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INCREMENTAL ATTRIBUTE REDUCTION ALGORITHM IN PROBABILISTIC ROUGH SETS UNDER THE VARIATION OF ATTRIBUTES

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Abstract. *First, a heuristic attribute reduction method based on probabilistic rough sets (PRSs) is introduced, and the incremental calculation method is used to improve the calculation methods of probabilistic approximation accuracy (PAA) and modified probabilistic approximation accuracy (MPAA), which improves the efficiency of the attribute reduction algorithm. Based on PAA and MPAA, a fast method for calculating attribute core and minimal attribute reduction is presented in PRSs under the variation of attributes. Then, we focus on the new strategy of dynamically updating the attribute core and calculating the minimum attribute reduction when the attribute set changes in PRS. Based on PRS, two incremental calculation attribute cores and one incremental calculation algorithm for minimum attribute reduction when adding and deleting attributes are proposed respectively. Finally, the feasibility and effectiveness of the proposed method are illustrated by an example. Compared with the non-incremental algorithm, the incremental algorithm has lower time complexity.*

Keywords: Probabilistic rough set, Attribute reduction, Incremental learning, Updating approximations

1. **Introduction.** Rough set theory (RS) is an effective mathematical tool to deal with fuzzy and uncertain information. It has been successfully applied in many research fields such as intelligent data analysis, knowledge discovery, data mining and machine learning [1, 2, 3, 4, 5]. Attribute reduction is one of the most important research directions in RS, which can avoid overfitting of data and improve the performance of classification algorithms, etc. [6, 7, 8, 9].

The classification of Pawlak RS model must be completely correct or definite. Therefore, it cannot effectively deal with noisy data and cannot mine its potentially useful information. In order to solve these problems, many researchers have proposed a lot of extended PRS models, such as variable precision rough set (VPRS), 0.5 PRS model and decision rough set (DRS) [10, 11, 12]. Yao and Zhao [13] presented a method of attribute reduction in DRS. Chen et al. [14] defined an attribute core of VPRS, and proposed a heuristic reduction algorithm based on attribute core to solve the minimal reduction problem. To minimize the cost of decision making, Jia et al. [15] proposed genetic algorithm, heuristic algorithm and simulated annealing algorithm to optimize the attribute reduction in RS. Jia et al. [16] designed an adaptive learning algorithm to get the cost function and threshold of RS, and proposed a method of computing attribute reduction

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by particle swarm optimization. Because the influence of the change of attribute set in the PRS on the classification area is no longer monotonous, the attribute reduction method of Pawlak RS is no longer applicable in the PRS. Many researchers have studied the problem deeply. Mandal and Ranadive [17] introduced general framework of multi-granulation bipolar-valued fuzzy (MGBVF) PRSs in MGBVF probabilistic approximation space over two universes. Ma and Yao [18] studied three types of class-specific attribute reduction in PRS and their relationships. Lang et al. [19] designed incremental algorithms for attribute reduction of dynamic covering decision systems (DSs) in terms of attribute adding and deleting.

For the classic RS, the classification of the positive domain decreases monotonically when the attribute is deleted. However, for the PRS, the probability domain is introduced because of the definition of the positive domain, so that when the attribute is deleted, the positive domain may become larger, smaller, or unchanged. Therefore, some new phenomena appear in the attribute reduction of PRSs, that is, the non-monotonicity of the positive domain. In fact, the attribute reduction of PRSs has a very large uncertainty, which has surpassed the basic framework of attribute reduction in classic RSs, thus bringing new difficulties and challenge. Wang et al. [20] proposed a monotonic uncertainty measure based on PRS attribute reduction. However, when a single attribute is deleted or added, the method needs to compute the PAA and the MPAA frequently. Such frequent iterative calculation will make the algorithm have higher time complexity and space complexity, making the algorithm inefficient. In addition, under the probability rough set, when the attribute value varies, the attribute core is updated incrementally, and the existing calculation results can be used to improve the efficiency of the algorithm. Incremental learning method is a kind of technology which makes full use of the historical training results in the current sample training, thus significantly reducing the time of follow-up training, which can greatly improve the efficiency of classification. Developing an effective knowledge acquisition method based on incremental learning has become a hot issue in RS research. For example, Chan [21] proposed the definition of the boundary set in the Pawlak RS, and used the update computation of the boundary set to realize the incremental computation of the Pawlak rough approximation set. Li [22] proposed a method to compute the approximate set by using the boundary set in the characteristic relation rough set. Chen et al. [23] presented an incremental algorithm for approximation sets when the attribute values are coarse or refined. Luo et al. [24] proposed a dynamic updating method for the approximation set in the set-valued ordered DS. Zhang et al. [25] presented an incremental algorithm for the approximation set when the object changes in the composite rough set.

To solve the problem of attribute reduction in PRS model, we propose a method for incrementally calculating attribute core and minimum attribute reduction. The research contributions of this paper mainly include the following.

1) The attribute reduction algorithm based on PRSs proposed by Wang et al. [20] is improved, the incremental calculation methods for PAA and MPAA are proposed, and theoretical analysis and proof are carried out. It solves the problem of high algorithm time complexity caused by the frequent calculation of PAA and MPAA in Wang et al.'s method, thereby improving the efficiency of the algorithm.

2) On the basis of 1), introducing the method of incrementally updating the upper and lower approximation sets in the PRS into the attribute reduction method, we further propose an incremental attribute reduction method when the attribute is adding or deleting. And conduct theoretical analysis and proof. It provides a new idea for the attribute reduction method of PRS.

3) Under the PRS model, an algorithm for incrementally calculating attribute cores is proposed when the attribute set varies, and on this basis, an algorithm for calculating the minimum attribute reduction is further proposed. Introduce the idea of incremental update into the calculation algorithm of attribute core, and improve the efficiency of the algorithm.

The rest of the paper is arranged as follows. In Section 2, the basic concepts of Pawlak RS and PRS are first introduced, then the definitions of PAA and MPAA are introduced, and finally the definition of attribute core is given. In Section 3, first, the relevant theorems and concepts of incremental calculation of PAA and MPAA are given. Then, two fast incremental algorithms for calculating the attribute cores and one algorithm for incrementally calculating the minimum attribute reduction are proposed under the PRS model when adding attributes and deleting attributes. An example is given to illustrate the methods of incremental attribute reduction when the attribute set is adding or deleting in Section 4. Section 5 compares and analyzes the non-incremental attribute reduction algorithm with the incremental attribute reduction algorithm proposed in this paper on the running time efficiency. Finally, in Section 6, the work of the paper is summarized and prospected.

2. **Rough Set Models.**

In this section, the basic concepts of RS and PRS are first introduced [10, 12, 27]. Then, the definition and calculation method of PAA, MPAA and attribute core are introduced [20, 26].

2.1. **Pawlak rough set model.**

The approximate space $AS = (U, R)$ is defined by a binary relationship R and a universe U. Suppose U is a finite and non-empty set, and R is the binary relationship of U. When R is an equivalent relationship, $AS = (U, R)$ is defined as the Pawlak approximate space. Let $Y \subseteq U$, the lower approximation and upper approximation are defined as follows [27]:

$\underline{AS}_R(Y) = \{y \in U | [y]_R \subseteq Y\}$; $\overline{AS}_R(Y) = \{y \in U | [y]_R \cap Y \neq \emptyset\}$.

Obviously, the upper and lower approximations divide the universe U into a positive region $POS_R(Y)$, a boundary region $BND_R(Y)$ and a negative region $NEG_R(Y)$, and these three regions are disjoint from each other. They are defined as follows:

$POS_R(Y) = \underline{AS}_R(Y)$; $BND_R(Y) = \overline{AS}_R(Y) - \underline{AS}_R(Y)$; $NEG_R(Y) = U - \overline{AS}_R(Y)$.

2.2. **PRS model.**

$I = (U, A, V, f)$ is an information system (IS), where $U = \{y_1, \ldots, y_n\}$ is a collection of finite and non-empty objects, A is a non-empty and finite attribute set, $V = \bigcup_{a \in A} V_a$, where V_a is called the domain of a, which is a non-empty value set of attribute a, f: $U \times A \to V$ is a function that maps the objects in U to a value in V_a, so that $\forall a \in A$, $y \in U$, $f(y, a) \in V_a$ established. For simplicity, $I = (U, A, V, f)$ is abbreviated as $I = (U, A)$.

Definition 2.1. [26] *Given an IS $I = (U, A)$, $Y \subseteq U$, $R \subseteq A$ and $0 \leq \beta < \alpha \leq 1$, relative to R, (α, β)-upper approximation and (α, β)-lower approximation of Y are defined as follows, respectively.*

$\underline{AS}_R^{(\alpha,\beta)}(Y) = \{y \in U | P_r(Y | [y]_R) \geq \alpha\}$, $\overline{AS}_R^{(\alpha,\beta)}(Y) = \{y \in U | P_r(Y | [y]_R) > \beta\}$.

And, the (α, β)-probabilistic positive region $POS_R^{(\alpha,\beta)}(Y)$, boundary region $BND_R^{(\alpha,\beta)}(Y)$ and negative region $NEG_R^{(\alpha,\beta)}(Y)$ are defined as follows:

$POS_R^{(\alpha,\beta)}(Y) = \{y \in U | P_r(Y | [y]_R) \geq \alpha\}$, $BND_R^{(\alpha,\beta)}(Y) = \{y \in U | \beta < P_r(Y | [y]_R) < \alpha\}$, $NEG_R^{(\alpha,\beta)}(Y) = \{y \in U | P_r(Y | [y]_R) \leq \beta\}$, *where* $P_r(Y | [y]_R) = \frac{|[y]_R \cap Y|}{|[y]_R|}$.

Definition 2.2. [20] *Given a DS* $D = (U, C \cup D)$*,* $0 \leq \beta < \alpha \leq 1$*,* C *and* D *are a conditional attribute set and a decision attribute set,* $R \subseteq C$*,* $U/D = \{T_1, T_2, \ldots, T_M\}$ *is a classification of* U*. The probabilistic upper approximation* $\overline{AS}_R^{(\alpha,\beta)}(U/D)$ *and probabilistic lower approximation* $\underline{AS}_R^{(\alpha,\beta)}(U/D)$ *of* U/D *are defined as follows.*

$$\underline{AS}_R^{(\alpha,\beta)}(U/D) = \underline{AS}_R^{(\alpha,\beta)}(T_1) \cup \underline{AS}_R^{(\alpha,\beta)}(T_2) \cup \cdots \cup \underline{AS}_R^{(\alpha,\beta)}(T_M), \tag{1}$$

and

$$\overline{AS}_R^{(\alpha,\beta)}(U/D) = \overline{AS}_R^{(\alpha,\beta)}(T_1) \cup \overline{AS}_R^{(\alpha,\beta)}(T_2) \cup \cdots \cup \overline{AS}_R^{(\alpha,\beta)}(T_M). \tag{2}$$

Definition 2.3. [20] *Given a DS* $D = (U, C \cup D)$*,* $R \subseteq C$*,* $0 \leq \beta < \alpha \leq 1$*,* $U/D = \{T_1, T_2, \ldots, T_M\}$ *is a classification of* U*. The PAA of* U/D *is defined as follows.*

$$\Gamma_R^{(\alpha,\beta)}(U/D) = \frac{\sum_{T_j \in U/D} \left| \underline{AS}_R \left(\underline{AS}_C^{(\alpha,\beta)}(T_j) \right) \right|}{\sum_{T_j \in U/D} \left| \overline{AS}_R \left(\overline{AS}_C^{(\alpha,\beta)}(T_j) \right) \right|}, \tag{3}$$

where $j \in \{1, 2, \ldots, M\}$*.*

Definition 2.4. [28] *Suppose* $\pi = \{Y_1, Y_2, \ldots, Y_K\}$ *is a partition of* U *and* m*:* $2^U \to \Re$ *is the granularity measure of subsets of* U*. The expected granularity of* π *blocks is defined as follows:*

$$EG_m(\pi) = E_{P_\pi}(m(\cdot)) = \sum_{i=1}^{K} m(Y_i)p(Y_i), \tag{4}$$

where $P_\pi = (p(Y_1), p(Y_2), \ldots, p(Y_K)) = \left(\frac{|Y_1|}{|U|}, \frac{|Y_2|}{|U|}, \ldots, \frac{|Y_K|}{|U|} \right)$ *is the probability distribution defined by* π*, and* $E_{P_\pi}(\cdot)$ *is the mathematical expectation of the distribution* P_π*.*

Definition 2.5. [20] *Given a DS* $D = (U, C \cup D)$*,* $R \subseteq C$*,* $0 \leq \beta < \alpha \leq 1$*,* $U/D = \{T_1, T_2, \ldots, T_M\}$ *is a classification of* U*. The MPAA of* U/D *is defined as follows.*

$$\Omega_R^{(\alpha,\beta)}(U/D, m(\cdot)) = EG_m(\Pi_1) - \left(1 - \Gamma_R^{(\alpha,\beta)}(U/D)\right) EG_m(U/R), \tag{5}$$

where $\Pi_1 = \{U\}$*.*

Definition 2.6. [20] *Given a DS* $D = (U, C \cup D)$*,* $R \subseteq C$*,* $0 \leq \beta < \alpha \leq 1$*,* R *is an* (α, β)*-PAA reduce of* D *iff*

1) $\Gamma_R^{(\alpha,\beta)}(U/D) = \Gamma_C^{(\alpha,\beta)}(U/D)$*; 2)* $\Gamma_{R-\{a\}}^{(\alpha,\beta)}(U/D) \neq \Gamma_C^{(\alpha,\beta)}(U/D)$ *for* $\forall a \in R$*.*

Definition 2.7. [20] *Let* $D = (U, C \cup D)$ *is a DS. The core of an* (α, β)*-PRS attribute reduction is defined as follows, where* $0 \leq \beta < \alpha \leq 1$*.*

$$CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(C) = \left\{ c \in C \middle| \Gamma_{C-\{c\}}^{(\alpha,\beta)}(U/D) \neq \Gamma_C^{(\alpha,\beta)}(U/D) \right\}. \tag{6}$$

3. **Incremental Attribute Reduction Algorithms.** Under the PRS model, Wang et al. [20] proposed a heuristic algorithm for attribute reduction, in which you need to calculate the attribute core. Based on attribute core, we calculate the attribute reduction through the newly added attributes. Then we get rid of the redundant attributes in the attribute reduction and get the minimum attribute reduction. In this procedure, you need calculate the PAA $\Gamma_{C-\{c\}}^{(\alpha,\beta)}(U/D)$ when a single attribute is deleted in a given attribute set frequently. And also you need calculate the MPAA $\Omega_{R\cup\{a\}}^{(\alpha,\beta)}(U/D, m(\cdot))$ when a single attribute is added in a given attribute set frequently. These factors will lead to the inefficiency of the algorithm. In order to improve the efficiency of the algorithm, we present an incremental method to calculate PAA and MPAA. Then we design an incremental attribute reduction algorithm in PRS when the attribute set varies.

3.1. Theoretical analysis.

Definition 3.1. [21] *Given an IS* $I=(U,A)$, $R\subseteq A$, $\underline{\Delta}_R(Y)$ *is the lower boundary of* Y *in* (U,R), $\overline{\Delta}_R(Y)$ *is the upper boundary of* Y *in* (U,R), $BND_R(Y)$ *is boundary region, and the* $\underline{\Delta}_R(Y)$ *and* $\overline{\Delta}_R(Y)$ *are defined as*
$BND_R(Y)=\overline{\Delta}_R(Y)\cup\underline{\Delta}_R(Y)$, $\underline{\Delta}_R(Y)=Y-\underline{R}(Y)$, $\overline{\Delta}_R(Y)=\overline{R}(Y)-Y$.

Definition 3.1 gives the concept of boundary sets in Pawlak rough set. The boundary region $BND_R(Y)$ is divided into upper boundary sets $\overline{\Delta}_R(Y)$ and lower boundary sets $\underline{\Delta}_R(Y)$.

Theorem 3.1. [21] *Given an IS* $I=(U,A)$, $P\subseteq A$, $Y\subseteq U$. *Let* $a\in A$, *and* $a\notin P$. *By adding* a *to* P, *the lower approximation of* Y *in the Pawlak rough set can be updated with the following parts:* $\underline{AS}_{\{a\}}(Y)$, $\underline{\Delta}_{\{a\}}(Y)$, $\underline{AS}_P(Y)$ *and* $\underline{\Delta}_P(Y)$.

$$\underline{AS}_{P\cup\{a\}}(Y)=\underline{AS}_P(Y)\cup\underline{AS}_{\{a\}}(Y)\cup H, \tag{7}$$

where $H=\{y\in\underline{\Delta}_P(Y)\cap\underline{\Delta}_{\{a\}}(Y)\,|\bigcap_{b\in P\cup\{a\}}[y]_b\subseteq Y\}$.

Lemma 3.1. *Given an IS* $I=(U,A)$, $Y\subseteq U$, $0\le\beta<\alpha\le1$, $P\subseteq A$. *Let* a *be an attribute in* A, *and* $a\notin P$. *When adding* a *to* P, *the lower approximation* $\underline{AS}_A^{(\alpha,\beta)}(Y)$ *of* Y *in PRS can be updated as follows:*

$$\underline{AS}_{P\cup\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)=\underline{AS}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\cup\underline{AS}_{\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\cup H', \tag{8}$$

where $H'=\left\{y\in\underline{\Delta}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\cap\underline{\Delta}_{\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\,\middle|\,\bigcap_{b\in P\cup\{a\}}[y]_b\subseteq\underline{AS}_A^{(\alpha,\beta)}(Y)\right\}$.

Proof: The lower approximation $\underline{AS}_A^{(\alpha,\beta)}(Y)$ is an object set, and $\underline{AS}_A^{(\alpha,\beta)}(Y)\subseteq U$. $\forall y\in\underline{AS}_{P\cup\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)$, if $y\notin\underline{AS}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\cup\underline{AS}_{\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)$, we have $y\in H'$. This is because if $y\notin\underline{AS}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\cup\underline{AS}_{\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)$, iff $y\in\underline{\Delta}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\cap\underline{\Delta}_{\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)$. And if $y\in\underline{AS}_{P\cup\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)$, iff $\bigcap_{b\in P\cup\{a\}}[y]_b\subseteq\underline{AS}_A^{(\alpha,\beta)}(Y)$. Therefore, we have $y\in H'$. So we can get $\underline{AS}_{P\cup\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)=\underline{AS}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\cup\underline{AS}_{\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\cup H'$.

Theorem 3.2. [21] *Given an IS* $I=(U,A)$, $Y\subseteq U$, $P\subseteq A$. *Let* $a\in A$, *and* $a\notin P$. *By adding* a *to* P, *the upper approximation of* Y *in Pawlak rough set can be updated as follows:*

$$\overline{AS}_{P\cup\{a\}}(Y)=\overline{AS}_P(Y)\cap\overline{AS}_{\{a\}}(Y)-Z, \tag{9}$$

where $Z=\left\{y\in\bigcap_{b\in P\cup\{a\}}\overline{\Delta}_{\{b\}}(Y)\,\middle|\bigcap_{b\in P\cup\{a\}}[y]_b\subseteq\bigcap_{b\in P\cup\{a\}}\overline{\Delta}_{\{b\}}(Y)\right\}$.

Lemma 3.2. *Given an IS* $I=(U,A)$, $Y\subseteq U$, $0\le\beta<\alpha\le1$, $P\subseteq A$. *Let* a *be an attribute in* A, *and* $a\notin P$. *By adding* a *to* P, *the upper approximation* $\overline{AS}_A^{(\alpha,\beta)}(Y)$ *of* Y *in PRS can be updated as follows:*

$$\overline{AS}_{P\cup\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)=\overline{AS}_P\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)\cap\overline{AS}_{\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)-Z', \tag{10}$$

where $Z'=\left\{y\in\bigcap_{b\in P\cup\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)\,\middle|\,\bigcap_{b\in P\cup\{a\}}[y]_b\subseteq\bigcap_{b\in P\cup\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)\right\}$.

Proof: Let $y\in\overline{AS}_{P\cup\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$ and $y\notin Y$. According to Definition 3.1, we have $y\in\overline{\Delta}_{P\cup\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$, namely $y\in\overline{\Delta}_P\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$ and $\bigcap_{b\in P\cup\{a\}}[y]_b\bigcap\overline{AS}_A^{(\alpha,\beta)}(Y)\neq$

$\emptyset$. Because of $\left(\bigcap_{b\in P\cup\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)\right)\bigcap\overline{AS}_A^{(\alpha,\beta)}(Y)=\emptyset$, however $\bigcap_{b\in P\cup\{a\}}[y]_b$ is not a subset of $\bigcap_{b\in P\cup\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$, $y\notin Z'$. We can get $\overline{AS}_{P\cup\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)=\overline{AS}_P\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)\cap\overline{AS}_{\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)-Z'$.

Theorem 3.3. [21] *Given an IS $I=(U,A)$, $Y\subseteq U$, $P\subseteq A$. Let $a\in P$. By deleting a from P, the lower approximation of Y can be updated with the following parts: $\underline{AS}_P(Y)$, and $\underline{\Delta}_{P-\{a\}}(Y)$.*

$$\underline{AS}_{P-\{a\}}(Y)=\underline{AS}_P(Y)-\underline{\Delta}_{P-\{a\}}(Y),\tag{11}$$

where $\underline{\Delta}_{P-\{a\}}(Y)=\left\{y\in\bigcap_{b\in P-\{a\}}\underline{\Delta}_{\{b\}}(Y)\middle|\bigcap_{b\in P-\{a\}}[y]_b\not\subset Y\right\}$.

Lemma 3.3. *Given an IS $I=(U,A)$, $Y\subseteq U$, $0\le\beta<\alpha\le1$, $P\subseteq A$. Let a be an attribute in P. By deleting a from P, the lower approximation $\underline{AS}_A^{(\alpha,\beta)}(Y)$ of Y in PRS can be updated as follows:*

$$\underline{AS}_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)=\underline{AS}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)-\underline{\Delta}'_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right),\tag{12}$$

where $\underline{\Delta}'_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)=\left\{y\in\bigcap_{b\in P-\{a\}}\underline{\Delta}_{\{b\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\middle|\bigcap_{b\in P-\{a\}}[y]_b\not\subset\underline{AS}_A^{(\alpha,\beta)}(Y)\right\}$.

Proof: In general, we have $\underline{AS}_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\subseteq\underline{AS}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)$. In terms of lower boundary sets, we have $\underline{\Delta}'_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\subseteq\underline{\Delta}'_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)$. The contribution of an attribute a to the lower approximation of Y by P can be characterized by the set $\underline{\Delta}'_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)-\underline{\Delta}'_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)=\left\{y\in U\middle|y\in\underline{\Delta}'_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\text{ and }y\notin\underline{\Delta}'_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\right\}$. Therefore, the effect of deleting attribute a from P to the lower approximation of Y is $\underline{AS}_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)=\underline{AS}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)-\left(\underline{\Delta}'_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)-\underline{\Delta}'_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\right)$. Because of $\underline{AS}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\cap\underline{\Delta}'_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)=\emptyset$, then we have $\underline{AS}_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)=\underline{AS}_P\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)-\underline{\Delta}'_{P-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)$.

Theorem 3.4. [21] *Given an IS $I=(U,A)$, $Y\subseteq U$, $P\subseteq A$. Let $a\in P$. By deleting a from P, the upper approximation of Y can be updated as follows:*

$$\overline{AS}_{P-\{a\}}(Y)=Y\cup\overline{\Delta}_P(Y)\cup W,\tag{13}$$

where $W=\left\{y\in\bigcap_{b\in P-\{a\}}\overline{\Delta}_{\{b\}}(Y)\middle|\bigcap_{b\in P-\{a\}}[y]_b\not\subset\bigcap_{b\in P-\{a\}}\overline{\Delta}_{\{b\}}(Y)\right\}$.

Lemma 3.4. *Given an IS $I=(U,A)$, $Y\subseteq U$, $0\le\beta<\alpha\le1$, $P\subseteq A$. Let a be an attribute in P. By deleting a from P, the upper approximation $\overline{AS}_A^{(\alpha,\beta)}(Y)$ of Y in PRS can be updated as follows:*

$$\overline{AS}_{P-\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)=\overline{AS}_A^{(\alpha,\beta)}(Y)\cup\overline{\Delta}_P\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)\cup W',\tag{14}$$

where $W'=\left\{y\in\bigcap_{b\in P-\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)\middle|\bigcap_{b\in P-\{a\}}[y]_b\not\subset\bigcap_{b\in P-\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)\right\}$.

Proof: Assume $y \in \overline{AS}_{P-\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$ and $y \notin Y$, then Y is in $\overline{\Delta}_{P-\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$ by Definition 3.1. In general, we can get $\overline{\Delta}_P\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right) \subseteq \overline{\Delta}_{P-\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$. Therefore, if $y \in \overline{AS}_{P-\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$, $y \notin Y$ and $y \notin \overline{\Delta}_P\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$, then y must be in W', because $\bigcap_{b\in P-\{a\}}[y]_b \subseteq \bigcap_{b\in P-\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$ if and only if $y \in \overline{AS}_{P-\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$. This will be contrary to the following assumption $y \in \overline{AS}_{P-\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)$. So $\overline{AS}_{P-\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right) = \overline{AS}_A^{(\alpha,\beta)}(Y) \cup \overline{\Delta}_P\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right) \cup W'$.

According to Lemmas 3.1-3.4 and Definition 2.3 we can get Theorems 3.5-3.8.

Theorem 3.5. *Given an IS $I = (U, A)$, $Y \subseteq U$, $0 \le \beta < \alpha \le 1$, $R \subseteq A$. Let a be an attribute in A, and $a \notin R$. The PAA by adding a to R can be updated as*

$$\Gamma_{R\cup\{a\}}^{(\alpha,\beta)}(Y) = \frac{\left|\underline{AS}_R\left(\underline{P}(Y)\right) \cup \underline{AS}_{\{a\}}\left(\underline{P}(Y)\right) \cup H'\right|}{\left|\overline{AS}_R\left(\overline{P}(Y)\right) \cap \overline{AS}_{\{a\}}\left(\overline{P}(Y)\right) - Z'\right|}, \tag{15}$$

where $H' = \left\{y \in \underline{\Delta}_R\left(\underline{P}(Y)\right) \cap \underline{\Delta}_{\{a\}}\left(\underline{P}(Y)\right) \,\middle|\, \bigcap_{b\in R\cup\{a\}}[y]_b \subseteq \underline{P}(Y)\right\}$, $\overline{P}(Y) = \overline{AS}_A^{(\alpha,\beta)}(Y)$, $Z' = \left\{y \in \bigcap_{b\in R\cup\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{P}(Y)\right) \,\middle|\, \bigcap_{b\in R\cup\{a\}}[y]_b \subseteq \bigcap_{b\in R\cup\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{P}(Y)\right)\right\}$ *and* $\underline{P}(Y) = \underline{AS}_A^{(\alpha,\beta)}(Y)$.

Proof: Because of $\Gamma_{R\cup\{a\}}^{(\alpha,\beta)}(Y) = \frac{\left|\underline{AS}_{R\cup\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\right|}{\left|\overline{AS}_{R\cup\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)\right|}$, let $\underline{P}(Y) = \underline{AS}_A^{(\alpha,\beta)}(Y)$ and $\overline{P}(Y) = \overline{AS}_A^{(\alpha,\beta)}(Y)$, then we have $\Gamma_{R\cup\{a\}}^{(\alpha,\beta)}(Y) = \frac{\left|\underline{AS}_{R\cup\{a\}}(\underline{P}(Y))\right|}{\left|\overline{AS}_{R\cup\{a\}}(\overline{P}(Y))\right|}$. $\underline{AS}_{R\cup\{a\}}\left(\underline{P}(Y)\right) = \underline{AS}_R\left(\underline{P}(Y)\right) \cup \underline{AS}_{\{a\}}\left(\underline{P}(Y)\right) \cup H'$ by Lemma 3.1. According to Lemma 3.2, we can get $\overline{AS}_{R\cup\{a\}}\left(\overline{P}(Y)\right) = \overline{AS}_R\left(\overline{P}(Y)\right) \cap \overline{AS}_{\{a\}}\left(\overline{P}(Y)\right) - Z'$, so $\Gamma_{R\cup\{a\}}^{(\alpha,\beta)}(Y) = \frac{\left|\underline{AS}_R(\underline{P}(Y)) \cup \underline{AS}_{\{a\}}(\underline{P}(Y)) \cup H'\right|}{\left|\overline{AS}_R(\overline{P}(Y)) \cap \overline{AS}_{\{a\}}(\overline{P}(Y)) - Z'\right|}$.

Theorem 3.6. *Given an IS $I = (U, A)$, $Y \subseteq U$, $0 \le \beta < \alpha \le 1$, $R \subseteq A$. Let a be an attribute in R. The PAA by deleting a from R can be updated as*

$$\Gamma_{R-\{a\}}^{(\alpha,\beta)}(Y) = \frac{\left|\underline{AS}_R\left(\underline{P}(Y)\right) - \underline{\Delta}'_{R-\{a\}}\left(\underline{P}(Y)\right)\right|}{\left|\overline{P}(Y) \cup \overline{\Delta}_R\left(\overline{P}(Y)\right) \cup W'\right|}, \tag{16}$$

where $\underline{\Delta}'_{R-\{a\}}\left(\underline{P}(Y)\right) = \left\{y \in \bigcap_{b\in R-\{a\}}\underline{\Delta}_{\{b\}}\left(\underline{P}(Y)\right) \,\middle|\, \bigcap_{b\in R-\{a\}}[y]_b \not\subset \underline{P}(Y)\right\}$, $W' = \left\{y \in \bigcap_{b\in R-\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{P}(Y)\right) \,\middle|\, \bigcap_{b\in R-\{a\}}[y]_b \not\subset \bigcap_{b\in R-\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{P}(Y)\right)\right\}$, $\underline{P}(Y) = \underline{AS}_A^{(\alpha,\beta)}(Y)$ *and* $\overline{P}(Y) = \overline{AS}_A^{(\alpha,\beta)}(Y)$.

Proof: Because of $\Gamma_{R-\{a\}}^{(\alpha,\beta)}(Y) = \frac{\left|\underline{AS}_{R-\{a\}}\left(\underline{AS}_A^{(\alpha,\beta)}(Y)\right)\right|}{\left|\overline{AS}_{R-\{a\}}\left(\overline{AS}_A^{(\alpha,\beta)}(Y)\right)\right|}$, let $\underline{P}(Y) = \underline{AS}_A^{(\alpha,\beta)}(Y)$ and $\overline{P}(Y) = \overline{AS}_A^{(\alpha,\beta)}(Y)$, then we have $\Gamma_{R-\{a\}}^{(\alpha,\beta)}(Y) = \frac{\left|\underline{AS}_{R-\{a\}}(\underline{P}(Y))\right|}{\left|\overline{AS}_{R-\{a\}}(\overline{P}(Y))\right|}$. We can get $\underline{AS}_{R-\{a\}}\left(\underline{P}(Y)\right) = \underline{AS}_R\left(\underline{P}(Y)\right) - \underline{\Delta}'_{R-\{a\}}\left(\underline{P}(Y)\right)$ by Lemma 3.3 and $\overline{AS}_{R-\{a\}}\left(\overline{P}(Y)\right) = \overline{P}(Y) \cup \overline{\Delta}_R\left(\overline{P}(Y)\right) \cup W'$ by Lemma 3.4. Then we have $\Gamma_{R-\{a\}}^{(\alpha,\beta)}(Y) = \frac{\left|\underline{AS}_R(\underline{P}(Y)) - \underline{\Delta}'_{R-\{a\}}(\underline{P}(Y))\right|}{\left|\overline{P}(Y) \cup \overline{\Delta}_R(\overline{P}(Y)) \cup W'\right|}$.

Theorem 3.7. *Given a DS $D = (U, C \cup D)$, $R \subseteq C$, $0 \le \beta < \alpha \le 1$. Let $a \in C$, and $a \notin R$. $U/D = \{T_1, T_2, \ldots, T_M\}$ is a classification of U. The PAA of U/D with respect* *to R by adding a to R can be updated as*

$$\Gamma^{(\alpha,\beta)}_{R\cup\{a\}}(U/D) = \frac{\sum_{T_j\in U/D}\left|\underline{AS}_R\left(\underline{P}(T_j)\right)\cup\underline{AS}_{\{a\}}\left(\underline{P}(T_j)\right)\cup H''\right|}{\sum_{T_j\in U/D}\left|\overline{AS}_R\left(\overline{P}(T_j)\right)\cap\overline{AS}_{\{a\}}\left(\overline{P}(T_j)\right)-Z''\right|}, \quad (17)$$

where $H''=\left\{y\in\underline{\Delta}_R\left(\underline{P}(T_j)\right)\cap\underline{\Delta}_{\{a\}}\left(\underline{P}(T_j)\right)\,\middle|\,\bigcap_{b\in R\cup\{a\}}[y]_b\subseteq\underline{P}(T_j)\right\}$, $\underline{P}(T_j)=\underline{AS}^{(\alpha,\beta)}_C(T_j)$, $Z''=\left\{y\in\bigcap_{b\in R\cup\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{P}(T_j)\right)\,\middle|\,\bigcap_{b\in R\cup\{a\}}[y]_b\subseteq\bigcap_{b\in R\cup\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{P}(T_j)\right)\right\}$ *and* $\overline{P}(T_j)=\overline{AS}^{(\alpha,\beta)}_C(T_j)$, $j=\{1,2,\ldots,M\}$.

Proof: According to Definition 2.3 we can get $\Gamma^{(\alpha,\beta)}_R(U/D)=\frac{\sum_{T_j\in U/D}\left|\underline{AS}_R\left(\underline{AS}^{(\alpha,\beta)}_C(T_j)\right)\right|}{\sum_{T_j\in U/D}\left|\overline{AS}_R\left(\overline{AS}^{(\alpha,\beta)}_C(T_j)\right)\right|}$, When attribute a is added to R, the values of $\underline{AS}^{(\alpha,\beta)}_C(T_j)$ and $\overline{AS}^{(\alpha,\beta)}_C(T_j)$ do not change. Only the values of $\underline{AS}_R\left(\underline{AS}^{(\alpha,\beta)}_C(T_j)\right)$ and $\overline{AS}_R\left(\overline{AS}^{(\alpha,\beta)}_C(T_j)\right)$ change with the change of R. Therefore, we have $\Gamma^{(\alpha,\beta)}_{R\cup\{a\}}(U/D)=\frac{\sum_{T_j\in U/D}\left|\underline{AS}_{R\cup\{a\}}\left(\underline{AS}^{(\alpha,\beta)}_C(T_j)\right)\right|}{\sum_{T_j\in U/D}\left|\overline{AS}_{R\cup\{a\}}\left(\overline{AS}^{(\alpha,\beta)}_C(T_j)\right)\right|}$. Let $\underline{P}(T_j)=\underline{AS}^{(\alpha,\beta)}_C(T_j)$ and $\overline{P}(T_j)=\overline{AS}^{(\alpha,\beta)}_C(T_j)$, then we can get $\underline{AS}_{R\cup\{a\}}\left(\underline{P}(T_j)\right)=\underline{AS}_R\left(\underline{P}(T_j)\right)\cup\underline{AS}_{\{a\}}\left(\underline{P}(T_j)\right)\cup H''$ by Lemma 3.1 and $\overline{AS}_{R\cup\{a\}}\left(\overline{P}(T_j)\right)=\overline{AS}_R\left(\overline{P}(T_j)\right)\cap\overline{AS}_{\{a\}}\left(\overline{P}(T_j)\right)-Z''$ by Lemma 3.2. So we have $\Gamma^{(\alpha,\beta)}_{R\cup\{a\}}(U/D)=\frac{\sum_{T_j\in U/D}\left|\underline{AS}_R\left(\underline{P}(T_j)\right)\cup\underline{AS}_{\{a\}}\left(\underline{P}(T_j)\right)\cup H''\right|}{\sum_{T_j\in U/D}\left|\overline{AS}_R\left(\overline{P}(T_j)\right)\cap\overline{AS}_{\{a\}}\left(\overline{P}(T_j)\right)-Z''\right|}$.

Theorem 3.8. *Given a DS $D=(U,C/D)$, $R\subseteq C$, $0\le\beta<\alpha\le1$. Let a be an attribute in R. $U/D=\{T_1,T_2,\ldots,T_M\}$ is a classification of U. The PAA of U/D with respect to R by deleting a from R can be updated as*

$$\Gamma^{(\alpha,\beta)}_{R-\{a\}}(U/D) = \frac{\sum_{T_j\in U/D}\left|\underline{AS}_R\left(\underline{P}(T_j)\right)-\underline{\Delta}''_{R-\{a\}}\left(\underline{P}(T_j)\right)\right|}{\sum_{T_j\in U/D}\left|\overline{P}(T_j)\cup\overline{\Delta}_R\left(\overline{P}(T_j)\right)\cup W''\right|}, \quad (18)$$

where $\underline{\Delta}''_{R-\{a\}}\left(\underline{P}(T_j)\right)=\left\{y\in\bigcap_{b\in R-\{a\}}\underline{\Delta}_{\{b\}}\left(\underline{P}(T_j)\right)\,\middle|\,\bigcap_{b\in R-\{a\}}[y]_b\not\subseteq\underline{P}(T_j)\right\}$, $W''=\left\{y\in\bigcap_{b\in R-\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{P}(T_j)\right)\,\middle|\,\bigcap_{b\in R-\{a\}}[y]_b\not\subseteq\bigcap_{b\in R-\{a\}}\overline{\Delta}_{\{b\}}\left(\overline{P}(T_j)\right)\right\}$, $\underline{P}(T_j)=\underline{AS}^{(\alpha,\beta)}_C(T_j)$ *and* $\overline{P}(T_j)=\overline{AS}^{(\alpha,\beta)}_C(T_j)$, $j=\{1,2,\ldots,M\}$.

Proof: According to Definition 2.3 we can get $\Gamma^{(\alpha,\beta)}_R(U/D)=\frac{\sum_{T_j\in U/D}\left|\underline{AS}_R\left(\underline{AS}^{(\alpha,\beta)}_C(T_j)\right)\right|}{\sum_{T_j\in U/D}\left|\overline{AS}_R\left(\overline{AS}^{(\alpha,\beta)}_C(T_j)\right)\right|}$. When attribute a is removed from R, the values of $\underline{AS}^{(\alpha,\beta)}_C(T_j)$ and $\overline{AS}^{(\alpha,\beta)}_C(T_j)$ do not change. Only the values of $\underline{AS}_R\left(\underline{AS}^{(\alpha,\beta)}_C(T_j)\right)$ and $\overline{AS}_R\left(\overline{AS}^{(\alpha,\beta)}_C(T_j)\right)$ change with the change of R. So $\Gamma^{(\alpha,\beta)}_{R-\{a\}}(U/D)=\frac{\sum_{T_j\in U/D}\left|\underline{AS}_{R-\{a\}}\left(\underline{AS}^{(\alpha,\beta)}_C(T_j)\right)\right|}{\sum_{T_j\in U/D}\left|\overline{AS}_{R-\{a\}}\left(\overline{AS}^{(\alpha,\beta)}_C(T_j)\right)\right|}$. Let $\underline{P}(T_j)=\underline{AS}^{(\alpha,\beta)}_C(T_j)$ and $\overline{P}(T_j)=\overline{AS}^{(\alpha,\beta)}_C(T_j)$, we have $\underline{AS}_{R-\{a\}}\left(\underline{P}(T_j)\right)=\underline{AS}_R\left(\underline{P}(T_j)\right)-\underline{\Delta}''_{R-\{a\}}\left(\underline{P}(T_j)\right)$ by Lemma 3.3. and $\overline{AS}_{R-\{a\}}\left(\overline{P}(T_j)\right)=\overline{P}(T_j)\cup\overline{\Delta}_R\left(\overline{P}(T_j)\right)\cup W''$ by Lemma 3.4. Therefore, $\Gamma^{(\alpha,\beta)}_{R-\{a\}}(U/D)=\frac{\sum_{T_j\in U/D}\left|\underline{AS}_R\left(\underline{P}(T_j)\right)-\underline{\Delta}''_{R-\{a\}}\left(\underline{P}(T_j)\right)\right|}{\sum_{T_j\in U/D}\left|\overline{P}(T_j)\cup\overline{\Delta}_R\left(\overline{P}(T_j)\right)\cup W''\right|}$.

According to Theorems 3.5-3.8 and Definition 2.5 we can get Theorem 3.9 and Theorem 3.10.

Theorem 3.9. *Given a DS $D=(U,C/D)$, $R\subseteq C$, $0\le\beta<\alpha\le1$. Let $a\in C$, and $a\notin R$. $U/D=\{T_1,T_2,\ldots,T_M\}$ is a classification of U. The MPAA of U/D with respect* *to R by adding a to R can be updated as follows:*

$$\Omega^{(\alpha,\beta)}_{R\cup\{a\}}\left(U/D, m(\cdot)\right) = EG_m\left(\Pi_1\right) - \left(1 - \Gamma^{(\alpha,\beta)}_{R\cup\{a\}}(U/D)\right) EG_m\left(U/R \cup \{a\}\right), \tag{19}$$

where $\Pi_1 = \{U\}$.

Theorem 3.10. *Given a DS* $D = (U, C/D)$*,* $R \subseteq C$*,* $0 \le \beta < \alpha \le 1$*. Let a be an attribute in R.* $U/D = \{T_1, T_2, \ldots, T_M\}$ *is a classification of U. The MPAA of* U/D *with respect to R by deleting a from R can be updated as follows:*

$$\Omega^{(\alpha,\beta)}_{R-\{a\}}\left(U/D, m(\cdot)\right) = EG_m\left(\Pi_1\right) - \left(1 - \Gamma^{(\alpha,\beta)}_{R-\{a\}}(U/D)\right) EG_m\left(U/R - \{a\}\right), \tag{20}$$

where $\Pi_1 = \{U\}$.

According to Lemmas 3.1-3.4 and Theorems 3.7-3.10 we can get that the incremental updating of PAA and MPAA can be realized by the boundary sets. It avoids repeated calculation and greatly reduces the running time of the algorithm. Therefore, in the case attribute varies, we give incremental algorithms for attribute core in the PRS model.

3.2. **Incremental algorithms for attribute reduction in PRS under the variation of attributes.** In this section, we will first design two algorithms under PRS that incrementally calculate the attribute core when the attribute set varies. As described by Zhang et al. in [30], an effective way to maintain knowledge dynamically is to apply an effective incremental algorithm to the incremental update scheme. This method can use previous results or data structures to avoid unnecessary calculations. Then, Zhang et al. [30] further proposed a block-based strategy to update the approximation. The idea of designing incremental algorithms in their strategy is based on dividing the objects in the universe into equivalence classes. According to the method given in [29], we can conclude that the method of updating the upper and lower approximations in PRS depends mainly on the variation of the boundary set. Therefore, Liu et al. [29] first divided the boundary set into equivalence classes, and then designed an algorithm for incrementally updating the upper and lower approximations by obtaining the update of the boundary set in PRS.

Suppose that the process of updating the attribute core continues at two different moments: time t and time $t+1$. At time $t+1$, there are two possible variables in the attribute set, namely adding attributes and deleting attributes. The update processes of the attribute core in two cases are discussed below. Liu et al. [29] were committed to a new strategy for dynamically updating the upper and lower approximations in PRS, and studied four propositions for updating the upper and lower approximations in PRS in the information system. We extended this method to the decision information system and applied the extended method to the incremental update calculation of the attribute core. In the following, we discuss the incremental calculation method and algorithm of the attribute core of the probability rough set from the perspective of adding attributes and deleting attributes, respectively. Finally, based on these two algorithms for incrementally updating attribute cores, we give an algorithm for calculating the minimum attribute reduction.

3.2.1. *Incremental algorithm for updating attribute core when adding attributes.* In this section, we introduce the incremental update method of the upper and lower approximations of the PRS when adding attributes.

Suppose a decision system $D = (U, P/D)$ at time t. When the attribute set Q is added to the attribute set P, the decision system will change to $D' = (U, C/D)$ at time $t+1$, where $C = P \cup Q$. $P \subseteq C$, $U/D = \{T_1, T_2, \ldots, T_M\}$, we have: $\underline{AS}_P(T_j) \subseteq \underline{AS}_C(T_j)$, $\overline{AS}_C(T_j) \subseteq \overline{AS}_P(T_j)$, $BND_C(T_j) \subseteq BND_P(T_j)$, where $j = \{1, 2, \ldots, M\}$. We update the approximation by calculating the changes in the boundary set, $BND_P(T_j)/P = \{BND_{P_1}(T_j), BND_{P_2}(T_j), \ldots, BND_{P_m}(T_j)\}$, $\forall i \in \{1, 2, \ldots, m\}$. When Q is added into P, $BND_{P_i}(T_j)$ is divided into some small parts, and the criteria for determining whether $BND_{P_i}(T_j)$ are refined are as follows.

1) $\forall y, z \in BND_{P_i}(T_j)$, if $\forall a \in Q$, $f(y,a) = f(z,a)$, the $BND_{P_i}(T_j)$ keeps constant;

2) $\forall y, z \in BND_{P_i}(T_j)$, if $\exists a \in Q$, $f(y,a) \neq f(z,a)$, the $BND_{P_i}(T_j)$ refines.

Suppose $BND_{P_i}(T_j)/Q = \{BND_{P_{i1}}(T_j), BND_{P_{i2}}(T_j), \ldots, BND_{P_{in_i}}(T_j)\}$ at time $t+1$, $\forall i \in \{1, 2, \ldots, m\}$, $BND_{P_{ik}}(T_j)$ is used to calculate $P_r(T_j|BND_{P_{ik}}(T_j))$, where $P_r(T_j| BND_{P_{ik}}(T_j)) = \frac{|BND_{P_{ik}}(T_j) \cap T_j|}{|BND_{P_{ik}}(T_j)|}$. $BND_{P_{ik}}(T_j)$ is a part of the lower approximation if $P_r(T_j| BND_{P_{ik}}(T_j)) \geq \alpha$, and it is not a part of the upper approximation if $P_r(T_j|BND_{P_{ik}}(T_j)) \leq \beta$. Therefore, according to the Strategy 2 given in [29] Section 3.3.1, we can get $\underline{AS}_C^{(\alpha,\beta)}(T_j)$ and $\overline{AS}_C^{(\alpha,\beta)}(T_j)$ as follows.

1) $\underline{AS}_C^{(\alpha,\beta)}(T_j) = \underline{AS}_P(T_j) \cup \{BND_{P_{ik}}(T_j)|P_r(T_j|BND_{P_{ik}}(T_j)) \geq \alpha\}$;

2) $\overline{AS}_C^{(\alpha,\beta)}(T_j) = \overline{AS}_P(T_j) - \{BND_{P_{ik}}(T_j)|P_r(T_j|BND_{P_{ik}}(T_j)) \leq \beta\}$,

where $i = \{1, 2, \ldots, m\}$, $k = \{1, 2, \ldots, n_i\}$, $j = \{1, 2, \ldots, M\}$.

We outline the process of incrementally updating the attribute core when adding attributes in the PRS, as shown in Algorithm 1.

3.2.2. *Incremental update algorithm of attribute core when deleting attributes.* In this section, we introduce the incremental update method of the upper and lower approximations of the PRS when deleting attributes. Suppose $D = (U, C' \cup D)$ at time t, the attribute set Q' is removed from the attribute set C' and the decision system will change to $D' = (U, (C' - Q') \cup D)$ at time $t+1$, where $P' = C' - Q'$. $P' \subseteq C'$, $U/D = \{T_1, T_2, \ldots, T_M\}$, we have: $\underline{AS}_{P'}(T_j) \subseteq \underline{AS}_{C'}(T_j)$, $\overline{AS}_{C'}(T_j) \subseteq \overline{AS}_{P'}(T_j)$, $BND_{C'}(T_j) \subseteq BND_{P'}(T_j)$ and $NEG_{P'}(T_j) \subseteq NEG_{C'}(T_j)$, where $j = \{1, 2, \ldots, M\}$. The deletion of the attribute set Q' may cause some of the sets in $\underline{AS}_{C'}(T_j)$ and $NEG_{C'}(T_j)$ to become $BND_{P'}(T_j)$. We divide U into three parts: $POS_{C'}(T_j) = \underline{AS}_{C'}(T_j)$, $BND_{C'}(T_j) = \overline{AS}_{C'}(T_j) - \underline{AS}_{C'}(T_j)$, $NEG_{C'}(T_j) = U - \overline{AS}_{C'}(T_j)$. $POS_{C'}(T_j)/C' = \{C'_1, C'_2, \ldots, C'_u\}$, $BND_{C'}(T_j)/C' = \{C''_1, C''_2, \ldots, C''_v\}$, $NEG_{C'}(T_j)/C' = \{C'''_1, C'''_2, \ldots, C'''_w\}$.

By the discussions in [29], an effective strategy for updating approximation is to calculate $\underline{AS}_{P'}(T_j)$ and $BND_{P'}(T_j)$, which can be divided into two steps as follows.

Firstly, Finding $\underline{AS}_{P'}(T_j)$. $\forall C_i \subseteq POS_{C'}(T_j)$, $\forall C_k \subseteq \{BND_{C'}(T_j) \cup NEG_{C'}(T_j)\}$, there are two cases here.

1) $\forall y \in C_i$, $\forall z \in C_k$, if $\exists a \in P'$, $f(y,a) \neq f(z,a)$, then $C_i \cap C_k = \emptyset$. So $C_i \subseteq T_j$, then $C_i \subseteq \underline{AS}_{P'}(T_j)$. We denote these sets $\underline{AS}_{C'_1}(T_j) \subseteq \underline{AS}_{C'}(T_j)$ satisfy this criterion as $\Theta_1(T_j)$, $\Theta_1(T_j) = \underline{AS}_{C'_1}(T_j)$.

2) $\forall y \in C_i$, $\forall z \in C_k$, if $\forall a \in P'$, $f(y,a) = f(z,a)$, then C_i and C_k are merging to one set. $C_i \subseteq T_j$, $C_k \not\subseteq T_j$, so $C_i \cup C_k \subseteq BND_{P'}(T_j)$. We denote $\Theta_2(T_j) = \bigcup [y]_{P'}$ with $y \in C_i$ which satisfy this criterion. Moreover, suppose $\underline{AS}_{C'_2}(T_j) \subseteq \underline{AS}_{C'}(T_j)$ satisfy this criterion, and $\underline{AS}_{C'_2}(T_j)$ are the emigrations from $\underline{AS}_{C'}(T_j)$ to $BND_{P'}(T_j)$ and $\underline{AS}_{C'_2}(T_j) \subseteq \Theta_2(T_j)$.

Obviously, $\underline{AS}_{P'}(T_j) = \Theta_1(T_j)$.

Secondly, Finding $BND_{P'}(T_j)$. We find the $BND_{P'}(T_j)$ with excluding $\Theta_2(T_j)$. $\forall C_i \subseteq BND_{C'}(T_j)$, $\forall C_k \subseteq NEG_{C'}(T_j)$, there are two scenarios.

1) $\forall y \in C_i$, $\forall z \in C_k$, if $\forall a \in P'$, $f(y,a) = f(z,a)$, then C_i and C_k are merging to one set. $C_i \subseteq BND_{C'}(T_j) \subseteq BND_{P'}(T_j)$, $C_i \cup C_k \subseteq BND_{P'}(T_j)$. We denote these sets $NEG_{C'}(T_j)$ satisfy this criterion as $\Theta_3(T_j)$, and $\Theta_3(T_j)$ are the immigrations from $NEG_{C'}(T_j)$ to $BND_{P'}(T_j)$.

Algorithm 1. Incremental update algorithm of attribute core when adding attributes.

Input (1) $D = (U, P \cup D)$ at time t, $U/D = \{T_1, T_2, \dots, T_M\}$, the thresholds α and β; (2) The approximations $\underline{AS}_P(T_j)$ and $\overline{AS}_P(T_j)$ at time t; the classifications with boundary sets: $BND_P(T_j)/P = \{BND_{P_1}(T_j), BND_{P_2}(T_j), \dots, BND_{P_m}(T_j)\}$ at time t; (3) The adding attribute set Q.

Output The updated attribute core at time $t+1$ as $CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(C)$, where $C = P \cup Q$.

begin $C \leftarrow P \cup Q$
for $i = 1$ to m; $j = 1$ to M **do**
  for each $BND_{P_i}(T_j)$, $\forall y, z \in BND_{P_i}(T_j)$ **do**
    if $\forall a \in Q$, $f(y, a) = f(z, a)$ **then** $BND_{P_i}(T_j)$ keeps constant;
    else $BND_{P_i}(T_j)/Q = \{BND_{P_{i1}}(T_j), BND_{P_{i2}}(T_j), \dots, BND_{P_{in_i}}(T_j)\}$
    end if end for end for
for $j = 1$ to M; $i = 1$ to m; $k = 1$ to n_i **do**
  $\underline{AS}_C^{(\alpha,\beta)}(T_j) = \underline{AS}_P(T_j) \cup \{BND_{P_{ik}}(T_j) | P_r(T_j | BND_{P_{ik}}(T_j)) \geq \alpha\}$;
  $\overline{AS}_C^{(\alpha,\beta)}(T_j) = \overline{AS}_P(T_j) - \{BND_{P_{ik}}(T_j) | P_r(T_j | BND_{P_{ik}}(T_j)) \leq \beta\}$.
end for
for $i = 1$ to n; $\forall a_i \in C$ **do** Calculate U/a_i **end for**
Calculate $U/C = \bigcap_{i=1}^{n} U/a_i$
Let $\underline{P}(T_j) = \underline{AS}_C^{(\alpha,\beta)}(T_j)$ and $\overline{P}(T_j) = \overline{AS}_C^{(\alpha,\beta)}(T_j)$, Calculate $\underline{AS}_C(\underline{P}(T_j))$ and $\overline{AS}_C(\overline{P}(T_j))$. Then, Calculate $\Gamma_C^{(\alpha,\beta)}(U/D)$.
for $i = 1$ to n; $j = 1$ to M **do** Calculate $\underline{AS}_{\{a_i\}}(\underline{P}(T_j))$ and $\overline{AS}_{\{a_i\}}(\overline{P}(T_j))$;
  Calculate $\underline{\Delta}_{\{a_i\}}(\underline{P}(T_j))$ and $\overline{\Delta}_{\{a_i\}}(\overline{P}(T_j))$;
  Calculate $\underline{AS}_{C-\{a_i\}}(\underline{P}(T_j))$ and $\overline{AS}_{C-\{a_i\}}(\overline{P}(T_j))$; Calculate $\Gamma_{C-\{a_i\}}^{(\alpha,\beta)}(U/D)$
end for
Let $CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(C) = \emptyset$
for $i = 1$ to n **do**
  if $\Gamma_{C-\{a_i\}}^{(\alpha,\beta)}(U/D) \neq \Gamma_C^{(\alpha,\beta)}(U/D)$ **then** $CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(C) = CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(C) \cup \{a_i\}$
  end if end for
Return $CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(C)$
end begin

2) $\forall y \in C_i$, $\forall z \in C_k$, if $\exists a \in P'$, $f(y, a) \neq f(z, a)$, then $C_i \cap C_k = \emptyset$. So $C_k \subseteq NEG_{C'}(T_j)$.

Intuitively, we have $BND_{P'}(T_j) = BND_{C'}(T_j) \cup \Theta_2(T_j) \cup \Theta_3(T_j)$, $\overline{AS}_{P'}(T_j) = BND_{C'}(T_j) \cup \Theta_1(T_j) \cup \Theta_2(T_j) \cup \Theta_3(T_j)$. Namely, $BND_{P'}(T_j)/P' = \{BND_{P_1'}(T_j), \dots, BND_{P_l'}(T_j)\}$.

According to the Strategy 2 given in [29] Section 3.3.2, we can get $\underline{AS}_{P'}^{(\alpha,\beta)}(T_j)$ and $\overline{AS}_{P'}^{(\alpha,\beta)}(T_j)$ as follows.

$\underline{AS}_{P'}^{(\alpha,\beta)}(T_j) = \underline{AS}_{P'}(T_j) \cup \{BND_{P_i'}(T_j) | P_r(T_j | BND_{P_i'}(T_j)) \geq \alpha\}$;
$\overline{AS}_{P'}^{(\alpha,\beta)}(T_j) = \overline{AS}_{P'}(T_j) - \{BND_{P_i'}(T_j) | P_r(T_j | BND_{P_i'}(T_j)) \leq \beta\}$, where $i = 1, \dots, l$.

We give the process of incrementally updating the attribute core when deleting attributes in the PRS, as shown in Algorithm 2.

Algorithm 2. Incremental update algorithm of attribute core when deleting attributes.

Input (1) $D = (U, C' \cup D)$ at time t, $C' = P' \cup Q'$ and $P' \cap Q' = \emptyset$, $U/D = \{T_1, T_2, \ldots, T_M\}$, the thresholds α and β; (2) The approximations $\underline{AS}_{C'}(T_j)$ and $\overline{AS}_{C'}(T_j)$; (3) The deleting attribute set Q'.

Output The updated attribute core at time $t+1$ as $CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(P')$, where $P' = C' - Q'$.

begin $P' \leftarrow C' - Q'$
for $j = 1$ to M **do** Calculate $POS_{C'}(T_j) = \underline{AS}_{C'}(T_j)$,
$BND_{C'}(T_j) = \overline{AS}_{C'}(T_j) - \underline{AS}_{C'}(T_j)$, $NEG_{C'}(T_j) = U - \overline{AS}_{C'}(T_j)$.
Calculate $POS_{C'}(T_j)/C' = \{C'_1(T_j), C'_2(T_j), \ldots, C'_u(T_j)\}$,
$BND_{C'}(T_j)/C' = \{C''_1(T_j), C''_2(T_j), \ldots, C''_v(T_j)\}$,
$NEG_{C'}(T_j)/C' = \{C'''_1(T_j), C'''_2(T_j), \ldots, C'''_w(T_j)\}$.
end for
for $j = 1$ to M **do**
for $s = 1$ to u, $t = 1$ to $v + w$, $\Theta_1(T_j) = \emptyset$, $\Theta_2(T_j) = \emptyset$ **do**
for $\forall y \in C_s$, $C_s \subseteq POS_{C'}(T_j)$; $\forall z \in C_t$, $C_t \subseteq \{BND_{C'}(T_j) \cup NEG_{C'}(T_j)\}$ **do**
if $\exists a \in P'$, $f(y, a) \neq f(z, a)$ **then** $\Theta_1(T_j) = \Theta_1(T_j) \cup C_s$ **else** $\Theta_2(T_j) \leftarrow \Theta_2(T_j) \cup (C_s \cup C_t)$ **end for end for**
end for
for $j = 1$ to M **do**
for $s = 1$ to v, $t = 1$ to w, $\Theta_3(T_j) = \emptyset$ **do**
for $\forall y \in C_s - \Theta_2(T_j)$, $C_s \subseteq BND_{C'}(T_j)$; $\forall z \in C_t - \Theta_2(T_j)$, $C_t \subseteq NEG_{C'}(T_j)$ **do**
if $\forall a \in P'$, $f(y, a) = f(z, a)$ **then** $\Theta_3(T_j) \leftarrow \Theta_3(T_j) \cup C_t$ **else** $C_t \in NEG'_P(T_j)$
end for end for end for
for $j = 1$ to M **do** Calculate $BND_{P'}(T_j) = BND_{C'}(T_j) \cup \Theta_2(T_j) \cup \Theta_3(T_j)$;
$\underline{AS}'_P(T_j) = \Theta_1(T_j)$; $\overline{AS}'_P(T_j) = \Theta_1(T_j) \cup BND_{P'}(T_j)$;
$BND_{P'}(T_j)/P' = \{BND_{P_1{}'}(T_j), BND_{P_2{}'}(T_j), \ldots, BND_{P_l{}'}(T_j)\}$.
for $i = 1$ to l **do**
$\underline{AS}^{(\alpha,\beta)}_{P'}(T_j) = \underline{AS}'_P(T_j) \cup \{BND_{P_i{}'}(T_j) | P_r(T_j | BND_{P_i{}'}(T_j)) \geq \alpha\}$,
$\overline{AS}^{(\alpha,\beta)}_{P'}(T_j) = \overline{AS}'_P(T_j) - \{BND_{P_i{}'}(T_j) | P_r(T_j | BND_{P_i{}'}(T_j)) \leq \beta\}$.
end for
end for
for $i = 1$ to n, $\forall a_i \in P'$ **do** Calculate U/a_i **end for**
Calculate $U/P' = \bigcap_{i=1}^{n} U/a_i$
Let $\underline{P}(T_j) = \underline{AS}^{(\alpha,\beta)}_{P'}(T_j)$ and $\overline{P}(T_j) = \overline{AS}^{(\alpha,\beta)}_{P'}(T_j)$, Calculate $\underline{AS}_{P'}(\underline{P}(T_j))$ and $\overline{AS}_{P'}(\overline{P}(T_j))$. Then, Calculate $\Gamma^{(\alpha,\beta)}_{P'}(U/D)$
for $i = 1$ to n; $j = 1$ to M **do** Calculate $\underline{AS}_{\{a_i\}}(\underline{P}(T_j))$ and $\overline{AS}_{\{a_i\}}(\overline{P}(T_j))$;
Calculate $\underline{\Delta}_{\{a_i\}}(\underline{P}(T_j))$ and $\overline{\Delta}_{\{a_i\}}(\overline{P}(T_j))$;
Calculate $\underline{AS}_{P'-\{a_i\}}(\underline{P}(T_j))$ and $\overline{AS}_{P'-\{a_i\}}(\overline{P}(T_j))$; Calculate $\Gamma^{(\alpha,\beta)}_{P'-\{a_i\}}(U/D)$
end for
Let $CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(P') = \emptyset$
for $i = 1$ to n **do**
if $\Gamma^{(\alpha,\beta)}_{P'-\{a_i\}}(U/D) \neq \Gamma^{(\alpha,\beta)}_{P'}(U/D)$ **then** $CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(P') = CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(P') \cup \{a_i\}$
end if end for
Return $CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(P')$
end begin

Algorithm 1 and Algorithm 2 described the methods of incremental updating of attribute core in PRS. First, the lower and upper approximations under the new attribute set are calculated when the attributes are added or deleted in PRSs. Then, the partitioning of the object sets under each attribute is calculated. The upper and lower approximation sets of all conditional attributes in PRS model are calculated. Then, calculate the PAA $\Gamma_{C-\{a_i\}}^{(\alpha,\beta)}(U/D)$ when a single attribute a_i is deleted from the conditional attribute set. Finally, if $\Gamma_{C-\{a_i\}}^{(\alpha,\beta)}(U/D) \neq \Gamma_{C}^{(\alpha,\beta)}(U/D)$, then a_i belongs to attribute core.

On the basis of Algorithm 1 and Algorithm 2, an accelerated algorithm of attribute reduction in PRS model is put forward according to the incremental calculation of the MPAA. We outline the incremental attribute reduction algorithm in Algorithm 3.

Algorithm 3. An incremental attribute reduction algorithm in PRS.

Input (1) $D = (U, C \cup D)$, $U/D = \{T_1, T_2, \ldots, T_M\}$, $C = \{a_1, a_2, \ldots, a_n\}$, the thresholds α and β; (2) The results already calculated in Algorithm 1, Include $\underline{P}(T_j)$, $\overline{P}(T_j)$, U/a_i and $\underline{AS}_{\{a_i\}}(\underline{P}(T_j))$, $\overline{AS}_{\{a_i\}}\left(\overline{P}(T_j)\right)$, $\underline{\Delta}_{\{a_i\}}(\underline{P}(T_j))$, $\overline{\Delta}_{\{a_i\}}\left(\overline{P}(T_j)\right)$, where $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, M\}$; (3) The attribute core $CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(C)$.

Output Minimal attribute reduction of PRS R.

begin Let $R = CORE_{\Gamma^{(\alpha,\beta)}(U/D)}(C)$, $CA = C - R$, Calculate $\Gamma_R^{(\alpha,\beta)}(U/D)$
while $\Gamma_R^{(\alpha,\beta)}(U/D) \neq \Gamma_C^{(\alpha,\beta)}(U/D)$ **do**
  for each $a_i \in CA$ **do** Calculate $\Omega_{R\cup\{a_i\}}^{(\alpha,\beta)}(U/D, m(\cdot))$ **end for**
  if the value of $\Omega_{R\cup\{a_i\}}^{(\alpha,\beta)}(U/D, m(\cdot))$ is greatest when the attribute a_i is added to the set R **then** Let $R = R \cup \{a_i\}$, $CA = CA - \{a_i\}$ **end if**
end while
// Calculate minimal attribute reduction
Let $CD = R$. **for** each $a_i \in CD$ **do** Calculate $\Omega_{\{a_i\}}^{(\alpha,\beta)}(U/D, m(\cdot))$ **end for**
Sort attributes in CD according to $\Omega_{\{a_i\}}^{(\alpha,\beta)}(U/D, m(\cdot))$ in an ascending order
while $CD \neq \emptyset$ **do** $CD = CD - \{a\}$, where a is the first element in CD
  if $\Gamma_{R-\{a\}}^{(\alpha,\beta)}(U/D) = \Gamma_C^{(\alpha,\beta)}(U/D)$ **then** $R = R - \{a\}$ **end if**
end while
Return R
end begin

Algorithm 3 describes an accelerated algorithm of attribute reduction in PRS model. Use the calculation results of Algorithm 1 (when adding attributes) or Algorithm 2 (when deleting attributes) to calculate the MPAA $\Omega_{R\cup\{a_i\}}^{(\alpha,\beta)}(U/D, m(\cdot))$. Find out the attribute a_i which makes the maximum accuracy of the MPAA and add it to the attribute set R until $\Gamma_R^{(\alpha,\beta)}(U/D) = \Gamma_C^{(\alpha,\beta)}(U/D)$, then R is an attribute reduction. At last, the MPAA $\Omega_{\{a_i\}}^{(\alpha,\beta)}(U/D, m(\cdot))$ of each attribute in R is calculated by using the results of Algorithm 1 (when adding attributes) or Algorithm 2 (when deleting attributes). The attributes are sorted according to their importance, and the redundant attributes in attribute reduction R are removed in this order to get the minimum attribute reduction.

According to Algorithms 1, 2 and 3, we can get the steps to calculate the minimum attribute reduction when the attribute set varies as follows.

Step 1. Calculate the attribute core according to Algorithm 1 when adding attributes. Or calculate the attribute core according to Algorithm 2 when deleting attributes.

Step 2. Calculate minimum attribute reduction. Based on the attribute core calculated in Step 1, minimum attribute reduction is calculated according to Algorithm 3.

4. **An Example.** In order to better explain the calculation process of the incremental minimum attribute reduction algorithm proposed in this paper, in this section, we use the example in [20] to illustrate. Given a DS $D = (U, C \cup D)$ as shown in Table 1, $U = \{y_1, y_2, \ldots, y_{10}\}$. We set $\alpha = 0.75$, $\beta = 0.60$.

TABLE 1. A complete information system

U	a_1	a_2	a_3	a_4	a_5	a_6	d
y_1	0	0	0	1	1	1	1
y_2	0	0	0	1	1	1	1
y_3	0	0	0	1	1	1	0
y_4	0	1	0	0	0	0	0
y_5	1	1	1	1	0	1	1
y_6	1	1	1	0	1	0	1
y_7	1	1	1	1	0	0	1
y_8	0	0	1	0	0	0	0
y_9	0	0	1	0	0	0	0
y_{10}	0	0	1	0	0	1	1

4.1. **Incremental calculation of attribute core when attribute set increases.** Let $P = \{a_1, a_2, a_3, a_4\}$, $Q = \{a_5, a_6\}$. $C = P \cup Q$. From Table 1, we have $U/P = \{\{y_1, y_2, y_3\}, \{y_4\}, \{y_5, y_7\}, \{y_6\}, \{y_8, y_9, y_{10}\}\}$, $U/D = \{T_1, T_2\} = \{\{y_1, y_2, y_5, y_6, y_7, y_{10}\}, \{y_3, y_4, y_8, y_9\}\}$. $\underline{AS}_P(T_1) = \{y_5, y_6, y_7\}$, $\overline{AS}_P(T_1) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}$, $\underline{AS}_P(T_2) = \{y_4\}$, $\overline{AS}_P(T_2) = \{y_1, y_2, y_3, y_4, y_8, y_9, y_{10}\}$.

So $BND_P(T_1) = \{y_1, y_2, y_3, y_8, y_9, y_{10}\}$, $BND_P(T_2) = \{y_1, y_2, y_3, y_8, y_9, y_{10}\}$. Then, $BND_P(T_1)/P = \{\{y_1, y_2, y_3\}, \{y_8, y_9, y_{10}\}\}$, $BND_P(T_2)/P = \{\{y_1, y_2, y_3\}, \{y_8, y_9, y_{10}\}\}$.

According to Algorithm 1, in the PRS model, when attribute set Q is added to attribute set P, the incremental calculation process and results of the upper and lower approximation sets of conditional attributes $C = P \cup Q$ are as follows: $BND_{P_1}(T_1)/Q = \{BND_{P_{11}}(T_1)\} = \{y_1, y_2, y_3\}$, $BND_{P_2}(T_1)/Q = \{BND_{P_{21}}(T_1), BND_{P_{22}}(T_1)\} = \{\{y_8, y_9\}, \{y_{10}\}\}$. $BND_{P_1}(T_2)/Q = \{BND_{P_{11}}(T_2)\} = \{y_1, y_2, y_3\}$, $BND_{P_2}(T_2)/Q = \{BND_{P_{21}}(T_2), BND_{P_{22}}(T_2)\} = \{\{y_8, y_9\}, \{y_{10}\}\}$.

We obtain: $P_r(T_1|BND_{P_{11}}(T_1)) = \frac{|\{y_1,y_2,y_3\} \cap \{y_1,y_2,y_5,y_6,y_7,y_{10}\}|}{|\{y_1,y_2,y_3\}|} = \frac{2}{3}$, $P_r(T_1|BND_{P_{21}}(T_1)) = \frac{|\{y_8,y_9\} \cap \{y_1,y_2,y_5,y_6,y_7,y_{10}\}|}{|\{y_8,y_9\}|} = 0$, $P_r(T_1|BND_{P_{22}}(T_1)) = \frac{|\{y_{10}\} \cap \{y_1,y_2,y_5,y_6,y_7,y_{10}\}|}{|\{y_{10}\}|} = 1$. $P_r\big(T_2|BND_{P_{11}}(T_2)\big) = \frac{|\{y_1,y_2,y_3\} \cap \{y_3,y_4,y_8,y_9\}|}{|\{y_1,y_2,y_3\}|} = \frac{1}{3}$, $P_r(T_2|BND_{P_{21}}(T_2)) = \frac{|\{y_8,y_9\} \cap \{y_3,y_4,y_8,y_9\}|}{|\{y_8,y_9\}|} = 1$, $P_r(T_2|BND_{P_{22}}(T_2)) = \frac{|\{y_{10}\} \cap \{y_3,y_4,y_8,y_9\}|}{|\{y_{10}\}|} = 0$.

So $\underline{AS}_C^{(0.75,0.60)}(T_1) = \{y_5, y_6, y_7, y_{10}\}$, $\overline{AS}_C^{(0.75,0.60)}(T_1) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_{10}\}$. $\underline{AS}_C^{(0.75,0.60)}(T_2) = \{y_4, y_8, y_9\}$, $\overline{AS}_C^{(0.75,0.60)}(T_2) = \{y_4, y_8, y_9\}$.

Let $\underline{P}(T_1) = \underline{AS}_C^{(0.75,0.60)}(T_1)$, $\overline{P}(T_1) = \overline{AS}_C^{(0.75,0.60)}(T_1)$, $\underline{P}(T_2) = \underline{AS}_C^{(0.75,0.60)}(T_2)$ and $\overline{P}(T_2) = \overline{AS}_C^{(0.75,0.60)}(T_2)$. According to Table 1, the equivalence classes of each condition attribute are calculated as follows:
$U/a_1 = \{\{y_1, y_2, y_3, y_4, y_8, y_9, y_{10}\}, \{y_5, y_6, y_7\}\}$, $U/a_2 = \{\{y_1, y_2, y_3, y_8, y_9, y_{10}\}, \{y_4, y_5, y_6, y_7\}\}$, $U/a_3 = \{\{y_1, y_2, y_3, y_4\}, \{y_5, y_6, y_7, y_8, y_9, y_{10}\}\}$, $U/a_4 = \{\{y_1, y_2, y_3, y_5, y_7\}, \{y_4, y_6, y_8, y_9, y_{10}\}\}$, $U/a_5 = \{\{y_1, y_2, y_3, y_6\}, \{y_4, y_5, y_7, y_8, y_9, y_{10}\}\}$, $U/a_6 = \{\{y_1, y_2, y_3, y_5, y_{10}\}, \{y_4, y_6, y_7, y_8, y_9\}\}$, $U/C = \bigcap_{i=1}^{6} U/a_i = \{\{y_1, y_2, y_3\}, \{y_4\}, \{y_5\}, \{y_6\}, \{y_7\}, \{y_8, y_9\}, \{y_{10}\}\}$.

The upper and lower approximation sets of condition attributes C in Pawlak rough set are calculated as follows: $\underline{AS}_C(\underline{P}(T_1)) = \{y_5, y_6, y_7, y_{10}\}$, $\underline{AS}_C(\underline{P}(T_2)) = \{y_4, y_8, y_9\}$. $\overline{AS}_C\big(\overline{P}(T_1)\big) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_{10}\}$, $\overline{AS}_C\big(\overline{P}(T_2)\big) = \{y_4, y_8, y_9\}$.

According to Definition 2.3, it is calculated $\Gamma_C^{(0.75,0.60)}(U/D) = \frac{7}{10}$. The upper and lower approximation sets of Pawlak rough set for a single attribute are as follows: $\underline{AS}_{\{a_1\}}(\underline{P}(T_1)) = \{y_5, y_6, y_7\}$; $\underline{AS}_{\{a_i\}}(\underline{P}(T_1)) = \emptyset$, where $i = \{2,3,4,5,6\}$; $\underline{AS}_{\{a_i\}}(\underline{P}(T_2)) = \emptyset$, where $i = \{1,2,3,4,5,6\}$; $\overline{AS}_{\{a_i\}}\left(\overline{P}(T_1)\right) = U$, where $i = \{1,2,3,4,5,6\}$; $\overline{AS}_{\{a_1\}}\left(\overline{P}(T_2)\right) = \{y_1, y_2, y_3, y_4, y_8, y_9, y_{10}\}$; $\overline{AS}_{\{a_2\}}\left(\overline{P}(T_2)\right) = \overline{AS}_{\{a_3\}}\left(\overline{P}(T_2)\right) = U$; $\overline{AS}_{\{a_4\}}\left(\overline{P}(T_2)\right) = \{y_4, y_6, y_8, y_9, y_{10}\}$; $\overline{AS}_{\{a_5\}}\left(\overline{P}(T_2)\right) = \{y_4, y_5, y_7, y_8, y_9, y_{10}\}$; $\overline{AS}_{\{a_6\}}\left(\overline{P}(T_2)\right) = \{y_4, y_6, y_7, y_8, y_9\}$.

According to Definition 3.1, the upper and lower boundary sets of a single attribute are as follows: $\underline{\Delta}_{\{a_1\}}(\underline{P}(T_1)) = \{y_{10}\}$; $\underline{\Delta}_{\{a_i\}}(\underline{P}(T_1)) = \{y_5, y_6, y_7, y_{10}\}$, where $i = \{2,3,4,5,6\}$; $\underline{\Delta}_{\{a_i\}}(\underline{P}(T_2)) = \{y_4, y_8, y_9\}$, where $i = \{1,2,3,4,5,6\}$; $\overline{\Delta}_{\{a_i\}}\left(\overline{P}(T_1)\right) = \{y_4, y_8, y_9\}$, where $i = \{1,2,3,4,5,6\}$; $\overline{\Delta}_{\{a_1\}}\left(\overline{P}(T_2)\right) = \{y_1, y_2, y_3, y_{10}\}$; $\overline{\Delta}_{\{a_2\}}\left(\overline{P}(T_2)\right) = \overline{\Delta}_{\{a_3\}}\left(\overline{P}(T_2)\right) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_{10}\}$; $\overline{\Delta}_{\{a_4\}}\left(\overline{P}(T_2)\right) = \{y_6, y_{10}\}$; $\overline{\Delta}_{\{a_5\}}\left(\overline{P}(T_2)\right) = \{y_5, y_7, y_{10}\}$; $\overline{\Delta}_{\{a_6\}}\left(\overline{P}(T_2)\right) = \{y_6, y_7\}$.

According to Lemma 3.3, the lower approximation set of attribute set after deleting an attribute is as follows: $\underline{AS}_{C-\{a_1\}}(\underline{P}(T_1)) = \{y_5, y_6, y_7, y_{10}\}$; $\underline{AS}_{C-\{a_6\}}(\underline{P}(T_1)) = \{y_5, y_6, y_7\}$; $\underline{AS}_{C-\{a_i\}}(\underline{P}(T_1)) = \{y_5, y_6, y_7, y_{10}\}$, where $i \in \{2,3,4,5\}$. $\underline{AS}_{C-\{a_i\}}(\underline{P}(T_2)) = \{y_4, y_8, y_9\}$, where $i \in \{1,2,3,4,5\}$; $\underline{AS}_{C-\{a_6\}}(\underline{P}(T_2)) = \{y_4\}$.

According to Lemma 3.4, the upper approximation set of attribute set after deleting an attribute is as follows: $\overline{AS}_{C-\{a_1\}}\left(\overline{P}(T_1)\right) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_{10}\}$, $\overline{AS}_{C-\{a_i\}}\left(\overline{P}(T_1)\right) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_{10}\}$, where $i \in \{2,3,4,5\}$; $\overline{AS}_{C-\{a_6\}}\left(\overline{P}(T_1)\right) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}$. $\overline{AS}_{C-\{a_i\}}\left(\overline{P}(T_2)\right) = \{y_4, y_8, y_9\}$, where $i \in \{1,2,3,4,5\}$; $\overline{AS}_{C-\{a_6\}}\left(\overline{P}(T_2)\right) = \{y_4, y_8, y_9, y_{10}\}$.

According to Theorem 3.8, we can get $\Gamma_{C-\{a_i\}}^{(0.75,0.60)}(U/D) = \frac{7}{10}$, where $i \in \{1,2,3,4,5\}$; $\Gamma_{C-\{a_6\}}^{(0.75,0.60)}(U/D) = \frac{4}{13}$. According to the judgment method of attribute core in Algorithm 1, the attribute core of decision system is $\{a_6\}$.

4.2. **Calculation of minimum attribute reduction.** According to Algorithm 3 and the calculation results in Section 4.1, let $R = \{a_6\}$, $CA = \{a_1, a_2, a_3, a_4, a_5\}$. According to Definition 2.3, we get $\Gamma_{\{a_6\}}^{(0.75,0.60)}(U/D) = 0 \neq \Gamma_C^{(0.75,0.60)}(U/D)$.

According to Theorem 3.9, we get the MPAA as follows: $\Omega_{R\cup\{a_1\}}^{(0.75,0.60)}(U/D, m(\cdot)) = EG_m(U) - \left(1 - \Gamma_{R\cup\{a_1\}}^{(0.75,0.60)}(U/D)\right) EG_m(U/R \cup \{a_1\})$, where $\Gamma_{R\cup\{a_1\}}^{(0.75,0.60)}(U/D) = \frac{\sum_{T_j \in U/D}\left|\underline{AS}_{R\cup\{a_1\}}(\underline{P}(T_j))\right|}{\sum_{T_j \in U/D}\left|\overline{AS}_{R\cup\{a_1\}}\left(\overline{P}(T_j)\right)\right|}$. Because according to Lemma 3.1, there are $\underline{AS}_{R\cup\{a_1\}}(\underline{P}(T_1)) = \underline{AS}_{\{a_6\}}(\underline{P}(T_1)) \cup \underline{AS}_{\{a_1\}}(\underline{P}(T_1)) \cup H$, where $H = \{y \in \underline{\Delta}_{\{a_6\}}(\underline{P}(T_1)) \cap \underline{\Delta}_{\{a_1\}}(\underline{P}(T_1)) \mid \bigcap_{b\in\{a_1,a_6\}} [y]_b \subseteq \underline{P}(T_1)\} = \emptyset$, $\underline{AS}_{R\cup\{a_1\}}(\underline{P}(T_1)) = \{y_5, y_6, y_7\}$.

In the same way, we can get $\underline{AS}_{R\cup\{a_2\}}(\underline{P}(T_1)) = \{y_5\}$, $\underline{AS}_{R\cup\{a_3\}}(\underline{P}(T_1)) = \{y_5, y_{10}\}$, $\underline{AS}_{R\cup\{a_4\}}(\underline{P}(T_1)) = \{y_7, y_{10}\}$, $\underline{AS}_{R\cup\{a_5\}}(\underline{P}(T_1)) = \{y_5, y_6, y_{10}\}$. $\underline{AS}_{R\cup\{a_1\}}(\underline{P}(T_2)) = \{y_4, y_8, y_9\}$, $\underline{AS}_{R\cup\{a_2\}}(\underline{P}(T_2)) = \{y_8, y_9\}$, $\underline{AS}_{R\cup\{a_3\}}(\underline{P}(T_2)) = \{y_4\}$, $\underline{AS}_{R\cup\{a_4\}}(\underline{P}(T_2)) = \emptyset$, $\underline{AS}_{R\cup\{a_5\}}(\underline{P}(T_2)) = \emptyset$.

According to Lemma 3.2, we can get: $\overline{AS}_{R\cup\{a_1\}}\left(\overline{P}(T_1)\right) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_{10}\}$, $\overline{AS}_{R\cup\{a_2\}}\left(\overline{P}(T_1)\right) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_{10}\}$, $\overline{AS}_{R\cup\{a_3\}}\left(\overline{P}(T_1)\right) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}$, $\overline{AS}_{R\cup\{a_4\}}\left(\overline{P}(T_1)\right) = \overline{AS}_{R\cup\{a_5\}}\left(\overline{P}(T_1)\right) = U$. $\overline{AS}_{R\cup\{a_1\}}\left(\overline{P}(T_2)\right) = \{y_4, y_8, y_9\}$, $\overline{AS}_{R\cup\{a_2\}}\left(\overline{P}(T_2)\right) = \overline{AS}_{R\cup\{a_3\}}\left(\overline{P}(T_2)\right) = \{y_4, y_6, y_7, y_8, y_9\}$, $\overline{AS}_{R\cup\{a_4\}}\left(\overline{P}(T_2)\right) = \{y_4, y_6, y_8, y_9\}$, $\overline{AS}_{R\cup\{a_5\}}\left(\overline{P}(T_2)\right) = \{y_4, y_7, y_8, y_9\}$. So $\Gamma_{R\cup\{a_1\}}^{(0.75,0.60)}(U/D) = \frac{3}{5}$, $\Gamma_{R\cup\{a_2\}}^{(0.75,0.60)}(U/D) = \frac{3}{13}$, $\Gamma_{R\cup\{a_3\}}^{(0.75,0.60)}(U/D) = \frac{3}{14}$, $\Gamma_{R\cup\{a_4\}}^{(0.75,0.60)}(U/D) = \frac{1}{7}$, $\Gamma_{R\cup\{a_5\}}^{(0.75,0.60)}(U/D) = \frac{3}{14}$.

According to Definition 2.4, we can get $EG_m(U) = 1$, $EG_m(U/R \cup \{a_1\}) = \frac{4^2+3^2+1^2+2^2}{100} = \frac{3}{10}$, $EG_m(U/R \cup \{a_2\}) = EG_m(U/R \cup \{a_3\}) = EG_m(U/R \cup \{a_5\}) = \frac{3}{10}$, $EG_m(U/R \cup \{a_4\}) = \frac{17}{50}$. Then $\Omega^{(0.75,0.60)}_{R\cup\{a_1\}}(U/D, m(\cdot)) = \frac{22}{25} = 0.88$, $\Omega^{(0.75,0.60)}_{R\cup\{a_2\}}(U/D, m(\cdot)) = \frac{10}{13} = 0.77$, $\Omega^{(0.75,0.60)}_{R\cup\{a_3\}}(U/D, m(\cdot)) = \frac{107}{140} = 0.76$, $\Omega^{(0.75,0.60)}_{R\cup\{a_4\}}(U/D, m(\cdot)) = \frac{124}{175} = 0.71$, $\Omega^{(0.75,0.60)}_{R\cup\{a_5\}}(U/D, m(\cdot)) = \frac{107}{140} = 0.76$. Here $\Omega^{(0.75,0.60)}_{R\cup\{a_1\}}(U/D, m(\cdot))$ has the largest value, so let $R = \{a_6\} \cup \{a_1\}$, $CA = \{a_2, a_3, a_4, a_5\}$.

Because $\Gamma^{(0.75,0.60)}_{\{a_1,a_6\}}(U/D) = \frac{3}{5} \neq \Gamma^{(0.75,0.60)}_{C}(U/D)$, according to the above calculation method of MPAA, the cyclic calculation results are as follows: $\Gamma^{(0.75,0.60)}_{R\cup\{a_2\}}(U/D) = \frac{3}{5}$, $\Gamma^{(0.75,0.60)}_{R\cup\{a_i\}}(U/D) = \frac{7}{10}$, where $i \in \{3, 4, 5\}$. $EG_m(U/R \cup \{a_2\}) = \frac{13}{50}$, $EG_m(U/R \cup \{a_3\}) = \frac{1}{5}$, $EG_m(U/R \cup \{a_4\}) = EG_m(U/R \cup \{a_5\}) = \frac{11}{50}$. And then by calculating, we can get $\Omega^{(0.75,0.60)}_{R\cup\{a_2\}}(U/D, m(\cdot)) = \frac{112}{125} = 0.90$, $\Omega^{(0.75,0.60)}_{R\cup\{a_3\}}(U/D, m(\cdot)) = \frac{47}{50} = 0.94$, $\Omega^{(0.75,0.60)}_{R\cup\{a_4\}}(U/D, m(\cdot)) = \Omega^{(0.75,0.60)}_{R\cup\{a_5\}}(U/D, m(\cdot)) = \frac{467}{500} = 0.93$.

The MPAA of object a_3 is the maximum, so add a_3 to attribute set R to form a new attribute reduction. It can be seen from the above calculation results, that $\Gamma^{(0.75,0.60)}_{R\cup\{a_3\}}(U/D) = \frac{7}{10} = \Gamma^{(0.75,0.60)}_{C}(U/D)$. So the attribute reduction of decision Table 1 from Algorithm 1 and Algorithm 3 is $R = \{a_1, a_3, a_6\}$.

4.3. **Incremental update attribute core when deleting attributes.** Let $P' = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $Q' = \{a_5, a_6\}$. $B = P' - Q'$. From Table 1, we have $U/P' = \{\{y_1, y_2, y_3\}, \{y_4\}, \{y_5\}, \{y_6\}, \{y_7\}, \{y_8, y_9\}, \{y_{10}\}\}$, $U/D = \{T_1, T_2\} = \{\{y_1, y_2, y_5, y_6, y_7, y_{10}\}, \{y_3, y_4, y_8, y_9\}\}$. $\underline{AS}_{P'}(T_1) = \{y_5, y_6, y_7, y_{10}\}$, $\overline{AS}_{P'}(T_1) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_{10}\}$, $\underline{AS}_{P'}(T_2) = \{y_4, y_8, y_9\}$, $\overline{AS}_{P'}(T_2) = \{y_1, y_2, y_3, y_4, y_8, y_9\}$.

So $POS_{P'}(T_1) = \{y_5, y_6, y_7, y_{10}\}$, $BND_{P'}(T_1) = \overline{AS}_{P'}(T_1) - \underline{AS}_{P'}(T_1) = \{y_1, y_2, y_3\}$, $NEG_{P'}(T_1) = U - \overline{AS}_{P'}(T_1) = \{y_4, y_8, y_9\}$. $POS_{P'}(T_2) = \{y_4, y_8, y_9\}$, $BND_{P'}(T_2) = \overline{AS}_{P'}(T_2) - \underline{AS}_{P'}(T_2) = \{y_1, y_2, y_3\}$, $NEG_{P'}(T_2) = U - \overline{AS}_{P'}(T_2) = \{y_5, y_6, y_7, y_{10}\}$.

Then, $POS_{P'}(T_1)/P' = \{\{y_5\}, \{y_6\}, \{y_7\}, \{y_{10}\}\}$, $BND_{P'}(T_1)/P' = \{y_1, y_2, y_3\}$, $NEG_{P'}(T_1)/P' = \{\{y_4\}, \{y_8, y_9\}\}$. $POS_{P'}(T_2)/P' = \{\{y_4\}, \{y_8, y_9\}\}$, $BND_{P'}(T_2)/P' = \{y_1, y_2, y_3\}$, $NEG_{P'}(T_2)/P' = \{\{y_5\}, \{y_6\}, \{y_7\}, \{y_{10}\}\}$.

According to Algorithm 2, We have $\Theta_1(T_1) = \underline{B}(T_1) = \{y_5, y_6, y_7\}$, $\{y_{10}\} \in \Theta_2(T_1)$. Since $[y_{10}]_B = \{y_8, y_9, y_{10}\}$, $\Theta_2(T_1) = \{y_8, y_9, y_{10}\}$, $\Theta_3(T_1) = \emptyset$. In the same way, we can get $\Theta_1(T_2) = \underline{B}(T_2) = \{y_4\}$, $\Theta_2(T_2) = \{y_8, y_9, y_{10}\}$, $\Theta_3(T_2) = \emptyset$. Hence, $BND_B(T_1) = BND_{P'}(T_1) \cup \Theta_2(T_1) \cup \Theta_3(T_1) = \{y_1, y_2, y_3, y_8, y_9, y_{10}\}$, $BND_B(T_2) = BND_{P'}(T_2) \cup \Theta_2(T_2) \cup \Theta_3(T_2) = \{y_1, y_2, y_3, y_8, y_9, y_{10}\}$. $\overline{B}(T_1) = \Theta_1(T_1) \cup BND_B(T_1) = \{y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}\}$, $\overline{B}(T_2) = \Theta_1(T_2) \cup BND_B(T_2) = \{y_1, y_2, y_3, y_4, y_8, y_9, y_{10}\}$. $BND_B(T_1)/B = \{BND_{B_1}(T_1), BND_{B_2}(T_1)\} = \{\{y_1, y_2, y_3\}, \{y_8, y_9, y_{10}\}\}$. $BND_B(T_2)/B = \{BND_{B_1}(T_2), BND_{B_2}(T_2)\} = \{\{y_1, y_2, y_3\}, \{y_8, y_9, y_{10}\}\}$.

Next, we can obtain: $P_r(T_1|BND_{B_1}(T_1)) = \frac{|\{y_1,y_2,y_3\}\cap\{y_1,y_2,y_5,y_6,y_7,y_{10}\}|}{|\{y_1,y_2,y_3\}|} = \frac{2}{3}$, $P_r(T_1|BND_{B_2}(T_1)) = \frac{|\{y_8,y_9,y_{10}\}\cap\{y_1,y_2,y_5,y_6,y_7,y_{10}\}|}{|\{y_8,y_9,y_{10}\}|} = \frac{1}{3}$, $P_r(T_2|BND_{B_1}(T_2)) = \frac{|\{y_1,y_2,y_3\}\cap\{y_3,y_4,y_8,y_9\}|}{|\{y_1,y_2,y_3\}|} = \frac{1}{3}$, $P_r(T_2|BND_{B_2}(T_2)) = \frac{|\{y_8,y_9,y_{10}\}\cap\{y_3,y_4,y_8,y_9\}|}{|\{y_8,y_9,y_{10}\}|} = \frac{2}{3}$. So $\underline{AS}^{(0.75,0.60)}_B(T_1) = \underline{B}(T_1) \cup \{BND_{B_i}|P_r(T_1|BND_{B_i}) \geq 0.75\} = \{y_5, y_6, y_7\}$, $\overline{AS}^{(0.75,0.60)}_B(T_1) = \overline{B}(T_1) - \{BND_{B_i}|P_r(T_1|BND_{B_i}) \leq 0.60\} = \{y_1, y_2, y_3, y_5, y_6, y_7\}$. $\underline{AS}^{(0.75,0.60)}_B(T_2) = \underline{B}(T_2) \cup \{BND_{B_i}|P_r(T_2|BND_{B_i}) \geq 0.75\} = \{y_4\}$, $\overline{AS}^{(0.75,0.60)}_B(T_2) = \overline{B}(T_2) - \{BND_{B_i}|P_r(T_2|BND_{B_i}) \leq 0.60\} = \{y_4, y_8, y_9, y_{10}\}$, where $i = \{1, 2\}$.

When the attributes decrease, after incrementally updating the upper and lower approximation sets of the PRS, the steps to calculate the attribute core and the minimum attribute reduction are the same as the steps in Sections 4.1 and 4.2, and will not be repeated here.

5. **Comparative Analysis.** In order to verify that the proposed incremental algorithm (IA) for attribute reduction in PRS under the variation of attributes is more efficient than the non-incremental algorithm (NIA) for attribute reduction. The four data sets and the examples in this paper are used to compare the proposed incremental method with the non-incremental method, and compare the time consumed. The hardware and software environment of the experiment is as follows: CPU Intel Celeron single core 2.6GHz, Memory 2.0GB, Win 7 system, C++ development platform of Visual Studio 2013. Table 2 shows the basic information of the four data sets and the data sets in the example in this paper. Then, Table 3 shows the comparison of the average running time between Algorithm 1 and NIA after adding attributes, and Table 4 shows the comparison of average running time between Algorithm 2 and NIA after deleting attributes.

Table 2. The basic information of the four data sets and the data sets in example in this paper

Data sets	Number of objects	Number of attributes
Cup	209	6
IRIS	150	4
Soybean	682	35
Balance-scale	625	4
Our example	10	6

Table 3. Comparison of the average running time between NIA and Algorithm 1 after adding attributes

Data sets	P	Q	NIA (ms)	Algorithm 1 (ms)
Cup	$\{a_1, a_2, a_3\}$	$\{a_4, a_5, a_6\}$	1725	805
IRIS	$\{a_1, a_2, a_3\}$	$\{a_4\}$	1023	455
Soybean	$\{a_4, a_5, \ldots, a_{18}\}$	$\{a_1, a_2, a_3\}$	9239	4158
Balance-scale	$\{a_1, a_2\}$	$\{a_3\}$	3546	1622
Our example	$\{a_1, a_2, a_3, a_4\}$	$\{a_5, a_6\}$	1401	730

Table 4. Comparison of the average running time between NIA and Algorithm 2 after deleting attributes

Data sets	P'	Q'	NIA (ms)	Algorithm 2 (ms)
Cup	$\{a_1, a_2, a_3, a_4, a_5, a_6\}$	$\{a_4, a_5, a_6\}$	651	562
IRIS	$\{a_1, a_2, a_3, a_4\}$	$\{a_4\}$	404	275
Soybean	$\{a_1, a_2, \ldots, a_{18}\}$	$\{a_1, a_2, a_3\}$	6077	3685
Balance-scale	$\{a_1, a_2, a_3\}$	$\{a_3\}$	2496	1254
Our example	$\{a_1, a_2, a_3, a_4, a_5, a_6\}$	$\{a_5, a_6\}$	658	470

By comparing with the existing literature, we can draw the following conclusions.

1) In the calculation results of the example, the attribute core is $\{a_6\}$ and the minimum attribute reduction is $R = \{a_1, a_3, a_6\}$, which is consistent with the calculation result in

[20], which proves that the incremental calculation method proposed in this paper is feasible.

2) It can be seen from Tables 3 and 4 that the algorithm proposed in this paper to incrementally calculate the minimum attribute reduction has lower time complexity.

6. **Conclusions.** This paper proposes an accelerated attribute reduction algorithm in PRS using incremental learning. The set of attributes in IS will change continuously over time. When the attribute set varies, the upper and lower approximations will change dynamically. This paper introduces a new strategy for dynamically updating the upper and lower approximations in PRS. Under the PRS model, based on adding attributes and deleting attributes, two methods for incrementally calculating attribute cores and a method for calculating minimum attributes are proposed, respectively. This method uses the method of incrementally updating the upper and lower approximations from the Pawlak rough set to calculate the upper and lower approximations when a single attribute increases or decreases in the PRS. Further obtain the PAA and MPAA. The attribute core is obtained by comparing the values of the PAA. Then, based on the attribute core, the attribute reduction in the PRS model is quickly solved by comparing the values of the MPAA. Theoretical analysis and examples show that the attribute reduction algorithm proposed in this paper is feasible. The next work will explore the incremental reduction algorithm when the object set changes under the PRS model.

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