its related Bayes' theory [18,20], and in this section, we can start discussing these quantum machines with multi channels with errors. Now we can imagine two types of channels, one is a classical system and the other means quantum one. Moreover, we have discussed a type model, which had multi-inputs $|A_K\rangle$ and single output $|B\rangle$.

We would like to expand the above model to a kind of multi-outputs $|B_S\rangle$ model. Figure 1 means that this quantum system has multi-initial states and single output. On the other hand, Figure 2 is multi-outputs' system.

$$\begin{array}{ll} |A_1\rangle & \\ \vdots & \Rightarrow |B\rangle \\ |A_K\rangle & \end{array}$$

FIGURE 1. Single output system

$$\begin{array}{lcl} |A_1\rangle & & |B_1\rangle \\ \vdots & \Rightarrow & |B_j\rangle \\ |A_K\rangle & & \vdots \\ |A_n\rangle & & |B_q\rangle \end{array}$$

FIGURE 2. Multi-output system

We showed the results of quantum Bayes' form of the entropy and his quantum Bayes' calculations as shown in Section 5. Now, we would like to expand to systems of multi-inputs & multi-outputs channels (quantum machines) with errors. The quantum machines are not problem whether its systems are single output channel or not. Consequently, quantum machines are acquired by translating these probabilities into the corresponding quantum operators, for example, operators, $\hat{P}(A_S)$ (generally $\hat{P}(A)$) and $\hat{P}(B_j|A_S)$. After all, a classical event $A_S$ corresponds to a state vector $|A_S\rangle$ in quantum machine. From the previous section, the simplest multi quantum channels and steps are described by Equation (46).

$$\left\langle A_K \left| \hat{P}(A|B) \right| B_j \right\rangle \equiv \frac{\left\langle B_j \left| \hat{P}(B|A)\cdot\hat{P}(A) \right| A_K \right\rangle}{\sum_j^n \left\langle B_j \left| \hat{P}(B|A)\cdot\hat{P}(A) \right| A_j \right\rangle}. \tag{46}$$

We introduce both classical and quantum expressions of error's propagating probability into our quantum case Equation (46) or classical Equation (23). The classical case is the probability of real number, which is described as probabilty $1 - P(B_j|A_S)$. On the other hand, quantum one is not real probability but it means the operator for probability amplitude $1 - \hat{P}(B_j|A_S)$. Thus, we define the similar rules to simplify our quantum machine.

1) Base set: the state vectors $|A_S\rangle$, $(s = 1 \sim n)$ make a complete set, and they are in pure state. State vectors $|B_j\rangle$, $(j = 1 \sim q)$ are not in pure states, but they belong to the mixed states of all pure states $|A_S\rangle$.

2) Orthonormality of base set: the pure state vectors have an orthonormality.

$$\langle A_j(t_j)|A_K(t_K)\rangle = \delta_{jK}\delta(t_j - t_s). \tag{47}$$

3) As for eigen function, eigen state, and propagating operators, an emergence probability of $A_S$ makes eigen state.

$$\hat{P}(A)|A_j\rangle = \beta_j|A_j\rangle. \tag{48}$$

The propagating operators of probability, in quantum channels, with errors or with correct: If the correct propagator of probability of the state $|A_S\rangle$ is $1 - \hat{P}(B_j|A_S)$, then the error probability's operator state $|A_S\rangle$ becomes $1 - \hat{P}(B_j|A_S)$. If we have $p$ correct channels, the rest ($n$-$p$) is in the wrong. We define the correct and wrong propagating case.

a) Correct case:

$$\hat{P}(B|A)|A_j\rangle \equiv \hat{\eta}(A)|A_j\rangle = \xi_j\eta_j|A_j\rangle, \quad 1 \le j \le p \tag{49}$$

b) Wrong case:

$$\left\{1 - \hat{P}(B|A)\right\}|A_j\rangle \equiv \hat{\eta}^w(A)|A_j\rangle = \xi_j(1-\eta_j)|A_j\rangle. \tag{50}$$

The operator $\hat{\eta}$ ordinary conveys probability amplitude of correct information, and so the $\xi$ means conduction-rate of propagating processes. However, it sometimes fails to transmit the correct information from $|A_S\rangle$ to $|B_j\rangle$. If we assume that the $p$ numbers' channels are in correct states and the other ($n$-$p$) channels have the wrong signals, we can define two cases. One is the correct case and the other corresponds to the wrong case. Our propagating operator of neuron model is having four effects, which are neural conductions, ephapse among axons, thermal noise, and interferences at synaptic junction. And errors are caused by various quantum interference and noise. Thus, the correct propagating operators $\hat{\eta}(A)$ should be composed of those factors:

$\eta(A) =$ (neural conduction)+(ephapse)+(noise & attenuation)+(synaptic interferences).

4) Each final state $|B_j\rangle$, ($j = 1 \sim q$) is given as summing up pure initial states. Thus, each $|B_j\rangle$, is mixed and superposed by many pure states $|A_S\rangle$. So, we are able to expand the final mixed states by using $n$-numbers bases of each orthonormal pure state. Then we have descriptions of each final state. Each final state of $B$ is expressed as superposition:

$$|B_j\rangle = \sum_s^q C_s^j\,|A_s\rangle, \quad 1 \le j \le q. \tag{51}$$

If we assume that the $p$ numbers' channels are in correct and the others ($n$-$p$) are in wrong state, then finally Equation (46) is written down by Equations (47)-(50). We obtain Equation (52)

$$P_Q(A_s|B_j) = \frac{\left|C_{s=1}^{*j}\beta_s\xi_s\eta_s\right|}{\left(\left|\sum_{s=1}^{p} C_{s=1}^{*j}\beta_s\xi_s\eta_s\right|^2 + \left|\sum_{s=1}^{n-p} C_{p+s}^{*j}\beta_{p+s}\xi_{p+s}(1-\eta_{p+s})\right|^2 + Z_q\right)^{1/2}}$$
$$Z_q \equiv \left(\sum_{s=1}^{p} C_{s=1}^{*j}\beta_s\xi_s\eta_s\right)\cdot\left(\sum_{s=1}^{n-p} C_{p+s}^{*j}\beta_{p+s}\xi_{p+s}(1-\eta_{p+s})\right), \tag{52}$$

where $Z_q$ means quantum fluctuations between correct channels and wrong ones. So, we can obtain amplitude of entropy $\sigma_{tA}(B_j|A)$ for all paths from an initial state $A_S$, ($s = 1 \sim n$) to final state $B_j$, ($j = 1 \sim q$). Thus, we reach total probability amplitude of entropy $\sigma(B|A)$ for mixed vectors of all final state $B_j$. An amplitude of entropy $\sigma(B|A)$ becomes with Equation (23), so we can easily calculate the amplitude of entropy $\sigma_{tA}(B_j|A)$

$$\begin{aligned}\sigma_{tA}(B_j|A) &\equiv \sum_s^n \sigma(B_j|A_s) = \left\langle B_j \left| \hat{P}(B|A)\hat{P}(A) \cdot \log\left(\hat{P}(B|A)\hat{P}(A)\right) \right| A_s \right\rangle \\ &= -\sum_{s=1}^{p} C_s^{*j} \beta_s \xi_s \eta_s \cdot \log_2(\beta_s \eta_s) - \sum_{s=1}^{n-p} C_{p+s}^{*j} \beta_{p+s} \xi_{p+s} \bar{\eta}_{p+s} \cdot \log_2(\beta_{p+s}\bar{\eta}_{p+s}) \\ \because \bar{\eta}_{p+q} &\equiv 1 - \eta_{p+s}. \end{aligned} \tag{53}$$

Finally, total amplitude of $\sigma(B|A)$ becomes

$$\begin{aligned}\sigma(B|A) &\equiv \sum_j^q \sigma_{tA}(B_j|A) \\ &= -\sum_j^q \left( \sum_{s=1}^{p} C_s^{*j} \beta_s \xi_s \eta_s \cdot \log_2(\beta_s \eta_s) \right) \\ &\quad - \sum_j^q \left( \sum_{s=1}^{n-p} C_{p+s}^{*j} \beta_{p+s} \xi_{p+s} \bar{\eta}_{p+s} \cdot \log_2\left(\beta_{p+s}\bar{\eta}_{p+s}\right) \right), \end{aligned} \tag{54}$$

where $\xi$ means a conduction-rate of propagating processes. The total entropy $H(B|A)$, not complex amplitude, from state $A$ to state $B$ is given as

$$\begin{aligned}H(B|A) &\equiv -\left[\sigma^*(B|A) \cdot \sigma(B|A)\right]^{1/2} \\ &= Abs\left\{ \sum_j^q \left( \sum_{s=1}^{p} C_s^{*j} \beta_s \xi_s \eta_s \cdot \log_2(\beta_s \eta_s) \right) \right. \\ &\quad \left. + \sum_j^q \left( \sum_{s=1}^{n-p} C_{p+s}^{*j} \beta_{p+s} \xi_{p+s} \bar{\eta}_{p+s} \cdot \log_2\left(\beta_{p+s}\bar{\eta}_{p+s}\right) \right) \right\}. \end{aligned} \tag{55}$$

From Equation (55), we notice that there are quantum interference between correct channels and wrong ones. Those two kinds of channels mix each other, and they build mixed state.

7. **Differences and Similarities.** We have discussed Bayesian theorem, descriptions of entropy and multi-channel propagators based on both classical and quantum information theory. Theme of entropy, propagation of information and probability are start points of information theory, and belong to a part of fundamental concepts. We showed these contents by two ways: the one is in terms of classical and well-known method, and the other is quantum description.

However, the differences between classical and quantum rules are whether quantum fluctuation and interferences are included in the mechanical systems. Quantum systems have these fluctuations and their interferences. The classical ones do not have these fluctuations and interferences, which are excluded by standardizing these physical characteristics.

In this section, we intend to mention between the differences of these two neurons, and refer to formal similarities of the two systems [18]. And we should be noticed that classical systems are special limits of quantum ones, and essentially all systems should be expressed by quantum language. So, we are able to express that the classical systems almost imitate quantum ones. For example, one of the classical networks is represented as following equations:

$$U_K = \sum_{j}^{N} W_{Kj} X_j - h_j \tag{56}$$

where inputs signal is $X_j$ ($j = 1 \sim N$), weight is $W_{KJ}$ which means to be added to the $K$-th neuron. The $U_K$ is output of the $K$-th neuron. As you know, the output of $K$-th neuron $Y_K$ is described as

$$Y_K = f(U_K) = \frac{1}{1 + \exp(-aU_K)} \tag{57}$$

with control function $f(X)$ and $U_K$. The imitated quantum system has mathematical similar forms with state vectors and wave function: the state vector $\left|A_K^B(t)\right\rangle$ is written as

$$\begin{aligned} &\left|A_K^B(t)\right\rangle = \sum_{j}^{N} C_{Kj}(t)\left|A_j(t)\right\rangle = \sum_{j}^{N}\left\{C_{Kj}(t)\exp\left(\frac{-i\varepsilon_j t}{\hbar}\right)\left|A_j\right\rangle - h_j\left|A_j(0)\right\rangle\right\}, \\ &\because \left|A_j(0)\right\rangle = const. \end{aligned} \tag{58}$$

The weight $C_{Kj}$ (coefficient) is of superposition of the state vector $|A_j(t)\rangle$, and $\left|A_K^B(t)\right\rangle$ is the final state vector and it corresponds to the potential term $U_K$ of classical system. So, if we project these state vectors of Equation (58) on coordinate space $x$, then we obtain normal coordinate expression. It should be noticed that the quantum $U_K$, whose meaning is $\Phi_K$ in quantum language, is described by superposition of many wave functions. The constant wave function $C_{K0}$ plays a role of threshold value in classical system. So we have Equation (59), which is projection to a coordinate system

$$\begin{aligned} \Phi_K &= \left\langle x\middle|A_K^B(t)\right\rangle = \sum_{j}^{N} C_{Kj}(t)\left\langle x\middle|A_j(t)\right\rangle - h_j C_{j0} \\ &= \sum_{j}^{N} C_{Kj}(t)\exp\left(\frac{-i\varepsilon_j t}{\hbar}\right)\left\langle x\middle|A_j\right\rangle - h_j C_{j0} \\ &= \sum_{j}^{N}\left\{C_{Kj}(t)\exp\left(\frac{-i(\mathbf{px} + \varepsilon_j t)}{\hbar}\right) - h_j C_{j0}\right\}, \quad \because C_{j0} = \left\langle x\middle|A_j(0)\right\rangle. \end{aligned} \tag{59}$$

Thus, the quantum controlled output $\Psi_K$, which is the controlled wave function, is expressed as

$$\Psi_K = f(\Phi_K) = \frac{1}{1 + \exp(-a\Phi_K)}. \tag{60}$$

Then, both classical and quantum control expressions are so similar to each other. However, the essential differences are whether the interferences and fluctuations are included or not. Quantum system has interferences, superposition, fluctuations and probability amplitude because of complex number. On the other hand, classical one does not have them. We should notice that if we pay attention to similarities of both systems, we are able to re-write easily the classical control function into quantum expression with replacing $U_K$ and $Y_K$ into $\Phi_K$ and $\Psi_K$. This similarity is convenient merit because we are able to utilize our most knowledge of classical systems.

We point out the second similarity and difference: We would like to refer to thermal noises between classical signals and quantum ones (examples: polariton, photon, quantum particle).

The thermal noise is against neural conduction of electric signals, and heart is a kind of troublesome electric current being called noise current. J. B. Johnson, as you know, found out the noise electric current by observations of some resistors. The noise currents are called thermal noises and hot resisters generate noise power. The most of the noise power is described as

$$N = k_B T W. \tag{61}$$

$k_B$ is Boltzmann constant, and $T$ is absolute temperature. The symbol $W$ means the bandwidth of noise in cycles per second. The noise determines the power required to send messages (conduct on axon). When we transmit information $C$ bits/s, the signal power $P$ should be related to noise power $N$. $C$ is

$$C = W \log\left(\frac{1+P}{N}\right) = W \log\left(\frac{1+P}{k_B T W}\right). \tag{62}$$

If the $P/k_B TW$ becomes very small compared with unity then

$$P = 0.693 k_B T C. \tag{63}$$

Equation (64) shows that we utilize very wide bandwidth, and we spend at least its power $0.693 k_B T$ joule/s to send 1 bit/s. At 300 Kelvin, the signal power becomes $1.7\times10^{-2}$ eV (per/s)/(bit/s) as shown in Equation (63). At $T = 300$ K, the thermal noise energy is almost estimated,

$$3/2 k_B T = 6.3 \times 10^{-21}\ \mathrm{J} = 3.9 \times 10^{-2}\ \mathrm{eV}. \tag{64}$$

It should be noticed that the value of the thermal noise $3.9 \times 10^{-2}$ eV is lager than that of $0.639 k_B T$, $1.7 \times 10^{-2}$ eV (per/s)/(bit/s). In classical mechanics, the energy of particles should be larger than that of the thermal noise to convey the information of neuron. However, quantum particle's polariton obeys to quantum rules, and we should re-write classical form Equation (62) into quantum ones. Herry Nyquist gave relations between thermal noise and all frequencies of light. He mentioned that the thermal noise with bandwidth $W_i$ was described as

$$N_i = \frac{\hbar\omega_i W_i}{\exp\left(\hbar\omega_i/k_B T\right) - 1}. \tag{65}$$

Then the most noise power $N_i$ is given as

$$N_i \approx x(1 + x + x^2 + \cdots)\hbar\omega_i W_i = \left[\frac{\hbar\omega_i}{k_B T}\exp\left(-\frac{\hbar\omega_i}{k_B T}\right)\right] k_B T W_i, \quad x \equiv \exp\left(-\frac{\hbar\omega_i}{k_B T}\right). \tag{66}$$

We take sum of the suffix $i$ and calculate the average of Equation (66) according to statistical mechanics.

$$\langle N_i\rangle = \langle r_i W_i\rangle k_B T, \qquad \langle r_i W_i\rangle = \sum_i W_i\left[\frac{\hbar\omega_i}{k_B T}\exp\left(-\frac{\hbar\omega_i}{k_B T}\right)\right]. \tag{67}$$

If all $W_i$ are the same values and the average $r_i$ is nearly to 1, then Equation (67) becomes

$$\langle N\rangle = \langle r_i\rangle k_B T W \Rightarrow \langle N\rangle = k_B T W. \tag{68}$$

It should be noticed that Equation (68) is quantum description, and Equation (62) is classical one. Both forms of these two equations are so much similar to each; however, the classical form Equation (7) has only the value at low frequency. The energy per polariton is $h\nu$, and ideally the energy per bit is $0.639k_BT$. We are able to estimate amount of information which polariton conveys at 300 Kelvin.

$$\frac{\hbar\omega}{0.693k_BT} = 2.31 \times 10^{-3}\nu \text{ (bits/polariton)}. \tag{69}$$

If we can use frequency of thermal noise, then the polariton carries amount of information, $9.38 \times 10^{12}$ bits/polariton, at 300 Kelvin. We would like to propose the third example, which is similarities and differences between fuzzy set theory and quantum system [18]. To simplify these problems, we take up a fuzzy probability of dice. The *Set X* is defined as the set of numbers of dice:

$$Set\ X = \{1, 2, 3, 4, 5, 6\}. \tag{70}$$

The value of *Set X* is expressed to be nearly equal to 6, which is called as fuzzy probability $P_E(\approx 6)$. If membership function $A(X)$ is defined as $A(1) = 0$, $A(2) = 0.1$, $A(3) = 0.3$, $A(4) = 0.6$, $A(5) = 0.9$, $A(6) = 1$, then we obtain the fuzzy probability

$$P_E(\approx 6) = A(1)P(1) + A(2)P(2) + A(3)P(3) + A(4)P(4) + A(5)P(5) + A(6)P(6). \tag{71}$$

Expanding this fuzzy probability to continuous probability variable $X$, we can give an expression to the fuzzy probability $P_E(X)$ with membership function $F(X)$ and probability density

$$\rho_J(X)\colon P_E(\approx X_J) = \int_{all\ X} \rho_J(X)F_J(X)dX. \tag{72}$$

So, it is definition of fuzzy probability. On the other hand, expectation value of quantum mechanics is described as

$$\langle F_J\rangle = \int \psi_J^*(X)F_J(X)\psi_J(X)dX. \tag{73}$$

It should be noticed that Equation (72) is the expression of fuzzy probability, and the other equation Equation (73) means quantum expectation value of the physical observable $F_J$. When we compare Equation (72) with Equation (73), immediately we recognized that both equations are so much similar to each other. The probability density and the membership function correspond to quantum probability density $\psi^*(X)\psi(X)$ and the physical observable $F_j(X)$. However, the fuzzy probability is described with real number: Both membership function and probability density are real number. Generally, the probability amplitude $\psi(X)$ and the physical observable are often complex number, but the results of calculation should be real number. We never encounter complex number under its calculation process on fuzzy probability. Expectation value of quantum mechanics is essentially different from fuzzy probability except similarity of each other form. We assume that $\psi = \exp(-ikX)$ is plane wave and we adopt its complex conjugate wave function $\psi = \exp(ik+\Delta k)X$ with slight differences of momentum. So, we take momentum operator $F_j(X)$, and then its expectation value becomes

$$\begin{aligned}\langle F_J(\approx X_J)\rangle &= \int P_\rho(X)F_J(X)dX = \int \Psi^*(X)F_J(X)\Psi(X)dX \\ &= \int e^{ikX}\left(-i\hbar\frac{d}{dX}\right)e^{-i(k+\Delta k)X}dX = -\int_{-\infty}^{\infty} e^{-i\Delta k\,X}\cdot\hbar(k+\Delta k)dX \\ &= -2\pi\delta(\Delta k)\cdot\hbar(k+\Delta k),\end{aligned} \tag{74}$$

where $\delta(\Delta k)$, which has much sharp function, is named as Dirac delta function. We perform an integral to Equation (74) at near to zero, and then we obtain the following result:

$$\int \langle F_J \rangle d(\Delta k) = -\int_{-\varepsilon}^{\varepsilon} 2\pi\delta(\Delta k) \cdot \hbar(k + \Delta k) \cdot d(\Delta k) = -2\pi\hbar k. \tag{75}$$

We should pay attention to this result to have the finite value. On the other hand, the calculation of Equation (72) sometimes becomes infinite value or zero. If we adopt real function $\psi(X) = \cos(kX)$, $\psi(X) = \cos\{((k + \Delta k)X)\}$ and momentum operator $i\hbar d/dX$, then the fuzzy probability

$$\begin{aligned} P_E(\approx X_J) &= \int_{all\ X} P_\rho(X) F_J(X) dX = \int_{-\infty}^{\infty} \cos(kX) \cdot \left(i\hbar \frac{d}{dX}\right) \cos(k + \Delta k) X dX \\ &= \hbar^2 \int_{-\infty}^{\infty} \cos(kX) \cdot (k + \Delta k) \sin\{(k + \Delta k)X\} dX = 0. \end{aligned} \tag{76}$$

If the above membership takes following conditions
1) $AX + B$ $(-B/A \le X \le 0)$
2) $-AX + B$ $(0 \le X \le B/A)$
3) $0$ $(|X| > B/A)$,
where both $A$ and $B$ are positive numbers, we know that the fuzzy probability becomes divergence in limit of $\Delta k \to 0$. If $\Delta k$ is not near to zero, then we can obtain zero fuzzy probability. We think that these two cases are not significant under long-range integrals. Thus, if we take complex probability amplitude $\psi(X)$ in fuzzy probability, we will be able to reach quantum mechanical expectation or complex fuzzy theory in order to prevent the insignificant results.

8. **Summary and Conclusion.** We, at first, showed an equation of polariton motion based on Proca equation. And we have reduced Proca equation to quaternary Schrödinger equation. However, we can obtain the scalar potential $\phi$ by ignoring vector potential $\mathbf{A}$, if polariton's mass is so large and its propagating speed is so small. Interference among many neurons is described by path integrals, instead of wave equations, and the method of path integral is closely related to the propagation of polaritons. And then we have compared classical Bayesian theorem with quantum Bayes' cases. The quantum Bayes rule is given as operator's relation ($q$-number), though the classical Bayesian theorem is described by $c$-number's equation and the observable values are real number. An essential difference between the quantum description and the classical one in the Bayesian theorem is whether the incidental probability and the propagating probability take complex numbers or not. So, the classic system is related to the real number, which always directly means the probability. On the other hand, the quantum mechanical values are closely connected to complex number; however, the physical observable are always real number. So, it is said that physical quantities (momentum, Hamiltonian, potential, propagators, etc.) are always expressed as operators or probability amplitude; however, these observed physical values always should be probability of real number. For example, its wave function describes the probability amplitude, which always takes complex number, and so the probability is given as absolute values of the probability amplitude. Then, probability amplitude is complex number, but probability is real number. Thus, quantum Bayes' rule includes much interference among each quantum state. However, there is not interference in the classical Bayesian theorem. Then, the result of the quantum Bayes' form is equal to that of the classical Bayes' theorem, if it were not for the interferences among each quantum state. That result means that pure states are changed into many mixed states by interactions, interferences and potential scatterings. We calculated values of entropy by both types, which were classical system and quantum one. The quantum entropy, which is compared with classical one, has much interference term, and each interference

is combined with many states so as to make up new mixed states as well as quantum Bayes' theory. The quantum neurons can control their networks as the classical networks do. The only difference between both networks is whether there are interferences between each path or not. And the common probability density is governed by Fokker-Planck equation with the scalar density function $\phi$; however, the wave function of the polariton should be essentially described by the quaternary Schrödinger equation $A^{\mu}(\phi, \mathbf{A})$, except the slow change of magnetic field (i.e., vector potential $\mathbf{A}$ is almost constant). If $\mathbf{A}$ is near to zero then the polariton obeys the ordinary Schrödinger equation with only an electric field.

We showed some similarities and differences between classical systems and quantum ones. Both classical and quantum control expressions are so much similar to each other; however, the essential differences are whether the interferences are included or not. Quantum system has interferences, superposition and probability amplitude because of complex number. On the other hand, classical one does not have them. We should notice that if we pay attention to similarities of both systems, we are able to re-write easily the classical control function into quantum expression with replacing $U_K$ and $Y_K$ into $\Phi_K$ and $\Psi_K$. So, we mentioned relations between the classical noise theory (J. B. Johnson) and Herry Nyquist theorem. The classical theory gives correct results under low frequency, and the limits of quantum systems gain on classical ones when the frequencies are increasing. If we almost utilize the frequency of thermal noise, we see that the polariton carries amount of information, $9.38 \times 10^{12}$ bits/polariton, at room temperatures.

The form of fuzzy probabilities is similar to that of quantum mechanics. If we are able to find motion of equation, like as equations of membership function or equations of probability density, we recognize that both systems come to have more similarities. The both probability amplitude and the observable are almost complex variable, and their expectation values should be real number. On the other hand, parts of classical equations are essentially real number in classical systems. Thus, quantum systems have a tendency to converge to some value, when we perform integral. However, the classical ones sometimes go to zero or divergence. We would like to propose a kind of complex fuzzy system (or complex neural net works). Both systems (i.e., fuzzy system and quantum system) have something in common on these shapes.

If we can give the time-development equations, prescriptions, interferences and superposition to the real fuzzy system, then the real fuzzy system has complex variables and the fuzzy neural system becomes more similar to quantum calculation process of brain functions. One day in future, we expect that some parts of brain functions and its networks are able to describe with methods of complex fuzzy neural networks or quantum fuzzy processes.

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