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IMPROVED STABILITY CRITERIA FOR TIME-DELAYED LUR'E SYSTEMS WITH MARKOVIAN SWITCHING

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Abstract. *In this paper, we focus on the problem of the stochastic stability for the time-delayed Markov jump Lur'e system with time-invariant and time-varying nonlinearities. By using an improved relaxed integral inequality technologies, new delay-dependent stochastic stability criteria are proposed via Lyapunov-Krasovskii functional (LKF) approach. The stability conditions reduce the conservation of some previous ones. Some numerical examples are presented to show the effectiveness of the proposed approach.*
Keywords: Lur'e systems, Linear matrix inequality, Markov switch system, Time-delay, Robust stability

1. **Introduction.** The jump systems have the advantage of modeling the dynamic systems subject to abrupt variation in their structure, such as component failures or repairs, sudden environmental disturbance, changing subsystem interconnections, operating in different point of a plant. Many researchers have made a lot of progress in Markovian jump control theory, such as [1, 2, 3] and the references therein.

Most systems are nonlinear in practical engineering. It is well known that many of certain nonlinear systems, such as Chua's circuit and the Lorenz system, can be modeled as Lur'e system, which consists of a feedback connection of a linear dynamical system and a nonlinearity satisfying the sector bounded condition [4, 5]. Take the famous Chua's chaotic circuit for an example. Chua's circuit is the first electrical circuit to realize chaos in experiment, which exhibits very rich complex dynamical behavior, and yet has very high potential in real applications. By analyzing the absolute stability and robustness of the mathematical model for Lur'e systems, the stability characteristics related to Chua's chaotic circuit can be obtained easily. Thus, there is important theoretical significance to study Lur'e systems. Meanwhile, time delay inevitably appears in practical engineering [6, 7], and the stability problems for time-delayed Lur'e systems have been considered extensively [8, 9, 10, 11, 12, 13, 14, 15, 16], where lots of significant robust stability criteria had been given. [8, 10, 12] obtained the stability criteria based on Jensen's inequality; [15, 17] gave some improved robust stability criteria, where the general free-weighting matrix method was applied to illustrating the relationships of the terms appearing in the Leibniz-Newton formula. However, many variables must be decided, which makes the computation quite complex. In [13, 18, 19], by constructing sets of modified **LKF**s, improved robust

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stability criteria were derived, where the contribution in reducing conservation of the proposed stability criteria relied on the reciprocally convex method and Wirtinger-based inequality. Recently, [20, 21, 22, 23] obtained less conservative robust absolute stability criteria for uncertain Lur'e systems with time-varying delays. However, the variation interval of the time delay was directly considered in constructing the **LKF**s, which may neglect some information on the variation interval of the time delay according to the delay-fraction theory and piecewise analysis method.

Recently, more and more attention is paid to the control problems of discrete-time Markov jump Lur'e systems, for example, resilient dissipative dynamic output feedback control [24], distributed $\mathcal{H}_\infty$ filtering [25, 26, 27], and observer-based l_2-l_∞ control [28, 29] for discrete-time Markov jump Lur'e systems. However, to the best of the authors' knowledge, there exist only a few results available in the literatue [30, 31] to study the stochastic stability problem for the continuous-time Lur'e system with time-varying delays. In [30], by exploiting the free-weighting matrices approach, a delay-dependent stability criterion, which guarantees that the system is stochastically stable and robustly passive was developed for the continuous-time Lur'e singular systems with time-varying delays and time-invariant nonlinearities.

Motivated by the aforementioned factors, in this paper, we study the stochastic stability for uncertain continuous-time Lur'e systems with time-varying delays. Both time-invariant and time-varying nonlinearities are considered. The parameter uncertainties are assumed to be time-varying but norm-bounded. By choosing an LKF, employing an improved relaxed integral inequality technology, some new delay-dependent stochastic stability criteria are derived in terms of LMI without using the general free-weighting matrix method. In addition, when the system mode is one, our criteria are less conservative than some ones recently proposed. The contribution in reducing conservation of the proposed stability criteria relies on the meanwhile using of the delay fractionating method and the improved Wirtinger-based inequality proposed in [24, 32]. Finally, some detailed illustrative examples are provided to illustrate the effectiveness of the proposed method.

This paper is organized as follows. Section 2 gives the problem statement and provides some useful definitions, assumptions and lemmas. Section 3 presents the stable criteria and robust stable criteria for the nominal form without uncertainties and uncertain form, respectively, including theorems and corollaries. Section 4 shows numerical examples. Conclusions are drawn in Section 5.

Notation: Throughout this paper, the notations are standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices; for $P \in \mathbb{R}^{n\times n}$, $P > 0$ (respectively, $P < 0$) mean that P is a positive (respectively, negative) definite matrix. $\mathrm{diag}\{a_1, a_2, \ldots, a_n\}$ denotes an n-order diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$. e_i ($i = 1, \ldots, m$) are block entry matrices. For example, $e_2^T = \begin{bmatrix} 0 & I & \underbrace{0 \cdots 0}_{m-2} \end{bmatrix}$. For a real matrix B and two real symmetric matrices A and C of appropriate dimensions, $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ denotes a real symmetric matrix, where $*$ denotes the entries implied by symmetry, $Sym\{B\} = B + B^T$. $E[X]$ denotes the expectation of X.

2. **Problem Formulation.**

Consider the following Lur'e system with Markovian jumping parameters, time-varying delays and parametric uncertainties:

$$\begin{cases} \dot{x}(t) = (\mathit{\Delta} A(r_t) + A(r_t))x(t) + (\mathit{\Delta} A_1(r_t) + A_1(r_t)) \times x(t - h(t)) \\ \qquad\quad + (\mathit{\Delta} B(r_t) + B(r_t))w(t), \\ z(t) = M(r_t)x(t) + N(r_t)x(t - h(t)), \\ w(t) = -\varphi(t, z(t)), \\ x(s) = \phi(s), \quad s \in [-h, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^m$ are the state, input and output vectors of the system, respectively. $\phi(s)$ is an $\mathbb{R}^n$-valued continuous initial functional specified on $[-h, 0]$ with known positive scalars h. $A(r_t)$, $A_1(r_t)$, $B(r_t)$, $M(r_t)$ and $N(r_t)$ are real constant matrices with appropriate dimensions, and r_t is a continuous-time Markov process with a right continuous trajectory taking values finite set $\mathbf{S} = 1, 2, \ldots, N$ with transition probabilities as

$$P[r_{t+\Delta t} = j | r_t = i] = \begin{cases} \pi_{ij}\Delta + o(\Delta), & \text{if } j \neq i; \\ 1 + \pi_{ij}\Delta + o(\Delta), & \text{if } j = i \end{cases}$$

where $\lim_{\Delta \to 0} \frac{o(\Delta)}{\Delta} = 0$, $\pi_{ij} > 0$, $j \neq i$ and $\pi_{ii} = -\sum_{j \neq i} \pi_{ij}$ for each $i \in \mathbf{S}$.

Remark 2.1. *In practical engineering control, the system will inevitably encounter various mutations, such as the external environment disturbance, system internal parts failure, system internal subsystem jump, and the change of parameter relationship after system linearization, which often leads system change, rendering more structure. The Markov system describes this mutation better than the ordinary time-invariant system. Therefore, this paper focuses on the stochastic stability of Lur'e system with Markovian jumping parameters, which is more general and complex than the time-invariant system.*

The time-varying delay $h(t)$ is continuous-time functional and satisfies the following conditions:

$$0 \leq h(t) \leq h, \quad \mu_1 \leq \dot{h}(t) \leq \mu_2, \quad \forall t \geq 0, \tag{2}$$

where h, μ_1 and μ_2 are constants.

Remark 2.2. *From the point of view of system theory, the past state of any practice system will inevitably affect the current state, that is, the evolution trend of the system depends not only on the current state of the system, but also on the state of a certain time or several times in the past. Such a case can be described in terms of time-delayed systems. There are many examples for time-varying delay, such as the measurement of system variables, slow chemical reaction process, storage and release process of energy storage components. Time-varying delays are commonly found in circuits, optics, neural networks, biological environment and medicine, architectural structures, machinery and other fields. Time delay includes time-varying delay and time-invariant delay, while time-varying delay is more general and less conservative than time-invariant delay. Therefore, time-varying delays are considered in this paper.*

For notational simplicity, in the sequel, for each possible $r_t \in i$, $i \in \mathbf{S}$, a matrix $R(r_t)$ will be denoted by R_i. For example, $A(r_t)$ is denoted by A_i, and $A_1(r_t)$ is denoted by A_{1i}.

The time-varying structured uncertainties are of the form:

$$[\mathit{\Delta} A_i \ \mathit{\Delta} A_{1i} \ \mathit{\Delta} B_i] = D_i F_i(t) \left[E_{ai} \ E_{1i} \ E_{bi}\right], \quad i \in \mathbf{S}, \tag{3}$$

D_i, E_{ai}, E_{1i} and E_{bi} are known matrices, and $F_i(t)$ are unknown time-varying matrices that satisfy $F_i^T(t)F_i(t) \leq I$.

Remark 2.3. *Due to external disturbance, mathematical modeling and model linearization, some parameters of the practice system are lost or ignored. Thus, the robust stability of the system with uncertain parameters is more general in system analysis. The general assumptions of uncertain parameters should be time-varying and bounded, which is also the case considered in this paper.*

The nonlinear $\varphi(t, z(t))$ is formulated as $\varphi_i(t, z_i(t)) = \left[\varphi_{i1}^T(t, z_{i1}(t))\, \varphi_{i2}^T(t, z_{i2}(t)) \cdots \varphi_{im}^T(t, z_{im}(t))\right]^T$, where $\varphi_i(t, z_i(t)) \in \mathbb{R}^m$ is a memoryless, possibly time-varying, nonlinear functional, which is piecewise continuous in t, globally Lipschitz in $z_i(t)$, and satisfies the following inequation for $\forall t \geq 0$, $\varphi_i(t, 0) = 0$, $i \in \mathbf{S}$,

$$\varphi_i^T(t, z_i(t))[\varphi_i(t, z_i(t)) - K_i z_i(t)] < 0 \tag{4}$$

or

$$[\varphi_i(t, z_i(t)) - K_{i1} z_i(t)]^T [\varphi_i(t, z_i(t)) - K_{i2} z_i(t)] < 0, \tag{5}$$

where K_{i1} and K_{i2} are real matrices of appropriate dimensions, and $K_i = K_{i2} - K_{i1}$ is a symmetric positive definite matrix. In other words, the nonlinear function $\varphi_i(t, z_i(t))$ satisfying (5) is said to belong to a sector $[K_{i1}, K_{i2}]$. If the nonlinear function $\varphi_i(t, z_i(t))$ belongs to a sector $[0, K_i]$, then $\varphi_i(t, z_i(t))$ satisfies (4).

Definition 2.1. (Stochastic Stability) [33]. *The system* (1) *is said to be stochastically stable in the sector* $[K_{i1}, K_{i2}]$ *(or* $[0, K_i]$*) if, for* $\varphi_i(t, z_i(t))$ *satisfying* (5) *(or* (4)*), the following inequation is satisfied* $\lim\limits_{t\to\infty} E\left\{\int_0^t \|x^T(s, \phi, r_0)\| ds\right\} < \infty$*, where* $x(t, \phi, r_0)$ *denotes the solution to the considered system at time* t *under the initial conditions* $\phi(t)$ *and* r_0*.*

The main purpose of this paper is to investigate stochastic stability for the system (1) satisfying conditions (2)-(5) by using Lyapunov stability theory. Throughout this paper, results will be derived based on the following lemmas.

Lemma 2.1. [34]. *For a given matrix* $Z > 0$*, the following inequality holds for all continuously differentiable function* $x(t)$ *in* $[a, b] \in \mathbb{R}^n$*:* $(b-a)\int_a^b \dot{x}^T(s) Z \dot{x}(s) ds \geq [x(b) - x(a)]^T Z [x(b) - x(a)] + 3\omega^T Z \omega$*, where* $\omega = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s) ds$*.*

Lemma 2.2. [24]. *For symmetric matrices* $R_1 > 0$*,* $R_2 > 0$ *and any matrices* S_1*,* S_2 *with an appropriate dimension, then the following inequality holds for a real scalar* $\alpha \in (0, 1)$*:*

$$\begin{bmatrix} \frac{1}{\alpha} R_1 & 0 \\ 0 & \frac{1}{1-\alpha} R_2 \end{bmatrix} \geq \begin{bmatrix} R_1 + (1-\alpha) T_1 & (1-\alpha) S_1 + \alpha S_2 \\ & R_2 + \alpha T_2 \end{bmatrix}.$$

Lemma 2.3. [35]. *Given matrices* Γ*,* Ξ *and* $\Omega = \Omega^T$*, the following inequality* $\Omega + \Gamma F(\sigma) \Xi + \Xi^T F^T(\sigma) \Gamma^T < 0$ *holds for any* $F(\sigma)$ *satisfying* $F^T(\sigma) F(\sigma) \leq I$*, if and only if there exists a scalar* $\varepsilon > 0$ *such that* $\Omega + \varepsilon^{-1} \Gamma \Gamma^T + \varepsilon \Xi^T \Xi < 0$*.*

3. **Main Results.** In this section, we will give some stochastic stability criteria and robustly stochastic stability criteria for time-delay Lur'e system with Markovian jumping parameters.

3.1. **Stochastic stability criteria for nominal form.** In this subsection, first, by applying the loop transformation [20], we get the nominal form of system (1) without parametric uncertainties in the sector $[K_{i1}, K_{i2}]$ is equal to that of the following system in the sector $[0, K_i]$:

$$\begin{cases} \dot{x}(t) = \bar{A}(r_t)x(t) + \bar{A}_1(r_t)x(t-h(t)) + B(r_t)w(t), \\ z(t) = M(r_t)x(t) + N(r_t)x(t-h(t)), \\ w(t) = -\varphi(t, z(t)), \\ x(s) = \phi(s), \quad s \in [-h, 0]. \end{cases} \tag{6}$$

Here, $\bar{A}_i = A_i - B_i K_{i1} M_i$ and $\bar{A}_{1i} = A_{1i} - B_i K_{1i} N_i$.

For the sake of simplicity on matrix representation, the notations of several symbols and matrices are defined as

$$\xi_1^T(t) = \left[x^T(t) \ \ x^T(t-h(t)) \ \ x^T\left(t - \frac{h}{2}\right) \ \ x^T(t-h) \ \ \dot{x}^T(t) \ \ w^T(t) \right.$$
$$\left. \int_{t-h(t)}^{t} \frac{x^T(s)ds}{h(t)} \ \ \int_{t-\frac{h}{2}}^{t-h(t)} \frac{x^T(s)ds}{\frac{h}{2} - h(t)} \ \ \frac{h}{2} \int_{t-h}^{t-\frac{h}{2}} x^T(s)ds \right],$$
$$\xi_2^T(t) = \left[x^T(t) \ \ x^T\left(t - \frac{h}{2}\right) \ \ x^T(t-h(t)) \ \ x^T(t-h) \ \ \dot{x}^T(t) \ \ w^T(t) \right.$$
$$\left. \int_{t-\frac{h}{2}}^{t-h(t)} \frac{x^T(s)ds}{h(t) - \frac{h}{2}} \ \ \int_{t-h}^{t-h(t)} \frac{x^T(s)ds}{h - h(t)} \ \ \frac{h}{2} \int_{t-\frac{h}{2}}^{t} x^T(s)ds \right],$$
$$\zeta^T(t) = \left[x^T(t) \ \ \int_{t-\frac{h}{2}}^{t} x^T(s)ds \ \ \int_{t-h}^{t-\frac{h}{2}} x^T(s)ds \right].$$

Theorem 3.1. *The system* (6) *satisfying the conditions* (2) *and* (4) *is stochastically stable for given values of* h*,* μ_1*,* μ_2 *and* K_i *if there exist real positive definite matrices* $L_{3n\times 3n}$*,* P_i*,* $X_{2n\times 2n}$*,* $Y_{2n\times 2n}$*,* R_j *and any matrices* S_k*,* U_{ir} *(*$j = 1, 2$*;* $k = 1, 2, 3, 4$*;* $r = 1, 2, 3$*) with appropriate dimensions such that the following* ***LMIs*** *hold for* $\dot{h}(t) \in \{\mu_1, \mu_2\}$*,* $i \in \mathbf{S}$*:*

$$\begin{bmatrix} \varPi_i^1|_{h(t)=0} - \varTheta_{i1} & E_1 S_2 \\ * & -\tilde{R}_1 \end{bmatrix} < 0, \tag{7}$$

$$\begin{bmatrix} \varPi_i^1|_{h(t)=\frac{h}{2}} - \varTheta_{i1} & E_2 S_1^T \\ * & -\tilde{R}_1 \end{bmatrix} < 0, \tag{8}$$

$$\begin{bmatrix} \varPi_i^2|_{h(t)=\frac{h}{2}} - \varTheta_{i1} & E_4 S_4 \\ * & -\tilde{R}_2 \end{bmatrix} < 0, \tag{9}$$

$$\begin{bmatrix} \varPi_i^2|_{h(t)=h} - \varTheta_{i1} & E_5 S_3^T \\ * & -\tilde{R}_2 \end{bmatrix} < 0 \tag{10}$$

with

$$\varPi_i^1 = \varOmega_i + Sym\Bigg\{ \varSigma^1 L \varGamma^T + e_1 P_i e_5^T + \varPi_i^3 \varPi_i^4 + h(t)[e_7 \ \ e_1 - e_7] X [0 \ \ e_5]^T$$
$$+ \left(\frac{h}{2} - h(t)\right) [e_8 \ \ e_1 - e_8] Y [0 \ \ e_5]^T + \frac{h}{2} [e_9 \ \ e_1 - e_9] Y [0 \ \ e_5]^T \Bigg\}$$
$$- E_3 \tilde{R}_2 E_3^T - \begin{bmatrix} E_1^T \\ E_2^T \end{bmatrix}^T \begin{bmatrix} \frac{2h-2h(t)}{h}\tilde{R}_1 & \frac{h-2h(t)}{h} S_1 + \frac{2h(t)}{h} S_2 \\ * & \frac{h+2h(t)}{h}\tilde{R}_1 \end{bmatrix} \begin{bmatrix} E_1^T \\ E_2^T \end{bmatrix};$$

$$\Pi_i^2 = \Omega_i + Sym\left\{\Sigma^2 L\Gamma^T + e_1 P_i e_5^T + \Pi_i^3\Pi_i^4 + \left(h(t) - \frac{h}{2}\right)[e_7 \;\; e_1 - e_7]X[0 \;\; e_5]^T \right.$$
$$\left. + (h - h(t))[e_8 \;\; e_1 - e_8]Y[0 \;\; e_5]^T + \frac{h}{2}[e_9 \;\; e_1 - e_9]X[0 \;\; e_5]^T\right\}$$
$$- E_6\tilde{R}_1E_6^T - \begin{bmatrix} E_4^T \\ E_5^T \end{bmatrix}^T \begin{bmatrix} \frac{3h-2h(t)}{h}\tilde{R}_2 & \frac{2h-2h(t)}{h}S_3 + \frac{2h(t)-h}{h}S_4 \\ * & \frac{2h(t)}{h}\tilde{R}_2 \end{bmatrix} \begin{bmatrix} E_4^T \\ E_5^T \end{bmatrix},$$

where

$$\Omega_i = [e_1 \; 0]X[e_1 \; 0]^T + \left(1 - \dot{h}(t)\right)[e_2 \; e_1 - e_2](Y - X)[e_2 \; e_1 - e_2]^T$$
$$- [e_4 \; e_1 - e_4]Y[e_4 \; e_1 - e_4]^T + [e_1 \; e_3]G[e_1 \; e_3]^T - [e_3 \; e_4]G[e_3 \; e_4]^T + e_1\sum_{j\in\mathbf{S}}\pi_{ij}P_je_1^T$$
$$+ \frac{h^2}{4}e_5(R_1 + R_2)e_5^T;$$

$$\Sigma^1 = \left[e_1 \;\; h(t)e_7 + \left(\frac{h}{2} - h(t)\right)e_8 \;\; \frac{h}{2}e_9\right];$$
$$\Sigma^2 = \left[e_1 \;\; \frac{h}{2}e_9 \;\; \left(h(t) - \frac{h}{2}\right)e_7 + (h - h(t))e_8\right];$$
$$\Gamma = [e_5 \;\; e_1 - e_3 \; e_3 - e_4]; \quad \tilde{R}_j = \text{diag}\{R_j, 3R_j\}; \quad j = 1, 2;$$
$$\Pi_i^3 = [e_1U_{i1} + e_2U_{i2} + e_5U_{i3}]; \quad \Pi_i^4 = \left[A_ie_1^T + A_{1i}e_2^T + B_ie_6^T - e_5^T\right];$$
$$E_1 = [e_1 - e_2 \quad e_1 + e_2 - 2e_7]; \quad E_2 = [e_2 - e_3 \quad e_2 + e_3 - 2e_8];$$
$$E_4 = [e_2 - e_3 \quad e_2 + e_3 - 2e_7]; \quad E_5 = [e_3 - e_4 \quad e_3 + e_4 - 2e_8];$$
$$E_3 = [e_3 - e_4 \quad e_3 + e_4 - 2e_9]; \quad E_6 = [e_1 - e_2 \quad e_1 + e_2 - 2e_9];$$
$$\Theta_{i1} = Sym\left\{e_6\left(e_6^T + K_iM_ie_1^T + K_iN_ie_2^T\right)\right\}.$$

Proof: Construct an **LKF** candidate as

$$V(x(t), r_t, t) = \sum_{k=1}^{3} V_k(x(t), r_t, t) \tag{11}$$

with

$$V_1(x(t), r_t, t) = x^T(t)P(r_t)x(t) + \zeta^T(t)L\zeta(t),$$
$$V_2(x(t), r_t, t) = \int_{t-\frac{h}{2}}^{t}\begin{bmatrix} x(s) \\ x\left(s - \frac{h}{2}\right)\end{bmatrix}^T G \begin{bmatrix} x(s) \\ x\left(s - \frac{h}{2}\right)\end{bmatrix} ds$$
$$+ \int_{t-h(t)}^{t}\begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)du \end{bmatrix}^T X \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)du \end{bmatrix} ds$$
$$+ \int_{t-h}^{t-h(t)}\begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)du \end{bmatrix}^T Y \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)du \end{bmatrix} ds,$$
$$V_3(x(t), r_t, t) = \frac{h}{2}\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds + \frac{h}{2}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds.$$

Let $\mathcal{L}$ be the weak infinitesimal generator [9] of the random process $\{x(t), r_t\}$ along system (6).

Case I. For any $t > 0$, $h(t) \in \left[0, \frac{h}{2}\right]$, we have

$$\begin{aligned}
\mathcal{L}V_1(x(t), r_t, t) &= 2x^T(t)P_i\dot{x}(t) + x^T(t)\sum_{j\in\mathbf{S}}\pi_{ij}P_jx(t) + 2\zeta^T(t)L\dot{\zeta}(t) \\
&= \xi_1^T(t)\left[Sym\left\{\varSigma^1 L\varGamma^T + e_1P_ie_5^T\right\} + e_1\sum_{j\in\mathbf{S}}\pi_{ij}P_je_1^T\right]\xi_1(t), \\
\mathcal{L}V_2(x(t), r_t, t) &= \begin{bmatrix} x(t) \\ x\left(t-\frac{h}{2}\right)\end{bmatrix}^T G \begin{bmatrix} x(t) \\ x\left(t-\frac{h}{2}\right)\end{bmatrix} - \begin{bmatrix} x\left(t-\frac{h}{2}\right) \\ x(t-h)\end{bmatrix}^T G \begin{bmatrix} x\left(t-\frac{h}{2}\right) \\ x(t-h)\end{bmatrix} \\
&\quad + \begin{bmatrix} x(t) \\ 0 \end{bmatrix}^T X \begin{bmatrix} x(t) \\ 0 \end{bmatrix} \\
&\quad + \left(1-\dot{h}(t)\right)\begin{bmatrix} x(t-h(t)) \\ \int_{t-h(t)}^t \dot{x}(u)du \end{bmatrix}^T [Y - X] \begin{bmatrix} x(t-h(t)) \\ \int_{t-h(t)}^t \dot{x}(u)du \end{bmatrix} \\
&\quad - \begin{bmatrix} x(t-h) \\ \int_{t-h}^t \dot{x}(u)du \end{bmatrix}^T Y \begin{bmatrix} x(t-h) \\ \int_{t-h}^t \dot{x}(u)du \end{bmatrix} \\
&\quad + 2\int_{t-h(t)}^t \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)du \end{bmatrix}^T X \begin{bmatrix} 0 \\ \dot{x}(t) \end{bmatrix} ds \\
&\quad + 2\int_{t-h}^{t-h(t)} \begin{bmatrix} x(s) \\ \int_s^t \dot{x}(u)du \end{bmatrix}^T Y \begin{bmatrix} 0 \\ \dot{x}(t) \end{bmatrix} ds, \\
\mathcal{L}V_3(x(t), r_t, t) &= \frac{h^2}{4}\dot{x}^T(t)(R_1 + R_2)\dot{x}(t) - \frac{h}{2}\int_{t-\frac{h}{2}}^t \dot{x}^T(s)R_1\dot{x}(s)ds \\
&\quad - \frac{h}{2}\int_{t-h}^{t-\frac{h}{2}} \dot{x}^T(s)R_2\dot{x}(s)ds.
\end{aligned}$$

It is easy to get the following inequalities from Wirtinger-based inequality [34] and Lemmas 2.1 and 2.2 that:

$$-\frac{h}{2}\int_{t-h}^{t-\frac{h}{2}} \dot{x}^T(s)R_2\dot{x}(s)ds < -\xi_1^T(t)E_3\tilde{R}_2E_3^T\xi_1(t); \tag{12}$$

$$\begin{aligned}
&-\frac{h}{2}\int_{t-\frac{h}{2}}^t \dot{x}^T(s)R_1\dot{x}(s)ds \\
&= -\frac{h}{2}\int_{t-h(t)}^t \dot{x}^T(s)R_1\dot{x}(s)ds - \frac{h}{2}\int_{t-\frac{h}{2}}^{t-h(t)} \dot{x}^T(s)R_1\dot{x}(s)ds \\
&\le -\frac{\frac{h}{2}}{h(t)}\xi_1^T(t)E_1\tilde{R}_1E_1^T\xi_1(t) - \frac{\frac{h}{2}}{\frac{h}{2} - h(t)}\xi_1^T(t)E_2\tilde{R}_1E_2^T\xi_1(t) \\
&\le -\xi_1^T(t)\begin{bmatrix} E_1^T \\ E_2^T \end{bmatrix}^T \begin{bmatrix} \tilde{R}_1 + (1-\alpha(t))T_1 & (1-\alpha(t))S_1 + \alpha(t)S_2 \\ * & \tilde{R}_1 + \alpha(t)T_2 \end{bmatrix} \begin{bmatrix} E_1^T \\ E_2^T \end{bmatrix} \xi_1(t),
\end{aligned} \tag{13}$$

where $T_1 = \tilde{R}_1 - E_1S_2\tilde{R}_1^{-1}S_2^TE_1^T$, $T_2 = \tilde{R}_1 - E_2S_1^T\tilde{R}_1^{-1}S_1E_2^T$ and $\alpha(t) = \frac{2h(t)}{h}$.

It follows from (4) that:

$$-2\omega_i^T(t)\omega_i(t) - 2\omega_i^T(t)K_i\left[M_ix(t) + N_ix(t-h(t))\right] > 0. \tag{14}$$

For any appropriately dimensioned matrices U_{i1}, U_{i2} and U_{i3} it is true that

$$0 = [x(t)U_{i1} + x(t-h(t))U_{i2} + \dot{x}(t)U_{i3}]\,[A_i x(t) + A_{1i}x(t-h(t)) + B_i w(t) - \dot{x}(t)]\,. \quad (15)$$

Then, it follows from Equalities (12)-(15) that

$$\begin{aligned}&\mathcal{L}V(x(t), r_t, t)\\ \leq &\sum_{k=1}^{3}\mathcal{L}V_k(x(t),t,i) - 2\omega_i^T(t)\omega_i(t) - 2\omega_i^T(t)K_i\,[M_i x(t) + N_i x(t-h(t))]\\ &+ 2[x(t)U_{i1} + x(t-h(t))U_{i2} + \dot{x}(t)U_{i3}][A_i x(t) + A_{1i}x(t-h(t)) + B_i w(t) - \dot{x}(t)]\\ \leq &\,\xi_1^T(t)\left[\Pi_i^1 - \Theta_{i1} + (1-\alpha(t))E_1 S_2\tilde{R}_1^{-1}S_2^T E_1^T + \alpha(t)E_2 S_1^T\tilde{R}_1^{-1}S_1 E_2^T\right]\xi_1(t).\end{aligned}$$

Therefore, LMIs (7)-(8) hold for $\dot{h}(t) \in \{\mu_1, \mu_2\}$, $i \in \mathbf{S}$, which together with **Schur complement equivalence** imply that $\mathcal{L}V(x(t), r_t, t) < 0$.

Case II. For any $t > 0$, $h(t) \in \left[\frac{h}{2}, h\right]$, by following a similar line of arguments as that in **Case I**, we have

$$\mathcal{L}V_1(x(t), r_t, t) = \xi_2^T(t)\left[Sym\left\{\Sigma^2 L\Gamma^T + e_1 P_i e_5^T\right\} + e_1\sum_{j\in\mathbf{S}}\pi_{ij}P_j e_1^T\right]\xi_2(t); \quad (16)$$

$$-\frac{h}{2}\int_{t-\frac{h}{2}}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds < -\xi_2^T(t)E_6\tilde{R}_1E_6^T\xi_2(t); \quad (17)$$

$$\begin{aligned}&-\frac{h}{2}\int_{t-h}^{t-\frac{h}{2}}\dot{x}^T(s)R_2\dot{x}(s)ds\\ = &-\frac{h}{2}\int_{t-h(t)}^{t-\frac{h}{2}}\dot{x}^T(s)R_2\dot{x}(s)ds - \frac{h}{2}\int_{t-h}^{t-h(t)}\dot{x}^T(s)R_2\dot{x}(s)ds\\ \leq &-\frac{\frac{h}{2}}{h(t)-\frac{h}{2}}\xi_2^T(t)E_4\tilde{R}_2E_4^T\xi_2(t) - \frac{\frac{h}{2}}{h-h(t)}\xi_2^T(t)E_5\tilde{R}_2E_5^T\xi_2(t)\\ \leq &-\xi_2^T(t)\begin{bmatrix}E_4^T\\ E_5^T\end{bmatrix}^T\begin{bmatrix}\tilde{R}_2 + (1-\beta(t))T_3 & (1-\beta(t))S_3 + \beta(t)S_4\\ * & \tilde{R}_2 + \beta(t)T_4\end{bmatrix}\begin{bmatrix}E_4^T\\ E_5^T\end{bmatrix}\xi_2(t),\end{aligned} \quad (18)$$

where $T_3 = \tilde{R}_2 - E_4 S_4\tilde{R}_2^{-1}S_4^T E_4^T$, $T_4 = \tilde{R}_2 - E_5 S_3^T\tilde{R}_2^{-1}S_3 E_5^T$ and $\beta(t) = \frac{2h(t)-h}{h}$.

Then, it follows from Equalities (14)-(18) that

$$\begin{aligned}&\mathcal{L}V(x(t), r_t, t)\\ \leq &\sum_{k=1}^{3}\mathcal{L}V_k(x(t),t,i) - 2\omega_i^T(t)\omega_i(t) - 2\omega_i^T(t)K_i\,[M_i x(t) + N_i x(t-h(t))]\\ &+ 2[x(t)U_{i1} + x(t-h(t))U_{i2} + \dot{x}(t)U_{i3}][A_i x(t) + A_{1i}x(t-h(t)) + B_i w(t) - \dot{x}(t)]\\ \leq &\,\xi_2^T(t)\left[\Pi_i^2 - \Theta_{i1} + (1-\beta(t))E_4 S_4\tilde{R}_2^{-1}S_4^T E_4^T + \beta(t)E_5 S_3^T\tilde{R}_2^{-1}S_3 E_5^T\right]\xi_2(t).\end{aligned}$$

Therefore, LMIs (9)-(10) hold for $\dot{h}(t) \in \{\mu_1, \mu_2\}$, $i \in \mathbf{S}$, which together with **Schur complement equivalence** imply that $\mathcal{L}V(x(t), r_t, t) < 0$.

Now, we are in a position to deal with the stochastic stability of system (6). In view of $\mathcal{L}V(x(t), r_t, t) < 0$, we conclude that there exist scalars $\theta > 0$ such that

$$\mathcal{L}V(x(t), r_t, t) \leq -\theta\|x(t)\|^2 < 0$$

for all $x(t) \neq 0$.

Therefore, it is readily seen from Dynkin's formula that

$$E\left\{V(x(t),r_t,t)\right\}-E\left\{V(x(h),h,r_h)\right\}\leq-\theta E\left\{\int_h^t x^T(s)x(s)ds\right\}$$

which can be used to deduce that

$$E\left\{\int_h^t x^T(s)x(s)ds\right\}\leq\theta^{-1}E\{V(x(h),h,r_h)\}.$$

Then, using a similar method given in [33], we have that there exists a scalar $\rho_1>0$ such that, for any $t\geq 0$,

$$\sup_{-h\leq t\leq h}\|x(t)\|^2\leq 4e^{4\rho_1 h}\sup_{-h\leq t\leq 0}\|\phi(t)\|^2.$$

Based on the stochastic Lyapunov-Krasovskii functional $V(x(t),r_t,t)$, it can be verified that there exists a scalar $\rho_2>0$ such that

$$V(x(h),h,r_h)\leq\rho_2\sup_{-h\leq t\leq h}\|x(t)\|^2\leq 4\rho_2e^{4\rho_1 h}\sup_{-h\leq t\leq 0}\|\phi(t)\|^2.$$

Hence,

$$\begin{aligned}E\left\{\int_0^t x^T(s)x(s)ds\right\}&=E\left\{\int_0^h x^T(s)x(s)ds+\int_h^t x^T(s)x(s)ds\right\}\\&\leq 4\left(h+\rho_2\theta^{-1}\right)e^{4\rho_1 h}E\left\{\sup_{-h\leq t\leq 0}\|\phi(t)\|^2\right\}.\end{aligned}$$

Taking limit as $t\to\infty$, it is clear that $\lim\limits_{t\to\infty}E\left\{\int_0^t x^T(s,\phi,r_0)x(s,\phi,r_0)ds\right\}<\infty$ which shows that system (6) is stochastically stable from Definition 2.1. This completes the proof.

Remark 3.1. *A novel stochastic stability criterion is obtained in Theorem 3.1 via the augmented LKF application. Especially, X- and Y-dependent integral terms are introduced in $V_2(x(t),r_t,t)$, where more delay-dependent state information is considered. On the other hand, two cases based on time delay intervals, $\left[0,\frac{h}{2}\right]$ and $\left[\frac{h}{2},h\right]$, are estimated respectively via piecewise analysis method application. An improved relaxed integral inequality technique is used to bound the derivative of the LKF (11). By combining the above methods, the stochastic stability criterion proposed in this paper is less conservative than some previous.*

Remark 3.2. *Recently, the FWM-based method has been used to reduce conservatism of a stability criterion for time-delay Lur'e system in* [19]. *However, many slack matrices bring heavy computation complexity. Therefore, Lemma 2.2 is used in this paper to overcome the difficulties without increasing conservatism. This can make the calculation convenient and fast. Moreover, Lemma 2.2 and the delay decomposition method are used simultaneously to further reduce the conservation of stability criteria for time-delay Lur'e system.*

Remark 3.3. *When the nonlinear function in systems* (1) *and* (6) *is time-invariant decentralized, the absolute stability problem for the class of Lur'e systems with time-varying delay has been considered in* [12, 13, 15, 23]. *In the following discussion of this paper, we consider the stochastic absolute stability problem for the class of time-delay Lur'e system with Markovian jumping parameters described by*

$$\begin{cases}\dot{x}(t)=[A(r_t)+\varDelta A(r_t)]x(t)+[A_1(r_t)+\varDelta A_1(r_t)]x(t-h(t))\\ \qquad\quad+[B(r_t)+\varDelta B(r_t)]f(z(t)),\\ z(t)=H(r_t)x(t),\\ x(s)=\psi(s),\quad s\in[-h,0],\end{cases}\tag{19}$$

where the nonlinear feedback part $f(z_i(t))$ belongs to the set of nonlinear functions with bounded sector constraints, that is, $f(z_i(t))$ satisfies the following condition [10, 12, 13, 36]*:*

$$\begin{aligned}\phi = : \big\{ & f : \mathbb{R}^m \mapsto \mathbb{R}^m : f(z_i) = [f_1(z_{i1})\ f_2(z_{i2})\ \cdots\ f_m(z_{im})]^T, \\ & \alpha_{ik} z_{ik}^2 \le z_{ik} f_k(z_{ik}) \le \beta_{ik} z_{ik}^2 \text{ for } z_{ik} \ne 0,\ \alpha_{ik}, \beta_{ik} \ge 0,\ k = 1, 2, \ldots, m; i \in \mathbf{S} \big\}, \end{aligned} \tag{20}$$

where α_{ik} and β_{ik} are given constants. Here, for simplicity, let us define $J_{i1} = \mathrm{diag}\{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{im}\}$ and $J_{i2} = \mathrm{diag}\{\beta_{i1}, \beta_{i2}, \ldots, \beta_{im}\}$.

For a normal form of Lur'e system (19) with Markovian jumping parameters, we have the following delay-dependent stochastic stability results.

Theorem 3.2. *The normal form of Lur'e system* (19) *satisfying conditions* (4) *and* (20) *is stochastically absolutely stable for given values of h, μ_1, μ_2 and diagonal matrices J_{i1}, J_{i2} if there exist real positive definite matrices $L_{3n\times 3n}$, P_i, $X_{2n\times 2n}$, $Y_{2n\times 2n}$, R_j and any matrices $\Delta_{ij} = \mathrm{diag}\{\delta_{ij1}, \delta_{ij2}, \ldots, \delta_{ijm}\} \ge 0$, $\Lambda_i = \mathrm{diag}\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{im}\} \ge 0$, S_k, U_{ir} ($j = 1, 2$; $k = 1, 2, 3, 4$; $r = 1, 2, 3$) with appropriate dimensions such that the following* ***LMI****s hold for $\dot{h}(t) \in \{\mu_1, \mu_2\}$, $i \in \mathbf{S}$:*

$$\begin{bmatrix} \Pi_i^1\big|_{h(t)=0} + \Theta_{i2} & E_1 S_2 \\ * & -\tilde{R}_1 \end{bmatrix} < 0, \tag{21}$$

$$\begin{bmatrix} \Pi_i^1\big|_{h(t)=\frac{h}{2}} + \Theta_{i2} & E_2 S_1^T \\ * & -\tilde{R}_1 \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} \Pi_i^2\big|_{h(t)=\frac{h}{2}} + \Theta_{i2} & E_4 S_4 \\ * & -\tilde{R}_2 \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} \Pi_i^2\big|_{h(t)=h} + \Theta_{i2} & E_5 S_3^T \\ * & -\tilde{R}_2 \end{bmatrix} < 0, \tag{24}$$

where $\Theta_{i2} = Sym\big\{e_1 \left[H_i^T(J_{i2}\Delta_{i1} - J_{i1}\Delta_{i2})H_i\right] e_5^T + e_5 H_i^T(\Delta_{i2} - \Delta_{i1})e_6^T - e_1 H_i^T[J_{i1}\Lambda_i J_{i2} + J_{i2}\Lambda_i J_{i1}]H_i e_1^T + e_1\left[H_i^T(J_{i2} + J_{i1})\Lambda_i\right] e_6^T - e_6 \Lambda_i e_6^T\big\}$.

Proof: Construct an **LKF** candidate as

$$\begin{aligned}&\bar{V}(x(t), r_t, t) \\ &= \sum_{k=1}^{3} V_k(x(t), r_t, t) + 2\sum_{k=1}^{m} \int_0^{z_{ik}(t)} [(f_{ik}(s) - \alpha_{ik}s)\delta_{i2k} + (\beta_{ik}s - f_{ik}(s))\delta_{i1k}]ds.\end{aligned} \tag{25}$$

The weak infinitesimal of $2\sum_{k=1}^m \int_0^{z_{ik}(t)}[(f_{ik}(s) - \alpha_{ik}s)\delta_{i2k} + (\beta_{ik}s - f_{ik}(s))\delta_{i1k}]ds$ is as follows:

$$2[f(z_i(t)) - J_{i1}H_i x(t)]^T \Delta_{i2} H_i \dot{x}(t) + 2[J_{i2}H_i x(t) - f(z_i(t))]^T \Delta_{i1} H_i \dot{x}(t). \tag{26}$$

According to the sector condition (20), for any $\Lambda_i = \mathrm{diag}\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{im}\} \ge 0$, it follows from (20) that

$$0 \le 2[f(z_i(t)) - J_{i1}H_i x(t)]^T \Lambda_i [J_{i2}H_i x(t) - f(z_i(t))]. \tag{27}$$

Only if we let the following matrices, the proof can be completed by following a similar line of arguments as that in Theorem 3.1.

$$\bar{\xi}_1^T(t) = \Bigg[x^T(t)\ \ x^T(t - h(t))\ \ x^T\left(t - \frac{h}{2}\right)\ \ x^T(t - h)\ \ \dot{x}^T(t)\ \ f^T(z(t))$$

$$\int_{t-h(t)}^{t} \frac{x^T(s)ds}{h(t)} \quad \int_{t-\frac{h}{2}}^{t-h(t)} \frac{x^T(s)ds}{\frac{h}{2}-h(t)} \quad \frac{h}{2}\int_{t-h}^{t-\frac{h}{2}} x^T(s)ds \Bigg],$$

$$\bar{\xi}_2^T(t) = \Bigg[x^T(t) \;\; x^T\left(t-\frac{h}{2}\right) \;\; x^T(t-h(t)) \;\; x^T(t-h) \;\; \dot{x}^T(t) \;\; f^T(z(t))$$

$$\int_{t-\frac{h}{2}}^{t-h(t)} \frac{x^T(s)ds}{h(t)-\frac{h}{2}} \quad \int_{t-h}^{t-h(t)} \frac{x^T(s)ds}{h-h(t)} \quad \frac{h}{2}\int_{t-\frac{h}{2}}^{t} x^T(s)ds \Bigg].$$

3.2. **Robust stochastic stability criteria for uncertain form.** In this subsection, we extend the obtained absolute stability conditions above to robust absolute stability problem for uncertain Lur'e systems (1) and (19) with time-varying parameter uncertainties satisfying (3)-(5) and (20).

Theorem 3.3. *The uncertain Lur'e system* (1) *satisfying conditions (2)-(4) is stochastically robustly absolutely stable for given values of $h \geq 0$, μ_1 and μ_2, if there exist real positive definite matrices $L_{3n\times 3n}$, P_i, $X_{2n\times 2n}$, $Y_{2n\times 2n}$, $G_{2n\times 2n}$, R_j, any matrices S_k, U_{ir} ($j = 1, 2$; $k = 1, 2, 3, 4$; $r = 1, 2, 3$) with appropriate dimensions and a scalar $\varepsilon_i > 0$ such that the following* ***LMIs*** *hold for $\dot{h}(t) \in \{\mu_1, \mu_2\}$, $i \in \mathbf{S}$:*

$$\begin{bmatrix} \Pi_i^1\big|_{h(t)=0} - \Theta_{i1} + \varepsilon_i \Phi_{21}^T \Phi_{21} & E_1 S_2 & \Phi_{11}^T \\ * & -\tilde{R}_1 & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0, \tag{28}$$

$$\begin{bmatrix} \Pi_i^1\big|_{h(t)=\frac{h}{2}} - \Theta_{i1} + \varepsilon_i \Phi_{21}^T \Phi_{21} & E_2 S_1^T & \Phi_{11}^T \\ * & -\tilde{R}_1 & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0, \tag{29}$$

$$\begin{bmatrix} \Pi_i^2\big|_{h(t)=\frac{h}{2}} - \Theta_{i1} + \varepsilon_i \Phi_{22}^T \Phi_{22} & E_4 S_4 & \Phi_{12}^T \\ * & -\tilde{R}_2 & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0, \tag{30}$$

$$\begin{bmatrix} \Pi_i^2\big|_{h(t)=h} - \Theta_{i1} + \varepsilon_i \Phi_{22}^T \Phi_{22} & E_5 S_3^T & \Phi_{12}^T \\ * & -\tilde{R}_2 & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0, \tag{31}$$

where

$$\Phi_{11} = [e_1 U_1 D + e_2 U_2 D + e_5 U_3 D]^T, \quad \Phi_{21} = \left[e_1 E_a^T + e_2 E_{a1}^T + e_6 E_b^T\right]^T,$$
$$\Phi_{12} = [e_1 U_1 D + e_3 U_2 D + e_5 U_3 D]^T, \quad \Phi_{22} = \left[e_1 E_a^T + e_3 E_{a1}^T + e_6 E_b^T\right]^T.$$

Proof: We only need to replace $\bar{A}_i$, $\bar{A}_{1i}$ and $\bar{B}_i$ in **LMI**s (7)-(10) with $\bar{A}_i + D_i F_i(t) E_{ai}$, $\bar{A}_{1i} + D_i F_i(t) E_{a1i}$, $\bar{B}_i + D_i F_i(t) E_{bi}$, respectively. And it follows from Lemma 2.3 that if and only if there exist positive scalars $\varepsilon_i > 0$, such that the following matrix inequalities hold:

$$\Pi_i^1|_{h(t)=0} - \Theta_{i1} + E_1 S_2 \tilde{R}_1^{-1} S_2^T E_1^T + \varepsilon_i^{-1} \Phi_{11} \Phi_{11}^T + \varepsilon_i \Phi_{21}^T \Phi_{21} < 0, \tag{32}$$
$$\Pi_i^1|_{h(t)=\frac{h}{2}} - \Theta_{i1} + E_2 S_1^T \tilde{R}_1^{-1} S_1 E_2^T + \varepsilon_i^{-1} \Phi_{11} \Phi_{11}^T + \varepsilon_i \Phi_{21}^T \Phi_{21} < 0, \tag{33}$$
$$\Pi_i^2|_{h(t)=\frac{h}{2}} - \Theta_{i1} + E_4 S_4 \tilde{R}_2^{-1} S_4^T E_4^T + \varepsilon_i^{-1} \Phi_{12} \Phi_{12}^T + \varepsilon_i \Phi_{22}^T \Phi_{22} < 0, \tag{34}$$
$$\Pi_i^2|_{h(t)=h} - \Theta_{i1} + E_5 S_3^T \tilde{R}_2^{-1} S_3 E_5^T + \varepsilon_i^{-1} \Phi_{12} \Phi_{12}^T + \varepsilon_i \Phi_{22}^T \Phi_{22} < 0, \tag{35}$$

which are equivalent to **LMI**s in (28)-(31), respectively, by **Schur complement equivalence**. From Definition 2.1, this completes the proof.

When the nonlinear function in systems (1) and (6) is time-invariant decentralized, the following corollary is obtained.

Corollary 3.1. *The uncertain system (19) satisfying conditions (2), (3) and (20) is stochastically robustly absolutely stable for given values of* $h \geq 0$*,* μ_1*,* μ_2*, diagonal matrices* J_{i1} *and* J_{i2} *if there exist real positive definite matrices* $L_{3n\times 3n}$*,* P_i*,* $X_{2n\times 2n}$*,* $Y_{2n\times 2n}$*,* $G_{2n\times 2n}$*,* R_j*, any matrices* $\varDelta_{ij} = \mathrm{diag}\{\delta_{ij1}, \delta_{ij2}, \ldots, \delta_{ijm}\} \geq 0$*,* $\varLambda_i = \mathrm{diag}\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{im}\} \geq 0$*,* S_k*,* U_{ir} *(*$j = 1, 2$*;* $k = 1, 2, 3, 4$*;* $r = 1, 2, 3$*) with appropriate dimensions and a scalar* $\varepsilon_i > 0$ *such that the following LMIs hold for* $\dot{h}(t) \in \{\mu_1, \mu_2\}$*,* $i \in \mathbf{S}$*:*

$$\begin{bmatrix} \varPi_i^1\big|_{h(t)=0} + \varTheta_{i2} + \varepsilon_i \varPhi_{21}^T \varPhi_{21} & E_1 S_2 & \varPhi_{11}^T \\ * & -\tilde{R}_1 & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0, \tag{36}$$

$$\begin{bmatrix} \varPi_i^1\big|_{h(t)=\frac{h}{2}} + \varTheta_{i2} + \varepsilon_i \varPhi_{21}^T \varPhi_{21} & E_2 S_1^T & \varPhi_{11}^T \\ * & -\tilde{R}_1 & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0, \tag{37}$$

$$\begin{bmatrix} \varPi_i^2\big|_{h(t)=\frac{h}{2}} + \varTheta_{i2} + \varepsilon_i \varPhi_{22}^T \varPhi_{22} & E_4 S_4 & \varPhi_{12}^T \\ * & -\tilde{R}_2 & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0, \tag{38}$$

$$\begin{bmatrix} \varPi_i^2\big|_{h(t)=h} + \varTheta_{i2} + \varepsilon_i \varPhi_{22}^T \varPhi_{22} & E_5 S_3^T & \varPhi_{12}^T \\ * & -\tilde{R}_2 & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0. \tag{39}$$

Remark 3.4. *In the above results of Theorems 3.1-3.3 and Corollary 3.1, the stochastic and robustly stochastic stability for uncertain continuous-time Lur'e systems with time-varying delays is given based on LMI. Both time-invariant and time-varying nonlinearities are considered. By solving for the corresponding LMI, the maximal admissible delay upper bounds (MADUBs) to guarantee the stochastic stability of the Lur'e system can be calculated, which can be made an intuitive comparison with the previous ones. The detailed illustrative examples are shown in next section.*

4. **Numerical Example.**

In this section, we give an example to show the effectiveness of the criteria proposed in this paper. Moreover, the conservatism of the criteria is checked based on the calculated MADUBs. And the index of the number of decision variables (NoV) is applied to showing the complexity of criteria.

Remark 4.1. *In order to illustrate the effectiveness of the stochastic stability criteria proposed in this paper, some numerical examples with one or two modes, i.e.,* $\mathbf{S} = \{1\}$ *or* $\mathbf{S} = \{1, 2\}$*, commonly used in much recent literature, such as* [8, 9, 10, 12, 13, 15, 19, 31]*, are chosen. Therefore, by comparing our results with others in the recent literature, the advancement of our stochastic stability criteria is shown conveniently based on the lager MADUBs than some recent literature.*

Example 4.1. *Consider the Markov jump Lur'e system* (1) *with two modes, i.e.,* $\mathbf{S} = \{1, 2\}$*, the system parameters are described as:*

$$A_1 = \begin{bmatrix} -2 & 0.5 \\ 0 & -1 \end{bmatrix}, \ A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \ A_{11} = \begin{bmatrix} 1 & 0.4 \\ 0.4 & -1 \end{bmatrix}, \ A_{12} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.5 \\ -0.75 \end{bmatrix},\ B_2 = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix},\ M_1 = \begin{bmatrix} 0.2 & 0.6 \end{bmatrix},\ M_2 = \begin{bmatrix} 0.3 & 0.1 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0.3 & 0.2 \end{bmatrix},\ N_2 = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix},\ D_2 = D_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ E_{1b} = E_{2b} = 0,$$

$$E_{1a} = E_{2a} = E_{11} = E_{21} = I,\ K_{11} = K_{21} = 0.2,\ K_{12} = K_{22} = 0.5.$$

We assume the following transition probability matrix $\Pi = \begin{bmatrix} -0.5 & 0.5 \\ 0.3 & -0.3 \end{bmatrix}$.

We now apply the proposed approach to estimating the upper bound h such that system (1) given in this example is stochastically absolutely stable for any delay satisfying condition (4). In Table 1, the upper bound h for different μ_1, μ_2 are listed by using Theorem 3.1. Figure 2 displays the responses of system state $x(t)$ for $h(t) = 3$, $\varphi(z) = (0.35+0.15\sin(t))z(t)$ associate with the jump mode r_t as shown in Figure 1. It is possible to see that, for this realization, the trajectory converges to the origin, as expected.

TABLE 1. **MADUB**s h for different μ_1 and μ_2 (Example 4.1)

$\mu_1 \setminus \mu_2$	Methods	0.0	0.3	0.6	0.9
0.0	[31]	2.0025	1.9588	0.9997	0.9510
	Theorem 3.1	2.5777	2.1805	1.1297	1.0771
−0.3	[31]	2.0023	2.1784	0.9957	0.8911
	Theorem 3.1	2.5777	2.3499	1.1227	1.0589
−0.6	[31]	3.5589	2.2569	1.0009	0.8822
	Theorem 3.1	3.7744	2.3997	1.1217	1.0554
−0.9	[31]	3.6300	2.3741	1.0025	0.8811
	Theorem 3.1	3.9259	2.4287	1.1219	1.0551

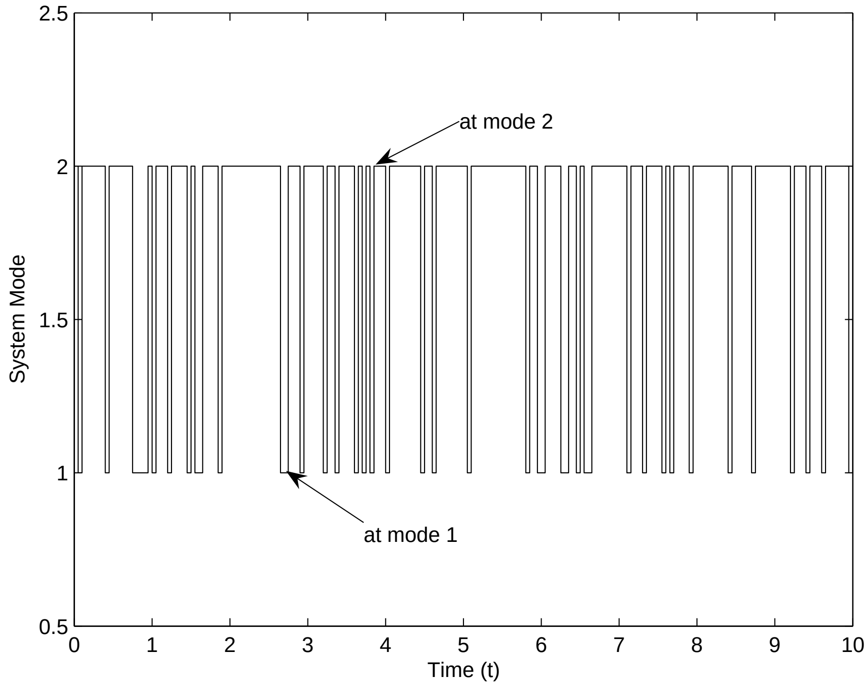

FIGURE 1. Jump mode r_t

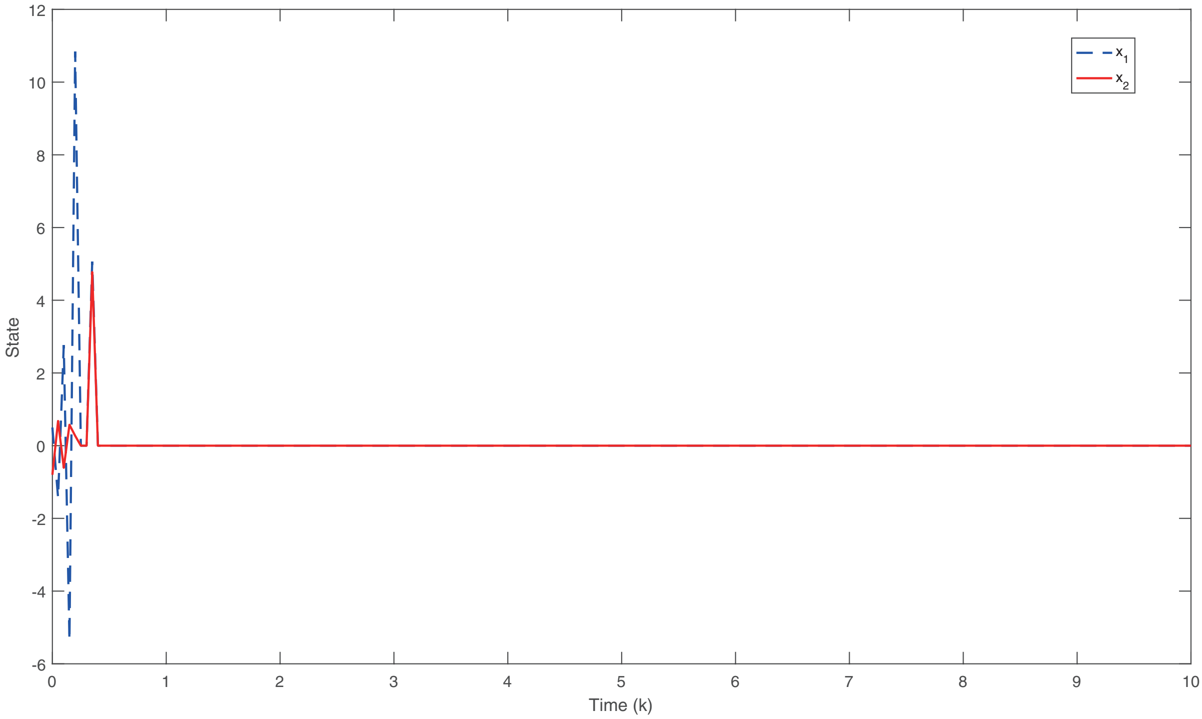

FIGURE 2. The state responses of system (1) in Example 4.1

Example 4.2. [8, 9]. *Consider the uncertain Lur'e system* (1) *without Markovian jumping parameters, that is* $\mathbf{S} = \{1\}$. *The system matrices are given by:*

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \ A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix},$$
$$M = \begin{bmatrix} 0.3 & 0.1 \end{bmatrix}, \ N = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \ K_1 = 0.2, \ K_2 = 0.5,$$
$$D = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ E_a = E_{a1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ E_b = 0.$$

In Table 2, the maximum allowed delay bounds MADUBs h of the Lur'e system (1) for different μ by using our results and methods in [8, 9, 19] are compared. From the tables, it is found that our results give better upper bounds on the delay h for robustly absolute stability of the Lur'e system (1) than some recent results. Here $-\mu_1 = \mu_2 = \mu$.

The NoV of Theorem 3.3 is $29.5n^2 + 4.5n$, which is $6.5n^2$ more than that of Theorem 2 [19] due to the piecewise analysis method. However, the MADUBs of Theorem 3.3 are larger than those of Theorem 2 [19], which means that the former further improve the results.

To confirm the obtained results ($h = 4.3929$), the simulation result is shown in Figure 3 which shows that the state responses of the Lur'e system (1) with $\varphi(t, z(t)) = 0.3\tanh(z(t))$ and $h(t) = 4.3929$ converge to zero under the random initial state.

TABLE 2. **MADUBs** h for different μ (Example 4.2)

Methods\\μ	0.0	0.3	0.6	0.9	NoVs
[8]	3.3057	2.0787	1.4195	0.9228	10
[9]	3.3057	2.2262	1.7409	1.4682	24
[19]	4.3332	2.6873	2.2021	1.9897	105
Theorem 3.3	4.3929	3.0001	2.4343	2.0248	127

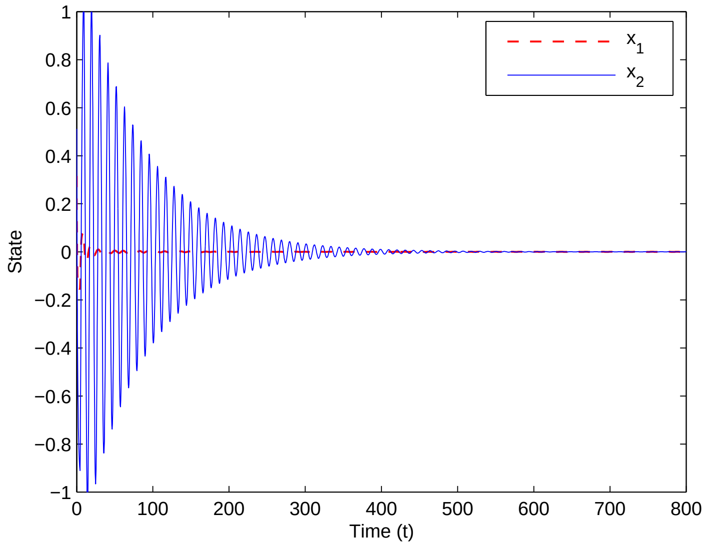

FIGURE 3. The state responses of system (1) in Example 4.2

Example 4.3. [10, 12, 13, 15]. *Consider the uncertain Lur'e system* (19) *without Markovian jumping parameters, that is* $\mathbf{S} = \{1\}$. *The system parameters are described as:*

$$A = \begin{bmatrix} -1.2 & 0 \\ 0.8 & -1 \end{bmatrix}, \; A_1 = \begin{bmatrix} -1 & 0.6 \\ -0.6 & -1 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, \; E_a = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$E_b = E_{a1} = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.03 \end{bmatrix}, \; K_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

In Table 3, the MADUBs h of the Lur'e system (19) for $\mu_1 = 0$ and different μ_2 by using Corollary 3.1 in this paper and methods in [10, 12, 13, 15] are compared. From the table, it is found that Corollary 3.1 gives better upper bounds on the delay h for robustly absolute stability of the Lur'e system (19) than some recent results. To confirm the obtained results ($h = 4.552$), the simulation result is shown in Figure 4 which shows that the state responses of the Lur'e system (19) with $f(z(t)) = \begin{bmatrix} |\sin(z_1(t))| \\ 3|\cos(z_2(t))| \end{bmatrix}$ and $h(t) = 4.552$ converge to zero under the random initial state.

TABLE 3. **MADUB**s with fixed $\mu_1 = 0$ (Example 4.3)

μ_2	Methods\\θ	0	0.2	0.4	0.6	0.8	1	NoVs
0	[10]	1.113	1.062	1.014	0.967	0.921	0.887	60
	[12]	3.325	3.128	2.849	2.780	2.651	2.522	153
	[13]	3.355	3.172	2.912	2.876	2.734	2.614	287
	[15]	4.372	3.840	3.456	3.160	2.921	2.723	86
	Corollary 3.1	4.552	3.985	3.578	3.264	3.011	2.800	133
0.1	[10]	1.026	0.984	0.940	0.898	0.857	0.818	60
	[12]	3.160	2.899	2.840	2.702	2.575	2.460	153
	[13]	3.224	3.046	2.900	2.804	2.603	2.554	287
	[15]	3.616	3.295	3.039	2.828	2.649	2.491	86
	Corollary 3.1	4.520	3.975	3.571	3.256	3.001	2.788	133

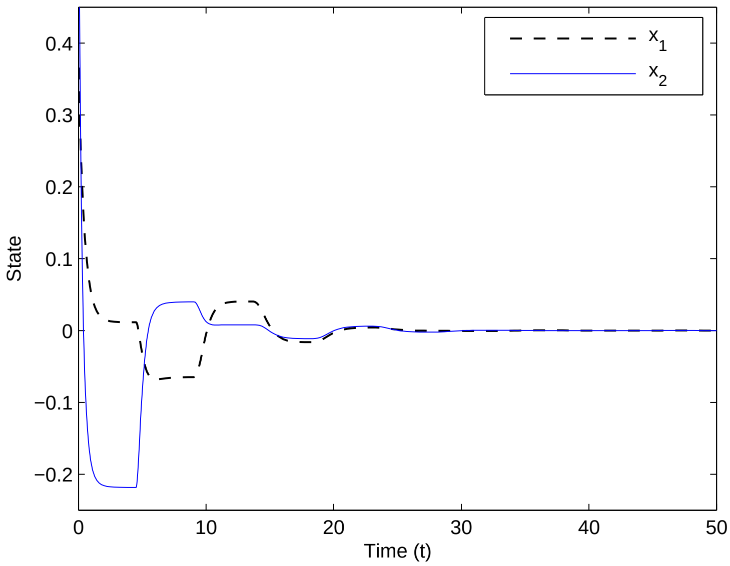

FIGURE 4. The state responses of system (19) under the conditions given in Example 4.3

Remark 4.2. *According to the above examples, it can be seen intuitively that the stochastic stability criteria proposed in this paper are less conservative than those proposed in* [8, 9, 10, 12, 13, 15, 19, 31]. *The maximum allowable delay upper bounds calculated are larger with guaranteeing stochastic stability of the Lur'e systems. The contribution in reducing conservation of the proposed stability criteria relies on the meanwhile using of the delay fractionating method, piecewise analysis method and the improved Wirtinger-based inequality, which matches the description in the Introduction and Remarks 3.1 and 3.2.*

5. **Conclusion.** In this paper, some new absolute and robustly absolute stochastic stability criteria are proposed for the uncertain Markov Lur'e systems with time-varying delays and sector bounded nonlinearities via a modified LKF. Some effective techniques, such as improved relaxed integral inequality method, and piecewise analysis method, are applied to reducing the conservation of the proposed criteria from some existing results. Finally, some numerical examples are used to illustrate the effectiveness of the proposed approaches.

In this paper, we only consider the case of zero lower bound of time-varying delay. In many cases, the lower bound of time-varying delay is not zero, while the stochastic stability criteria of this paper are conservative. On the other hand, the novel stability criteria can also be generalized to some other time-delayed control systems, for example, time-delayed neural networks, and time-delayed neutral-type systems. However, there are still some distances from practical application because of the complexity of the control theory, which can be something that we will continue to study in the future.

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