International Journal of Innovative Computing, Information and Control
Volume **16**, Number **6**, December **2020**

ICIC International ©2020 ISSN 1349-4198
pp. 2121–2130

SEMIGROUP CHARACTERIZED BY PICTURE FUZZY SETS

Pairote Yiarayong

Department of Mathematics
Faculty of Science and Technology
Pibulsongkram Rajabhat University
156 Moo 5, Tambon Phlaichumphon, Muang District, Phitsanulok 65000, Thailand
pairote0027@hotmail.com

Received June 2020; revised October 2020

Abstract. *In this paper, we apply the concept of picture fuzzy sets to semigroup theory. We characterize different classes regular (resp. intra-regular and semisimple) semigroups in terms of picture fuzzy left and right ideals (resp. picture fuzzy ideals). In this regard, we prove that a non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the picture characteristic function on S is the picture fuzzy subsemigroup (picture fuzzy left ideal, picture fuzzy right ideal, picture fuzzy two-sided ideal) on S.*

Keywords: Picture fuzzy set, Left (right) regular, Idempotent, Intra-regular, Semisimple

1. **Introduction.** A **picture fuzzy set** A on a universe X is an object having the form $A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle : x \in S\}$ where the functions $\mu_A : S \to [0, 1]$, $\eta_A : S \to [0, 1]$ and $\nu_A : S \to [0, 1]$ denote the degree of **positive membership**, **neutral membership** and the degree of **negative membership**, respectively, and $0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$ for all $x \in S$. The concept of picture fuzzy sets (PFS) was first introduced by Cuong in 2014 [2], which is a generalization of the concept of fuzzy sets and intuitionistic fuzzy sets. In 2019, Makhmudov and Ko [9] studied the interval transportation problem with multiple objectives and developed a compromise conflict resolution approach based on the fuzzy set theory.

In 2010, Xie and Tang [19] discussed some characterizations of regular ordered semigroups and intra-regular ordered semigroups in terms of fuzzy left ideals, fuzzy right ideals, fuzzy bi-ideals or fuzzy quasi-ideals of ordered semigroups. Many other researchers used the idea of fuzzy sets and gave several characterizations results in different branches of algebra, for example (see [5, 7, 8, 10, 11, 12, 13, 14, 15, 18, 21]). In 2011, Shabir et al. [16] characterized regular semigroup by their $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideals, fuzzy ideals, bi-ideals and generalized bi-ideals. In 2012, Khan et al. [6] introduced the concept of $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals in ordered semigroups, which is a generalization of the concept of (α, β)-fuzzy generalized bi-ideals in ordered semigroups and gave some basic properties of regular, left (resp. right) regular, completely regular and weakly regular ordered semigroups in terms of this notion. In 2013, Feng et al. [3] characterized regular semigroups and right weakly regular semigroups by the properties of $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals, $(\in, \in \vee q_k)$-fuzzy interior ideals, $(\in, \in \vee q_k)$-fuzzy left ideals and $(\in, \in \vee q_k)$-fuzzy right ideals. In 2014, Aslam et al. [1] defined the lower and upper parts of interval-valued fuzzy subsets of LA-semigroups and characterized regular

DOI: 10.24507/ijicic.16.06.2121

LA-semigroups by the properties of interval-valued $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy left ideals, interval-valued $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy quasi-ideals and interval-valued $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy generalized bi-ideals. The concept of $(\in, \in \vee q_k)$-fuzzy (left, right, bi-) ideals of ordered Abel Grassman's groupoids (AG-groupoid) by Yousafzai et al. [22] and they characterized intra-regular ordered AG-groupoids by the properties of generalized fuzzy ideals. In 2017, Khan et al. [4] initiated the notion of $\left(\in, \in \vee q_k^\delta\right)$-fuzzy left ideals and $\left(\in, \in \vee q_k^\delta\right)$-fuzzy right ideals of semigroups. They characterized semigroups (regular, left weakly regular, right weakly regular) by using the concept of $\left(\in, \in \vee q_k^\delta\right)$-fuzzy left ideals, $\left(\in, \in \vee q_k^\delta\right)$-fuzzy right ideals and $\left(\in, \in \vee q_k^\delta\right)$-fuzzy ideals. In 2018, Siripitukdet [17] characterized regular semigroups by the properties of fuzzy generalized bi-ideals, fuzzy bi-ideals, fuzzy left ideals, fuzzy right ideals and fuzzy ideals. In 2020, Yiarayong [20] discussed some characterizations of left (right) regular, completely regular, intra-regular, regular, left (right) quasi-regular, quasi-regular intra-regular, left quasi-regular and semisimple semigroups in terms of hesitant fuzzy ideals of semigroups.

The purpose of this paper is to deal with the algebraic structure of semigroups by applying picture fuzzy set theory. We characterize different classes regular (resp. intra-regular and semisimple) semigroups in terms of picture fuzzy left and right ideals (resp. picture fuzzy ideals). In this regard, we prove that a non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the picture characteristic function on S is the picture fuzzy subsemigroup (picture fuzzy left ideal, picture fuzzy right ideal, picture fuzzy two-sided ideal) on S.

2. **Preliminaries and Some Notations.** Firstly, we recollect some preliminaries and prerequisites of picture fuzzy sets on semigroups which we shall apply in the results of the main section.

We now recall the notion of picture fuzzy sets on semigroups.

Let S be a universe. Recall that a **picture fuzzy set** A on a semigroup S is an object having the form $A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle : x \in S\}$ where the functions $\mu_A : S \to [0,1]$, $\eta_A : S \to [0,1]$ and $\nu_A : S \to [0,1]$ denote the degree of **positive membership**, **neutral membership** and the degree of **negative membership**, respectively, and $\mu_A(x) + \eta_A(x) + \nu_A(x) \in [0,1]$ for all $x \in S$. Further, we denote by $PFS(S)$ the collection of picture fuzzy sets on a semigroup S with $\mathcal{S} = \{\langle x, 1, 0, 0\rangle : x \in S\}$ and $\emptyset = \{\langle x, 0, 0, 1\rangle : x \in S\}$.

Note that let $A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle : x \in S\}$ be a picture fuzzy set on a semigroup S. For the sake of simplicity, picture fuzzy set will be denoted by $A = (\mu_A, \eta_A, \nu_A)$. Thus, from the above definition, it is clear that picture fuzzy sets are a generalization of standard intuitionistic fuzzy subsets.

Recall that a picture fuzzy set $A = (\mu_A, \eta_A, \nu_A)$ on a semigroup S is called **picture fuzzy left (resp., right) ideal** if the following conditions hold:

(1) $\mu_A(xy) \geq \mu_A(y)\,(\mu_A(xy) \geq \mu_A(x))$,
(2) $\eta_A(xy) \leq \eta_A(y)\,(\eta_A(xy) \leq \eta_A(x))$,
(3) $\nu_A(xy) \leq \nu_A(y)\,(\nu_A(xy) \leq \nu_A(x))$

for all $x, y \in S$. Recall that a picture fuzzy set A on S is called a **picture fuzzy two-sided ideal or a picture fuzzy ideal** on S if it is both a picture fuzzy left and a picture fuzzy right ideal on S.

3. **Characterizing Regular Semigroups.** In this section, we prove that the concepts of picture characteristic functions on semigroup coincide. Also, we prove that a non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the picture characteristic function on S is the picture fuzzy subsemigroup

(picture fuzzy left ideal, picture fuzzy right ideal, picture fuzzy two-sided ideal) on S. In particular, we characterize (resp. left (right) regular) regular semigroups in terms of picture fuzzy right and left ideals.

Now we shall introduce the notion of picture characteristic functions on semigroups.

Let X be a subset of a semigroup S. The **picture characteristic function** of X is denoted by $\mathcal{C}^X = \{\langle x, \mu_{\mathcal{C}^X}(x), \eta_{\mathcal{C}^X}(x), \nu_{\mathcal{C}^X}(x)\rangle : x \in S\}$ and is defined as

$$\mu_{\mathcal{C}^X}(x) = \begin{cases} 1; & x \in X \\ 0; & \text{otherwise,} \end{cases}$$

$$\eta_{\mathcal{C}^X}(x) = \begin{cases} 0; & x \in X \\ 1; & \text{otherwise,} \end{cases}$$

and

$$\nu_{\mathcal{C}^X}(x) = \begin{cases} 0; & x \in X \\ 1; & \text{otherwise.} \end{cases}$$

Observe that if $X = S\,(X = \emptyset)$, then it is easy to see that $\mathcal{C}^X = \mathcal{S}\left(\mathcal{C}^\emptyset = \emptyset\right)$.

Now we investigate the operation properties of picture characteristic function on a semigroup S.

Lemma 3.1. *Suppose that $\mathcal{C}^X$ and $\mathcal{C}^Y$ are two picture fuzzy sets on a semigroup S. Then the following properties hold.*

(1) $\mathcal{C}^{X\cap Y} = \mathcal{C}^X \cap \mathcal{C}^Y$.

(2) $\mathcal{C}^{XY} = \mathcal{C}^X \odot \mathcal{C}^X$.

Proof: (1) Let x be any element of S. If $x \in X \cap Y$, then $x \in X$ and $x \in Y$. Thus $(\mu_{\mathcal{C}^X} \wedge \mu_{\mathcal{C}^Y})(x) = \mu_{\mathcal{C}^X}(x) \wedge \mu_{\mathcal{C}^Y}(x) = 1 = \mu_{\mathcal{C}^{X\cap Y}}(x)$ and $(\eta_{\mathcal{C}^X} \vee \eta_{\mathcal{C}^Y})(x) = \eta_{\mathcal{C}^X}(x) \vee \eta_{\mathcal{C}^Y}(x) = 0 = \eta_{\mathcal{C}^{X\cap Y}}(x)$. Now, if $x \notin X \cap Y$, then $x \notin X$ or $x \notin Y$, which means that

$$\begin{aligned}(\mu_{\mathcal{C}^X} \wedge \mu_{\mathcal{C}^Y})(x) &= \mu_{\mathcal{C}^X}(x) \wedge \mu_{\mathcal{C}^Y}(x) \\ &= 0 \\ &= \mu_{\mathcal{C}^{X\cap Y}}(x)\end{aligned}$$

and

$$\begin{aligned}(\eta_{\mathcal{C}^X} \vee \eta_{\mathcal{C}^Y})(x) &= \eta_{\mathcal{C}^X}(x) \vee \eta_{\mathcal{C}^Y}(x) \\ &= 1 \\ &= \eta_{\mathcal{C}^{X\cap Y}}(x).\end{aligned}$$

Similarly we obtain that $(\nu_{\mathcal{C}^X} \vee \nu_{\mathcal{C}^Y})(x) = \nu_{\mathcal{C}^{X\cap Y}}(x)$. Therefore, $\mathcal{C}^{X\cap Y} = \mathcal{C}^X \cap \mathcal{C}^Y$.

(2) Let x be any element of S. If $x \in XY$, then $x = yz$ for some $y \in X$ and $z \in Y$, which implies that

$$\begin{aligned}(\mu_{\mathcal{C}^X} \circ \mu_{\mathcal{C}^Y})(x) &= \bigvee_{x=ab} \{\mu_{\mathcal{C}^X}(a) \wedge \mu_{\mathcal{C}^Y}(b)\} \\ &\geq \mu_{\mathcal{C}^X}(y) \wedge \mu_{\mathcal{C}^Y}(z) \\ &= 1.\end{aligned}$$

We get $(\mu_{\mathcal{C}^X} \circ \mu_{\mathcal{C}^Y})(x) = 1$. Since $x \in XY$, we have $\mu_{\mathcal{C}^{XY}}(x) = 1$. In the case, when $x \notin XY$, we have $x \neq yz$ for all $y \in X$ and $z \in Y$. If $x = uv$ for some $u, v \in S$, then $(\mu_{\mathcal{C}^X} \circ \mu_{\mathcal{C}^Y})(x) = \bigvee_{x=ab} \{\mu_{\mathcal{C}^X}(a) \wedge \mu_{\mathcal{C}^Y}(b)\} = 0 = \mu_{\mathcal{C}^{XY}}(x)$. Now, if $x \neq uv$ for all $u, v \in S$, then $(\mu_{\mathcal{C}^X} \circ \mu_{\mathcal{C}^Y})(x) = 0 = \mu_{\mathcal{C}^{XY}}(x)$. In any case, we have $(\mu_{\mathcal{C}^X} \circ \mu_{\mathcal{C}^Y})(x) = \mu_{\mathcal{C}^{XY}}(x)$. Similarly, we can show that $(\eta_{\mathcal{C}^X} \circ \eta_{\mathcal{C}^Y})(x) = \eta_{\mathcal{C}^{XY}}(x)$ and $(\nu_{\mathcal{C}^X} \circ \nu_{\mathcal{C}^Y})(x) = \nu_{\mathcal{C}^{XY}}(x)$. Therefore, $\mathcal{C}^{XY} = \mathcal{C}^X \odot \mathcal{C}^X$. □

The following theorem shows that a non empty subset of a semigroup S is a subsemigroup (left ideal, right ideal, two-sided ideal) of S if and only if the picture characteristic function on S is the picture fuzzy subsemigroup (picture fuzzy left ideal, picture fuzzy right ideal, picture fuzzy two-sided ideal) on S.

Theorem 3.1. *Suppose that $\mathcal{C}^X$ is a picture fuzzy set on a semigroup S. Then the following properties hold.*

(1) X is a left ideal of S if and only if $\mathcal{C}^X$ is a picture fuzzy left ideal on S.
(2) X is a right ideal of S if and only if $\mathcal{C}^X$ is a picture fuzzy right ideal on S.
(3) X is an ideal of S if and only if $\mathcal{C}^X$ is a picture fuzzy ideal on S.

Proof: (1) First assume that X is a left ideal of S. Let x and y be any elements of S. If $y \notin A$, then $\mu_{\mathcal{C}^X}(y) = 0$ and $\eta_{\mathcal{C}^X}(y) = 1$, which implies that $\mu_{\mathcal{C}^X}(y) \leq \mu_{\mathcal{C}^X}(xy)$ and $\eta_{\mathcal{C}^X}(y) \geq \eta_{\mathcal{C}^X}(xy)$. Now if $y \in A$, then $\mu_{\mathcal{C}^X}(y) = 1$ and $\eta_{\mathcal{C}^X}(y) = 0$. It follows that $xy \in A$, since A is a subsemigroup of S. Thus $\mu_{\mathcal{C}^X}(xy) \geq \mu_{\mathcal{C}^X}(y)$ and $\eta_{\mathcal{C}^X}(xy) \leq \eta_{\mathcal{C}^X}(y)$. Similarly, we can show that $\nu_{\mathcal{C}^X}(xy) \leq \nu_{\mathcal{C}^X}(y)$. Therefore, $\mathcal{C}^X$ is a picture fuzzy left ideal on S.

Conversely, assume that $\mathcal{C}^X$ is a picture fuzzy subsemigroup on S. Let x and y be any element of S such that $y \in X$. We obtain $\eta_{\mathcal{C}^X}(y) = 0$. It follows that $\eta_{\mathcal{C}^X}(y) \geq \eta_{\mathcal{C}^X}(xy)$, since $\mathcal{C}^X$ is a picture fuzzy subsemigroup on S. This means that $\eta_{\mathcal{C}^X}(xy) = 0$. Therefore, $xy \in X$ and hence X is a subsemigroup of S.

(2)-(3) It can be proved similarly as (1). □

Recall that an element x is called **regular** if there exists an element s in a semigroup S such that $x = xsx$. A semigroup S is called **regular** if every element of S is regular. Observe that a semigroup S is regular if and only if $R \cap L = RL$ for every right ideal R of S and every left ideal L of S. Next, we define the operation "=" on a semigroup S. Let A and B be two picture fuzzy sets on a semigroup S. Then we have, $A = B$ if and only if $\mu_A(x) = \mu_B(x)$, $\eta_A(x) = \eta_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in S$. A picture fuzzy set A on a semigroup S is said to be **idempotent** if $A = A \odot A$.

Theorem 3.2. *Every picture fuzzy two-sided ideal on a regular semigroup is idempotent.*

Proof: Let A be a picture fuzzy two-sided ideal on a regular semigroup S. Then we have $A \odot A \subseteq A \odot \mathcal{S} \subseteq A$. On the other hand, let x be any element of S. Then, since S is regular, there exists an element s in S such that $x = xsx$. Therefore,

$$\begin{aligned}(\mu_A \circ \mu_A)(x) &= \bigvee_{x=ab} (\mu_A(a) \wedge \mu_A(b)) \\ &\geq \mu_A(xs) \wedge \mu_A(x) \\ &\geq \mu_A(x) \wedge \mu_A(x) \\ &= \mu_A(x)\end{aligned}$$

and

$$\begin{aligned}(\eta_A \circ \eta_A)(x) &= \bigwedge_{x=ab} (\eta_A(a) \vee \eta_A(b)) \\ &\leq \eta_A(xs) \vee \eta_A(x) \\ &\leq \eta_A(x) \vee \eta_A(x) \\ &= \eta_A(x).\end{aligned}$$

Similarly, we can prove that, $(\nu_A \circ \nu_A)(x) \leq \nu_A(x)$. Hence $A = A \odot A$. Consequently, A is idempotent. □

Example 3.1. *Let $S = \{apple, banana, grape, peach\}$ be a semigroup and consider an operation $\star$ which produces the following products:*

$$apple \star x = apple \text{ for all } x \in S,$$
$$banana \star x = apple \text{ for all } x \in S,$$

$$grape \star x = \begin{cases} apple; & x \in \{apple, banana, grape\} \\ banana; & otherwise, \end{cases}$$

$$peach \star x = \begin{cases} apple; & x \in \{apple, banana\} \\ apple; & x \in \{banana\} \\ banana; & otherwise. \end{cases}$$

Let $A = (\mu_A, \eta_A, \nu_A)$ be a picture fuzzy set defined as:

A	μ_A	η_A	ν_A
apple	0.3	0.1	0.1
banana	0.3	0.1	0.1
grape	0.1	0.3	0.3
peach	0.1	0.3	0.3

Then, clearly $A = (\mu_A, \eta_A, \nu_A)$ is a picture fuzzy ideal on S.

Now we give some characterizations of a regular semigroup in terms of picture fuzzy right and left ideals.

Theorem 3.3. *For a semigroup S the following conditions are equivalent.*

(1) S is regular.

(2) For every picture fuzzy right ideal A and every picture fuzzy left ideal B on S, $A \cap B = A \odot B$.

Proof: First assume that S is a regular semigroup. Let A be any picture fuzzy right ideal and B any picture fuzzy left ideal on S. Then we have $A \odot B \subseteq A \odot \mathcal{S} \subseteq A$ and $A \odot B \subseteq \mathcal{S} \odot B \subseteq B$. It follows that $A \odot B \subseteq A \cap B$. On the other hand, let x be any element of S. Then, since S is regular, there exists an element s in S such that $x = xsx$. Therefore,

$$\begin{aligned} (\mu_A \circ \mu_B)(x) &= \bigvee_{x=ab} (\mu_A(a) \wedge \mu_B(b)) \\ &\geq \mu_A(xs) \wedge \mu_B(x) \\ &\geq \mu_A(x) \wedge \mu_B(x) \\ &= (\mu_A \wedge \mu_B)(x) \end{aligned}$$

and

$$\begin{aligned} (\eta_A \circ \eta_B)(x) &= \bigwedge_{x=ab} (\eta_A(a) \vee \eta_B(b)) \\ &\leq \eta_A(xs) \vee \eta_B(x) \\ &\leq \eta_A(x) \vee \eta_B(x) \\ &= (\eta_A \vee \eta_B)(x). \end{aligned}$$

Similarly to the previous case, $(\nu_A \circ \nu_B)(x) \leq (\nu_A \vee \nu_B)(x)$. Hence $A \cap B = A \odot B$.

Conversely, assume that (2) holds. Let R and L be any right ideal and any left ideal of S, respectively. In order to see that $R \cap L \subseteq RL$ holds, let x be any element of $R \cap L$. Then by Theorem 3.1, the picture characteristic functions $\mathcal{C}^R$ and $\mathcal{C}^L$ of R and L are a picture fuzzy right ideal and a picture fuzzy left ideal on S, respectively. Then it follows from Lemma 3.1,

$$\begin{aligned} \eta_{\mathcal{C}^{RL}}(x) &= (\eta_{\mathcal{C}^R} \circ \eta_{\mathcal{C}^L})(x) \\ &= (\eta_{\mathcal{C}^R} \vee \eta_{\mathcal{C}^L})(x) \\ &= \eta_{\mathcal{C}^{R\cap L}}(x) \\ &= 0, \end{aligned}$$

which means that $x \in RL$. Thus $R \cap L \subseteq RL$. Since the inclusion in the other direction always holds, we obtain that $R \cap L = RL$. Hence S is a regular semigroup. $\square$

Recall that a semigroup S is called **right (left) zero** if $xy = y$ ($xy = x$) for all $x, y \in S$. Then we have the following result.

Theorem 3.4. *For a regular semigroup S, the following conditions are equivalent.*

(1) The set $\mathcal{E}(S)$ of all idempotents of S forms a left zero subsemigroup of S.

(2) For every picture fuzzy left ideal A on S, $\mu_A(x) = \mu_A(y)$, $\eta_A(x) = \eta_A(y)$ and $\nu_A(x) = \nu_A(y)$ for all $x, y \in \mathcal{E}(S)$.

Proof: First assume that (1) holds. Let A be a picture fuzzy left ideal on S and $x, y \in \mathcal{E}(S)$. Then, since $xy = x$ and $yx = y$, we have $\mu_A(x) = \mu_A(xy) \geq \mu_A(y)$ and $\mu_A(y) = \mu_A(yx) \geq \mu_A(x)$. Hence $\mu_A(x) = \mu_A(y)$. Also, $\eta_A(x) = \eta_A(xy) \leq \eta_A(y)$ and $\eta_A(y) = \eta_A(yx) \leq \eta_A(x)$, it follows that $\eta_A(x) = \eta_A(y)$. Similarly, we can prove that $\nu_A(x) = \nu_A(y)$. Thus (1) implies (2).

Conversely, assume that (2) holds. Since S is regular, $\mathcal{E}(S)$ is non empty. Thus it follows from Theorem 3.1(2) that the picture characteristic function $\mathcal{C}^{Sy}$ of the left ideal Sy of S is a picture fuzzy left ideal on S. Thus we have $\eta_{\mathcal{C}^{Sy}}(x) = \eta_{\mathcal{C}^{Sy}}(y) = 0$, which implies that $x \in Sy$. Hence for some $s \in S$, we have $x = sy = s(yy) = (sy)y = xy$. Thus $\mathcal{E}(S)$ is a left zero subsemigroup of S and so (2) implies (1). □

By Theorem 3.4, we immediately obtain the following corollary.

Corollary 3.1. *For a regular semigroup S, the following conditions are equivalent.*

(1) The set $\mathcal{E}(S)$ of all idempotents of S forms a right zero subsemigroup of S.

(2) For every picture fuzzy right ideal A on S, $\mu_A(x) = \mu_A(y)$, $\eta_A(x) = \eta_A(y)$ and $\nu_A(x) = \nu_A(y)$ for all $x, y \in \mathcal{E}(S)$.

Recall that a semigroup S is called **left (right) regular** if for each element x of S, there exists an element $s \in S$ such that $x = sx^2$ ($x = x^2 s$). From the above discussion, we can immediately obtain the following theorems.

Theorem 3.5. *For a semigroup S, the following conditions are equivalent.*

(1) S is left (right) regular.

(2) For every picture fuzzy left (right) ideal A on S, $\mu_A(x) = \mu_A(x^2)$, $\eta_A(x) = \eta_A(x^2)$ and $\nu_A(x) = \nu_A(x^2)$ for all $x \in S$.

Proof: First assume that (1) holds. Let A be any picture fuzzy left ideal on S and x any element of S. Then, since S is left regular, there exists an element s in S such that $x = sx^2$. Thus we have, $\mu_A(x) = \mu_A(sx^2) \geq \mu_A(x^2)$ and $\mu_A(x^2) \geq \mu_A(x)$, which implies $\mu_A(x) = \mu_A(x^2)$. Also, $\eta_A(x) = \eta_A(sx^2) \leq \eta_A(x^2)$ and $\eta_A(x^2) \leq \eta_A(x)$, it follows that $\eta_A(x) = \eta_A(x^2)$. Similarly to the previous case, $\nu_A(x) = \nu_A(x^2)$, meaning that (1) implies (2).

Conversely, assume that (2) holds. Let x be any element of S. Clearly, $x^2 \cup Sx^2$ is a left ideal of S. Then it follows from Theorem 3.1(1), the picture characteristic function $\mathcal{C}^{x^2 \cup Sx^2}$ of the principal left ideal $x^2 \cup Sx^2$ of S is a picture fuzzy left ideal on S. It follows that $\eta_{\mathcal{C}^{x^2 \cup Sx^2}}(x) = \eta_{\mathcal{C}^{x^2 \cup Sx^2}}(x^2) = 0$, since $x^2 \in x^2 \cup Sx^2$. This means that $x \in x^2 \cup Sx^2$. Hence S is left regular and so (2) implies (1). The case when S is right regular can be similarly proved. □

Recall that a subset A of a semigroup S is called **semiprime** if for all $x \in S$, $x^2 \in A$ implies $x \in A$. A picture fuzzy set A on S is called **picture fuzzy semiprime** if $\mu_A(x) \geq \mu_A(x^2)$, $\eta_A(x) \leq \eta_A(x^2)$ and $\nu_A(x) \leq \nu_A(x^2)$ for all $x \in S$. The following theorem shows that the concept of picture fuzzy semiprimality in a semigroup is an extension of semiprimality.

Theorem 3.6. *For a non empty subset A of a semigroup S, the following conditions are equivalent.*

(1) A is semiprime.
(2) The picture characteristic function $\mathcal{C}^A$ of A is a picture fuzzy semiprime set.

Proof: First assume that (1) holds. Let x be any element of S. If $x^2 \in A$, then, since P is semiprime, $x \in A$. This implies that $\mu_{\mathcal{C}^A}(x) = 1$, $\eta_{\mathcal{C}^A}(x) = 0$ and $\nu_{\mathcal{C}^A}(x) = 0$. Now, if $x^2 \notin A$, then $\mu_{\mathcal{C}^A}(x^2) = 0$, $\eta_{\mathcal{C}^A}(x^2) = 1$ and $\nu_{\mathcal{C}^A}(x^2) = 1$. In any case, we have $\mu_{\mathcal{C}^A}(x^2) \leq \mu_{\mathcal{C}^A}(x)$, $\eta_{\mathcal{C}^A}(x^2) \geq \eta_{\mathcal{C}^A}(x)$ and $\nu_{\mathcal{C}^A}(x^2) \geq \eta_{\mathcal{C}^A}(x)$ for all $x \in S$. Therefore, $\mathcal{C}^A$ is a picture fuzzy semiprime set and hence (1) implies (2).

Conversely, assume that (2) holds. Let x be any element of S such that $x^2 \in A$. Now, since $\mathcal{C}^A$ is a picture fuzzy semiprime set, we have $0 = \eta_{\mathcal{C}^A}(x^2) \geq \eta_{\mathcal{C}^A}(x)$. This implies that $\eta_{\mathcal{C}^A}(x) = 0$, that is, $x \in A$. Thus A is a semiprime set and hence (2) implies (1). □

The following theorem shows that a picture fuzzy subsemigroup A on a semigroup S is a picture fuzzy semiprime set on S if and only if $\mu_A(x) = \mu_A(x^2)$, $\eta_A(x) = \eta_A(x^2)$ and $\nu_A(x) = \nu_A(x^2)$ for all $x \in S$.

Theorem 3.7. *For a picture fuzzy subsemigroup A on a semigroup S, the following conditions are equivalent.*
(1) A is a picture fuzzy semiprime set.
(2) For every $x \in S$, $\mu_A(x) = \mu_A(x^2)$, $\eta_A(x) = \eta_A(x^2)$ and $\nu_A(x) = \nu_A(x^2)$.

Proof: It is clear that (2) implies (1). Assume that (1) holds. Let x be any element of S. It follows that $\mu_{\mathcal{C}^A}(x^2) \leq \mu_{\mathcal{C}^A}(x)$, $\eta_{\mathcal{C}^A}(x^2) \geq \eta_{\mathcal{C}^A}(x)$ and $\nu_{\mathcal{C}^A}(x^2) \geq \eta_{\mathcal{C}^A}(x)$ for all $x \in S$, since A is a picture fuzzy semiprime set on S. Now, since A is a picture fuzzy subsemigroup on S, we have $\mu_A(x) = \mu_A(x^2)$, $\eta_A(x) = \eta_A(x^2)$ and $\nu_A(x) = \nu_A(x^2)$. Thus (1) implies (2). □

4. **Characterizing Intra-Regular Semigroups.**

In this section, we investigate mainly the characterizations of intra-regular semigroups and semisimple semigroups by fuzzy ideals.

Recall that a semigroup S is called **intra-regular** if for each element x of S, there exist elements r and s in S such that $x = rx^2s$. Now we characterize an intra-regular semigroup in terms of picture fuzzy ideals.

Theorem 4.1. *For a semigroup S, the following conditions are equivalent.*
(1) S is intra-regular.
(2) For every picture fuzzy two-sided ideal on S is a picture fuzzy semiprime set.
(3) For every picture fuzzy two-sided ideal A on S, $\mu_A(x) = \mu_A(x^2)$, $\eta_A(x) = \eta_A(x^2)$ and $\nu_A(x) = \nu_A(x^2)$ for all $x \in S$.

Proof: First assume that (1) holds. Let A be any picture fuzzy two-sided ideal on S and x any element of S. Then, since S is intra-regular, there exist elements r and s in S such that $x = rx^2s$. Hence we have

$$\begin{aligned}\mu_A(x) &= \mu_A(rx^2s)\\ &\geq \mu_A(rx^2)\\ &\geq \mu_A(x^2)\end{aligned}$$

and

$$\begin{aligned}\eta_A(x) &= \eta_A(rx^2s)\\ &\leq \eta_A(rx^2)\\ &\leq \eta_A(x^2).\end{aligned}$$

Similarly, we can prove that $\nu_A(x) \leq \nu_A(x^2)$. Since the inclusion in the other direction always holds, we obtain that $\mu_A(x) = \mu_A(x^2)$, $\eta_A(x) = \eta_A(x^2)$ and $\nu_A(x) = \nu_A(x^2)$. Thus (1) implies (3).

It is clear that (2) and (3) are equivalent.

Assume that (2) holds. Let x be any element of S. It is clear that $x^2 \cup Sx^2 \cup x^2S \cup Sx^2S$ is an ideal of S. Then it follows from Theorem 3.1(3), the picture characteristic function $\mathcal{C}^{x^2 \cup Sx^2 \cup x^2S \cup Sx^2S}$ of the ideal $x^2 \cup Sx^2 \cup x^2S \cup Sx^2S$ of S is a picture fuzzy ideal on S. It follows that $\left(\eta_{\mathcal{C}^{x^2 \cup Sx^2 \cup x^2S \cup Sx^2S}}\right)(x) = \left(\eta_{\mathcal{C}^{x^2 \cup Sx^2 \cup x^2S \cup Sx^2S}}\right)(x^2) = 0$, since $x^2 \in x^2 \cup Sx^2 \cup x^2S \cup Sx^2S$. Thus $x \in x^2 \cup Sx^2 \cup x^2S \cup Sx^2S$. Therefore, it is easily seen that S is intra-regular and hence (2) implies (1). □

Now we characterize the intra-regular semigroup in terms of picture fuzzy left ideals and picture fuzzy right ideals.

Theorem 4.2. *For a semigroup S, the following conditions are equivalent.*

(1) S is intra-regular.

(2) $A \cap B \subseteq A \odot B$ for every picture fuzzy left ideal A and every picture fuzzy right ideal B on S.

Proof: First assume that (1) holds. Let A and B be any picture fuzzy left ideal and any picture fuzzy right ideal on S, respectively. Let x be any element of S. Then, since S is intra-regular, there exist elements r and s in S such that $x = rx^2s$. Hence we have,

$$\begin{aligned}(\mu_A \circ \mu_B)(x) &= \bigvee_{x=yz} \{\mu_A(y) \wedge \mu_B(z)\} \\ &\geq \mu_A(rx) \wedge \mu_B(xs) \\ &\geq \mu_A(x) \wedge \mu_B(x) \\ &= (\mu_A \wedge \mu_B)(x)\end{aligned}$$

and

$$\begin{aligned}(\eta_A \circ \eta_B)(x) &= \bigwedge_{x=yz} \{\eta_A(y) \vee \eta_B(z)\} \\ &\leq \eta_A(rx) \vee \eta_B(xs) \\ &\leq \eta_A(x) \vee \eta_B(x) \\ &= (\eta_A \vee \eta_B)(x).\end{aligned}$$

Similarly, one proves that $(\nu_A \circ \nu_B)(x) \leq (\nu_A \vee \nu_B)(x)$. Therefore, $A \cap B \subseteq A \odot B$ and hence (1) implies (2).

Conversely, assume that (2) holds. Let L and R be any left ideal and any right ideal of S, respectively. Now, let x be any element of S such that $x \in L \cap R$. It is clear that $x \in L$ and $x \in R$. Since by Theorem 3.1 $\mathcal{C}^L$ and $\mathcal{C}^R$ are a picture fuzzy left ideal and a picture fuzzy right ideal on S, respectively, by Lemma 3.1 we have

$$\begin{aligned}\eta_{\mathcal{C}^{LR}}(x) &= (\eta_{\mathcal{C}^L} \circ \eta_{\mathcal{C}^R})(x) \\ &\leq (\eta_{\mathcal{C}^L} \vee \eta_{\mathcal{C}^R})(x) \\ &= \eta_{\mathcal{C}^{L \cap R}}(x) \\ &= 0.\end{aligned}$$

It is obvious that $x \in LR$. Thus $L \cap R \subseteq LR$ and hence S is intra-regular. □

In the following theorem we give a characterization of a semigroup that is both regular and intra-regular in terms of picture fuzzy right ideals and picture fuzzy left ideals.

Theorem 4.3. *For a semigroup S, the following conditions are equivalent.*

(1) S is regular and intra-regular.

(2) $A \cap B \subseteq (A \odot B) \cap (B \odot A)$ for every picture fuzzy right ideal A and every picture fuzzy left ideal B on S.

Proof: First assume that (1) holds. Let A and B be any picture fuzzy right ideal and any picture fuzzy left ideal on S, respectively. Then it follows from Theorems 3.3, 4.2

that $A \cap B = B \cap A \subseteq B \odot A$ and $A \cap B \subseteq A \odot B$. Therefore, $A \cap B \subseteq (A \odot B) \cap (B \odot A)$ and hence (1) implies (2).

Conversely, assume that (2) holds. Let A and B be any picture fuzzy right ideal and any picture fuzzy left ideal on S, respectively. Then we have, $B \cap A = A \cap B \subseteq (A \odot B) \cap (B \odot A) \subseteq B \odot A$. Thus it follows from Theorem 4.2 that S is intra-regular. On the other hand, $A \odot B \subseteq A$ and $A \odot B \subseteq B$, which means that $A \odot B \subseteq A \cap B$. Since the inclusion $A \cap B \subseteq A \odot B$ always holds, we have $A \odot B = A \cap B$. Hence it follows from Theorem 3.3 that S is regular. This completes the proof. □

Recall that a semigroup S is called **semisimple** if every two-sided ideal of S is idempotent. It is clear that S is semisimple if and only if $x \in (SxS)(SxS)$ for every $x \in S$, that is, there exist elements q, r and s in S such that $x = qxrxs$. Now we give characterizations of semisimple semigroups by the properties of picture fuzzy ideals.

Theorem 4.4. *For a semigroup S, the following conditions are equivalent.*

(1) S is semisimple.

(2) Every picture fuzzy two-sided ideal on S is idempotent.

(3) $A \cap B = A \odot B$ for every picture fuzzy two-sided ideals A and B on S.

Proof: Assume that (1) holds. Let A and B be two picture fuzzy two-sided ideals on S. By assumption, $A \odot B \subseteq A$ and $A \odot B \subseteq B$, which implies that $A \odot B \subseteq A \cap B$. On the other hand, let x be any element of S. Since S is semisimple, there exist elements q, p, r and s in S such that $x = (qxp)(rxs)$. Hence we have,

$$\begin{aligned}(\mu_A \circ \mu_B)(x) &= \bigvee_{x=yz} \{\mu_A(y) \wedge \mu_B(z)\} \\ &\geq \mu_A(qxp) \wedge \mu_B(rxs) \\ &\geq \mu_A(xp) \wedge \mu_B(xs) \\ &\geq \mu_A(x) \wedge \mu_B(x) \\ &= (\mu_A \wedge \mu_B)(x)\end{aligned}$$

and

$$\begin{aligned}(\eta_A \circ \eta_B)(x) &= \bigwedge_{x=yz} \{\eta_A(y) \vee \eta_B(z)\} \\ &\leq \eta_A(qxp) \vee \eta_B(rxs) \\ &\leq \eta_A(xp) \vee \eta_B(xs) \\ &\leq \eta_A(x) \vee \eta_B(x) \\ &= (\eta_A \vee \eta_B)(x).\end{aligned}$$

Similarly, we obtain $(\nu_A \circ \nu_B)(x) \leq (\nu_A \vee \nu_B)(x)$. Therefore, $A \cap B \subseteq A \odot B$ and hence (3) implies (1).

It is clear that, (3) ⇒ (2).

Assume that (2) holds. Let x be any element of S. It is clear that $x \cup Sx \cup xS \cup SxS$ is an ideal of S. Then it follows from Theorem 3.1(3), the picture characteristic function $\mathcal{C}^{x \cup Sx \cup xS \cup SxS}$ of the ideal $x \cup Sx \cup xS \cup SxS$ of S is a picture fuzzy ideal on S. Hence by Lemma 3.1, we have

$$\begin{aligned}\left(\eta_{\mathcal{C}^{(x\cup Sx\cup xS\cup SxS)(x\cup Sx\cup xS\cup SxS)}}\right)(x) &= \left(\eta_{\mathcal{C}^{x\cup Sx\cup xS\cup SxS}} \circ \eta_{\mathcal{C}^{x\cup Sx\cup xS\cup SxS}}\right)(x) \\ &= \left(\eta_{\mathcal{C}^{x\cup Sx\cup xS\cup SxS}} \vee \eta_{\mathcal{C}^{x\cup Sx\cup xS\cup SxS}}\right)(x) \\ &= \left(\eta_{\mathcal{C}^{x\cup Sx\cup xS\cup SxS}}\right)(x) \vee \left(\eta_{\mathcal{C}^{x\cup Sx\cup xS\cup SxS}}\right)(x) \\ &= 0,\end{aligned}$$

which implies that $x \in (x \cup Sx \cup xS \cup SxS)(x \cup Sx \cup xS \cup SxS) \subseteq (SxS)(SxS)$. Thus S is semisimple and hence (2) implies (1). □

REFERENCES

[1] M. Aslam, S. Abdullah and S. Aslam, Characterization of regular LA-semigroups by interval-valued $(\bar{\alpha}, \bar{\beta})$-fuzzy ideals, *Afr. Mat.*, vol.25, pp.501-518, 2014.

[2] B. C. Cuong, Picture fuzzy sets, *J. Comput. Sci. Cybern.*, vol.30, pp.409-420, 2014.

[3] F. Feng, M. Khan, S. Anis and M. Qadeer, Right weakly regular semigroups characterized by their generalized fuzzy ideals, *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, vol.75, no.4, pp.53-66, 2013.

[4] M. S. A. Khan, S. Abdullah, Y. B. Jun and K. Rahman, Generalized fuzzy subsemigroups ideals in semigroups, *Honam Mathematical J.*, vol.39, no.4, pp.527-559, 2017.

[5] A. Khan, Y. B. Jun and M. Z. Abbas, Characterizations of ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy interior ideals, *Neural Comput. & Applic.*, vol.21, pp.433-440, 2012.

[6] A. Khan, Y. B. Jun, N. H. Sarmin and F. M. Khan, Ordered semigroups characterized by $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals, *Neural Comput. & Applic.*, vol.21, no.1, pp.S121-S132, 2012.

[7] A. Khan, N. H. Sarmin, B. Davvaz and F. M. Khan, New types of fuzzy bi-ideals in ordered semigroups, *Neural Comput. & Applic.*, vol.21, no.1, pp.S295-S305, 2012.

[8] H. U. Khan, N. H. Sarmin and A. Khan, A new form of fuzzy generalized bi-ideals in ordered semigroups, *Honam Mathematical J.*, vol.36, no.3, pp.569-596, 2014.

[9] M. Makhmudov and C. S. Ko, A compromise conflict resolution approach to the uncertain transportation problem based on fuzzy set theory, *ICIC Express Letters*, vol.13, no.4, pp.325-332, 2019.

[10] S. K. Sardar, B. Davvaz, S. K. Majumder and S. Kayal, Generalized fuzzy interior ideals in Γ-semigroups, *Hacettepe Journal of Mathematics and Statistics*, vol.41, no.2, pp.231-241, 2012.

[11] M. Shabir, Y. B. Jun and Y. Nawaz, Characterizations of regular semigroups by (α, β)-fuzzy ideals, *Computers and Mathematics with Applications*, vol.59, pp.161-175, 2010.

[12] M. Shabir, Y. B. Jun and Y. Nawaz, Semigroups characterized by $(\in, \in \vee q_k)$-fuzzy ideals, *Computers and Mathematics with Applications*, vol.60, pp.1473-1493, 2010.

[13] M. Shabir and A. Khan, Chains of archimedean ordered semigroups by terms of fuzzy subsets, *World Academy of Science, Engineering and Technology*, vol.56, pp.957-963, 2011.

[14] T. Shah and N. Kausar, Characterizations of non-associative ordered semigroups by their fuzzy bi-ideals, *Theoretical Computer Science*, vol.529, pp.96-110, 2014.

[15] M. Shabir and A. Khan, Characterizations of ordered semigroups by the properties of their fuzzy ideals, *Computers and Mathematics with Applications*, vol.59, pp.539-549, 2010.

[16] M. Shabir, Y. Nawaz and M. Ali, Characterizations of semigroups by $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy ideals, *World Applied Sciences Journal*, vol.14, no.12, pp.1866-1878, 2011.

[17] M. Siripitukdet, Characterizations of fuzzy ideals of semigroups by soft sets, *Thai J. Math.*, vol.16, no.1, pp.103-120, 2018.

[18] J. Tang and X. Y. Xie, On $(\in, \in \vee q_k)$-fuzzy ideals of ordered semigroups, *Fuzzy Inf. Eng.*, vol.1, pp.57-67, 2013.

[19] X. Y. Xie and J. Tang, Regular ordered semigroups and intra-regular ordered semigroups in terms of fuzzy subsets, *Iranian Journal of Fuzzy Systems*, vol.7, no.2, pp.121-140, 2010.

[20] P. Yiarayong, Applications of hesitant fuzzy sets to completely regular semigroups, *IAENG International Journal of Computer Science*, vol.47, no.1, pp.107-118, 2020.

[21] Y. Yin and J. Zhan, Characterization of ordered semigroups in terms of fuzzy soft ideals, *Bull. Malays. Math. Sci. Soc.*, vol.35, no.4, pp.997-1015, 2012.

[22] F. Yousafzai, N. Yaqoob and A. Zeb, On generalized fuzzy ideals of ordered AG-groupoids, *Int. J. Mach. Learn. & Cyber.*, vol.7, pp.995-1004, 2016.