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SIMULATION EFFECT OF NATURAL RESERVES IN PRESERVING THE ENVIRONMENTAL EQUILIBRIA

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Abstract. *The importance of natural reserves existence and how it affects the whole environment in terms of creating a natural environmental balance has encouraged us to propose and study an environmental model dealing with endangered species (Lemur animals) and two types of hunters (the black panthers and hyenas animals) which are linked together by a food web. Mature and immature Lemur animals are two compartments that form the two-stage structure. They have the ability to protect themselves against predation. In particular, we were interested in determining whether intra-specific competition exists within the Lemur animal populations and whether the inter-specific competition also exists between the hunter's populations, and this is due to the substantial differences between them. Different types of functional responses for describing predation are proposed, and the essential mathematical features of the model are analyzed thoroughly in terms of existence, uniqueness, and boundedness of the solution. Besides, the stability analysis of all possible equilibrium points is discussed. Finally, competition between living organisms on food and refuge was checked to reach the proposed system's natural ecological balance.*

Keywords: Food web, Refuge privacy, Stage-structure, Inter and intra-specific competition, Stability analysis, Lyapunov function

1. **Introduction.** Animal behaviors are described through ecological features such as foraging strategies and habitat usage, which form natural selection objects. Hence, it is essential to study the stability of an ecosystem, prone to environmental disturbances. In an ecosystem containing several interacting species, the behavior of one of them depends on the interactions with other individuals, which coexist within the ecosystem, as well as on its ecotype [1]. Despite the diverse ecological features and niches of the coexisting species, they share some traits such as sex and the corresponding sexual differences. Many eukaryotic species propagate sexually, especially when the life aspects between both sexes are different. Sexual species predominate in nature [2, 3, 4, 5]. This fact motivated us to determine the effect of evolution from an ecological perspective, such as the coexistence of species and society's structure. Many researchers in this field are attempting to generalize

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the pattern formation of predator-prey models and create a more involving and realistic system. This new system could then be used in many applications that could be of interest to explain the existence and the global asymptotic stability of predator-prey models. In 1920, two mathematicians, Lotka and Volterra [6], were the first to introduce such a model independently. Berryman [7] considered the dynamical interactions between predator and prey as one of the most essential and central subjects, which play a significant role in mathematical ecologies, natural, social and technological sciences, particularly in terms of ecological and biological research [8, 9, 10, 11, 12, 13].

Most species pass through both immature and mature stages during their life cycle. The juveniles of many species cannot reproduce or hunt for themselves and must rely on the adults for their food. Therefore, in this situation, the introduction of a state structure is inevitable and has been studied for different functional responses [14, 15, 16, 17, 18, 19, 20]. Besides, the refuges that members of a prey species use to protect themselves from predators play a vital role in preventing their numbers from shrinking too rapidly. Yang and Zhong [21] have proposed a system of two-stage-structured deterministic and stochastic predator-prey systems, where immature prey is predated with Beddington-DeAngelo functional responses, and mature prey is predated with Holling type-II functional responses. Majeed and Rahi [22] have studied food chain structures and dynamics across a refuge, stage-structured predator-prey model. Majeed in [23] has discussed the dynamics of the predator-prey model with prey refuge and stage structures in both populations. Determining the consequences of prey refuges on dynamic predator-prey interactions is a challenging area of research. It has been studied intensely by many researchers as it was considered as a fundamental problem in both mathematics and ecology [24, 25, 26, 27, 28, 29, 30, 31].

This study focused on analyzing of a three-different species of animals for a food web system. The system is proposed according to criteria that correspond to a real contemporary problem; its purpose is to simulate the vital role of natural reserves in preserving the existence and permanence of some endangered species. It is also designed to prevent its negative effects on other accompanying neighborhoods that may be harmed as their lives depend on homogenization and mixing with these organisms.

The proposed system consists of four phases:

- The first phase studies the structure of the aggregations of the endangered (Lemur animals) populations, with Lemur animals refuge, as a defensive property against predation.
- The second phase studies the dynamics of the two hunter species (the black panthers and hyenas animals) as they compete for the same Lemur animal species. Using the effect of the Lemur animals refuge on the dynamics of the system, two types of functional responses are determined: the Lotka-Volterra response and the Holling type-II response.
- The third phase studies the nature of competition between these organisms and their effect on the environment's natural stability.
- The fourth phase studies the role of nature reserves in preserving the presence of some organisms threatened with extinction from the risk of disappearing, such as overfishing, predation, organized displacement, and fire risk.

The model consists of four ordinary, nonlinear differential equations to describe the interaction between the mature and immature Lemur animals and both hunter species. The remaining sections of this study are set out as follows. Section 2 presents the mathematical model and some underlying assumptions. The nonnegativity, existence, and boundedness of all solutions of the proposed system are proved in this section. Section 3 discusses the necessary conditions for the existence of the equilibrium points. Section 4 presents the stability analysis of equilibrium points. Finally, the discussion and conclusion with some suggestions for future works are shown in Sections 5 and 6.

2. **The Basic Idea of the Model.** The food web model consists of a mid-level hunter (Hyenas), top-level hunter (Black panthers), and the Lemur animals stage structure. The Lemur animals species grow logistically in the absence of hunters, whereas the number of hunters declines exponentially in the absence of Lemur animal species. The Lemur animal populations are divided into the following categories: immature $X_1(T)$, which refers to the population size at time T, and the mature Lemur animal populations $X_2(T)$ at time T. $Y_1(T)$ represents the mid-level hunter populations size at time T, while $Y_2(T)$ denotes the top-level hunter at time T. Such a model can be described mathematically by the following hypotheses.

1) The immature Lemur animal species is completely dependent on the adults for its food, which denotes the logistic growth with an actual growth rate $r > 0$. The immature Lemur animal individuals become mature Lemur animal individuals with a growing rate of $\beta > 0$. The intra-specific competition rate, among the immature Lemur animal individuals is defined as $c_1 > 0$, while the intra-specific competition rate, between the mature Lemur animal individuals is $c_2 > 0$. In addition, all individuals of the immature and mature Lemur animals are facing death, with natural death rates $d_1 > 0$ and $d_2 > 0$ respectively.
2) The environment protects Lemur animal species against predation, with a refuge rate equal to $m \in (0, 1)$. Therefore, the number of Lemur animal species that predation can obtain is $(1 - m)$.
3) The mid-level hunter consumes Lemur animal individuals in both compartments (mature and immature) according to the Holling type-II functional response, with predation rates $\rho_1 > 0$, and $\rho_2 > 0$ respectively. $b_1 > 0$, and $b_2 > 0$ represent the half saturation constants, which contribute a portion of such food with the rates of conversion $0 < e_1 < 1$, and $0 < e_2 < 1$ respectively.
4) The top-level hunter also consumes Lemur animal individuals in both compartments (mature and immature) according to the Lotka-Volterra functional response, with predation rates $a_1 > 0$, and $a_2 > 0$ respectively, and according to the contribution of a portion of such food with the rates of conversion $0 < e_3 < 1$, and $0 < e_4 < 1$ respectively. Moreover, there is also inter-specific competition between the two predator species to consider, with competition rates $c_3 > 0$ and $c_4 > 0$, respectively.
5) Finally, the first and second hunters face death in the absence of Lemur animals, with rates $d_3 > 0$, and $d_4 > 0$, respectively.

Based on the assumptions above, the idea of the model was mathematically designed, and as shown in Figure 1.

Figure 1 simulates the virtual reality of the proposed ecological model, including a nature reserve containing (immature and mature Lemur animal populations). It is shaded in a green frame, with (top and mid-level hunter) outside the boundaries of this reserve; they compete with each other for food. The biological behavior of this problem is described as follows:

rX_2: *represent the growth of a mature Lemur animals*
βX_1: *represent the maturation of some Lemur animals*
$c_1X_1^2$: *represent the competition between the immature Lemur animals*
$c_2X_2^2$: *represent the competition between the mature Lemur animals*
d_1X_1: *represent the natural death of an immature Lemur animals*
d_2X_2: *represent the natural death of the mature Lemur animals*

$\frac{e_1\rho_1(1-m)X_1}{b_1+(1-m)X_1}Y_1$: *represent a predation of the mid-level hunter individuals for immature Lemur animals*

$\frac{e_2\rho_2(1-m)X_2}{b_2+(1-m)X_2}Y_1$: *represent a predation of the mid-level hunter individuals for mature Lemur animals*

d_3Y_1: *represent the natural death of the mid-level hunter individuals*

$e_3a_1(1-m)X_1Y_2$: *represent a predation of the top-level hunter individuals for immature Lemur animals*

$e_4a_2(1-m)X_2Y_2$: *represent a predation of the top-level hunter individuals for mature Lemur animals*

d_4Y_2: *represent the natural death of the top-level hunter individuals*

$c_{3,4}Y_1Y_2$: *represent the competition between the top and mid-level hunter individuals*

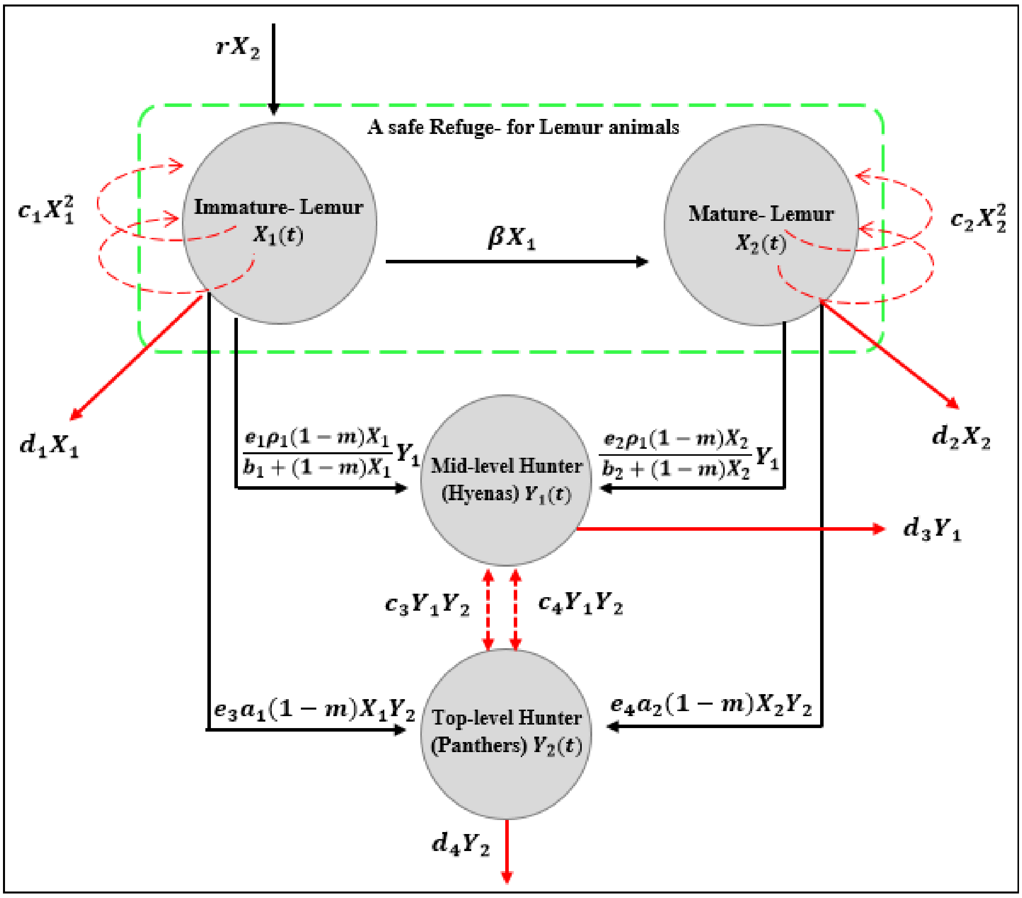

FIGURE 1. A graphical scheme of an ecological model, which includes a nature reserve with three-different species of animals shared in a food web

The general natural ecological interaction is described by the following system of four nonlinear differential equations,

$$\frac{dX_1}{dT} = rX_2 - \frac{\rho_1(1-m)X_1}{b_1+(1-m)X_1}Y_1 - a_1(1-m)X_1Y_2 - \beta X_1 - d_1X_1 - c_1X_1^2$$
$$\frac{dX_2}{dT} = \beta X_1 - \frac{\rho_2(1-m)X_2}{b_2+(1-m)X_2}Y_1 - a_2(1-m)X_2Y_2 - d_2X_2 - c_2X_2^2$$
$$\frac{dY_1}{dT} = \frac{e_1\rho_1(1-m)X_1}{b_1+(1-m)X_1}Y_1 + \frac{e_2\rho_2(1-m)X_2}{b_2+(1-m)X_2}Y_1 - d_3Y_1 - c_3Y_1Y_2$$

$$\frac{dY_2}{dT} = e_3a_1(1-m)X_1Y_2 + e_4a_2(1-m)X_2Y_2 - d_4Y_2 - c_4Y_1Y_2 \tag{1}$$

where the initial conditions are $X_i(0) \geq 0$ and $Y_i(0) \geq 0$, $i = 1, 2$. There are 21 parameters in the proposed system, which causes analyzing difficulty. Therefore, some of these parameters are reduced based on the following representations, which are used as dimensionless variables and help to simplify the system.

$$t = rT, \quad x = \frac{c_1}{r}X_1, \quad y = \frac{c_2}{r}X_2, \quad z = \frac{\rho_1 c_1}{r^2}Y_1 \text{ and } w = \frac{a_1(1-m)}{r}Y_2$$

The non-dimensional form of system (1) can be represented as:

$$\begin{aligned}
\frac{dx}{dt} &= u_1y - \frac{zx}{u_2+x} - wx - (u_3+u_4)x - x^2 = f_1(x,y,z,w); \quad x(0) \geq 0 \\
\frac{dy}{dt} &= u_5x - \frac{u_6zy}{u_7+y} - u_8wy - u_9y - y^2 = f_2(x,y,z,w); \quad y(0) \geq 0 \\
\frac{dz}{dt} &= \frac{u_{10}xz}{u_2+x} + \frac{u_{11}yz}{u_7+y} - u_{12}z - u_{13}wz = f_3(x,y,z,w); \quad z(0) \geq 0 \\
\frac{dw}{dt} &= u_{14}xw + u_{15}yw - u_{16}w - u_{17}zw = f_4(x,y,z,w); \quad w(0) \geq 0
\end{aligned} \tag{2}$$

Here:
$u_1 = \frac{c_1}{c_2}$; $u_2 = \frac{b_1c_1}{r(1-m)}$; $u_3 = \frac{\beta}{r}$; $u_4 = \frac{d_1}{r}$; $u_5 = \frac{\beta c_2}{rc_1}$; $u_6 = \frac{\rho_2c_2}{\rho_1c_1}$; $u_7 = \frac{b_2c_2}{r(1-m)}$; $u_8 = \frac{a_2}{a_1}$; $u_9 = \dfrac{d_2}{r}$; $u_{10} = \frac{e_1\rho_1}{r}$; $u_{11} = \frac{e_2\rho_2}{r}$; $u_{12} = \frac{d_3}{r}$; $u_{13} = \frac{c_2}{a_1(1-m)}$; $u_{14} = \frac{e_3a_1(1-m)}{c_1}$; $u_{15} = \frac{e_4a_2(1-m)}{c_2}$; $u_{16} = \frac{d_4}{r}$; $u_{17} = \frac{rc_4}{\rho_1c_1}$.

Since there are 21 parameters in system (1), we try to reduce the number of parameters in system (2) to 17. It is clear that system (2) has continuous interaction functions, and their partial derivatives are also continuous on $\mathbb{R}^4_+ = \{(x,y,z,w) \in \mathbb{R}^4_+ : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0 \text{ and } w(0) \geq 0\}$. It means that they are Lipschitzian functions, which leads us to conclude that the solution of system (2) exists and is unique. On the other hand, the existence and coexistence of the mid-level hunter in the system (2) are dependent on the following necessary and sufficient condition:

$$u_{12} < \min\{u_{10}, u_{11}\} \tag{3}$$

Furthermore, all other resolutions of this system are uniformly bounded with initial non-negative conditions. These parameters are formulated in the theorem shown below.

Theorem 2.1. *All the solutions of system (2), with the initial points in $\mathbb{R}^4_+$ are uniformly bounded.*

Proof: Let $N(t) = x(t) + y(t) + z(t) + w(t)$ be a solution of system (2), with the initial conditions $(x_0, y_0, z_0, w_0) \in \mathbb{R}^4_+$. Then the time derivative along N is given by

$$\frac{dN}{dt} \leqslant y(u_1 - y) + x(1-x) - \delta_1 N \leqslant \delta_2 - \delta_1 N \tag{4}$$

where
$\delta_1 = \min\{u_3 + u_4, u_9, u_{12}, u_{16}\}$.
$\delta_2 = \frac{1}{4}\left(u_1^2 + u_5^2\right)$.

Thus, using the standard comparison theorem [32], we get

$$0 < N(t) \leqslant \left(N_0\, e^{-\delta_1 t} + \frac{\delta_2}{\delta_1}\left(1 - e^{-\delta_1}t\right)\right) \tag{5}$$

So, for $t \to \infty$, $0 < N(t) \leqslant \frac{\delta_2}{\delta_1}$ is obtained, and all the solutions of system (2) in $\mathbb{R}^4_+$ are uniformly bounded. □

3. **The Existence of Equilibria.** This section simulates the existence of biological equilibrium. It also matches the existence mechanism reality of the nature of life and reproduction. In addition to the existence of death, birth, predation, etc., as basic variables, there are five equilibrium points in system (2), namely $E_i = (x, y, z, w)$, $i = 0, 1, 2, 3$, and 4. Each of these biologically feasible points has an existing condition, for example,

1) The vanishing point $E_0 = (0, 0, 0, 0)$, represents all organism extinction of system (2) for many different reasons that simulate the proposed natural environment, it always exists.
2) The hunters-free point $E_1 = (\bar{x}, \bar{y}, 0, 0)$, represents the extinction of top and mid-level hunters with emphasizing the presence of Lemur animals, this point also simulates a specific biological reality within system (2), where

$$\bar{x} = \frac{u_9\bar{y} + \bar{y}^2}{u_5} \tag{6}$$

and $\bar{y}$ represents a positive root of the third-order polynomial equation as:

$$A_1y^3 + A_2y^2 + A_3y + A_4 = 0 \tag{7}$$

where
$A_1 = 1.$
$A_2 = 2u_9.$
$A_3 = (u_5(u_3 + u_4) + u_9^2).$
$A_4 = (u_5u_9(u_3 + u_4) - u_1u_5^2).$
So, E_1 exists uniquely in the first quadrant of xy-plane provided that the following condition holds:

$$u_1 > \frac{u_9(u_3 + u_4)}{u_5} \tag{8}$$

3) The mid-level hunter-free point $E_2 = (\bar{\bar{x}}, \bar{\bar{y}}, 0, \bar{\bar{w}})$, represents the extinction of mid-level hunter with emphasizing the presence of Lemur animals and top-level hunter. It also simulates a specific biological reality within system (2), it exists uniquely in the first octant of the xyw-space, when there is a positive solution to the following set of algebraic Equations (9)-(11):

$$\frac{u_1y}{x} - ((u_3 + u_4) + x - w) = 0 \tag{9}$$

$$\frac{u_5x}{y} - (u_9 + y + u_8w) = 0 \tag{10}$$

$$u_{14}x + u_{15}y - u_{16} = 0 \tag{11}$$

From Equation(11), we have

$$x = \frac{u_{16} - u_{15}y}{u_{14}} \tag{12}$$

By substituting Equation (12) in (9) and (11), the following two isoclines are obtained:

$$\begin{aligned} f_1(y, w) = u_1u_{14}y - (u_{16} - u_{15}y)w - (u_3 + u_4)(u_{16} - u_{15}y) \\ - \left(\frac{u_{16}^2 - 2u_{15}u_{16} + u_{15}^2y^2}{u_{14}}\right) = 0 \end{aligned} \tag{13}$$

$$f_2(y, w) = u_5\left(\frac{u_{16} - u_{15}y}{u_{14}}\right) - u_8yw - u_9y - y^2 = 0 \tag{14}$$

From Equation (13), we notice that when $w \to 0$ then $y \to k_1$. k_1 represents the root of the polynomial Equation (15):

$$D_1 y^2 + D_2 y + D_3 = 0 \tag{15}$$

where
$D_1 = u_{15}^2$.
$D_2 = -\left(u_1 u_{14}^2 + (u_3 + u_4) u_{14} u_{15}\right)$.
$D_3 = -(u_{16}(2u_{15} - (u_{16} + (u_3 + u_4)u_{14})))$.
The sequential computing shows that Equation (15) has a unique positive root k_1 if the following condition holds:

$$u_{15} > \frac{u_{16} + (u_3 + u_4)u_{14}}{2} \tag{16}$$

Furthermore, Equation (14) shows that if $w \to 0$ then $y \to k_2$. k_2 is a positive root of polynomial (17),

$$H_1 y^2 + H_2 y + H_3 = 0 \tag{17}$$

where
$H_1 = u_{14}$.
$H_2 = u_{14}(u_5 u_{15} + u_9)$.
$H_3 = -u_5 u_{14} u_{15}$.
Sequential computing shows that Equation (17) has a unique positive root k_2.

From Equation (13), we have $\frac{dy}{dw} = -\left(\frac{\partial f_1}{\partial w}\right) \Big/ \left(\frac{\partial f_1}{\partial y}\right)$. So $\frac{dy}{dw} > 0$, if at least one of the following conditions holds:

$$\left(\frac{\partial f_1}{\partial w}\right) > 0, \left(\frac{\partial f_1}{\partial y}\right) < 0 \text{ or } \left(\frac{\partial f_1}{\partial w}\right) < 0, \left(\frac{\partial f_1}{\partial y}\right) > 0 \tag{18}$$

Furthermore, from Equation (14) we notice $\frac{dy}{dw} = -\left(\frac{\partial f_2}{\partial w}\right) \Big/ \left(\frac{\partial f_2}{\partial y}\right)$. So $\frac{dy}{dw} < 0$, if at least one of the following conditions holds:

$$\left(\frac{\partial f_2}{\partial w}\right) > 0, \left(\frac{\partial f_2}{\partial y}\right) > 0 \text{ or } \left(\frac{\partial f_2}{\partial w}\right) < 0, \left(\frac{\partial f_2}{\partial y}\right) < 0 \tag{19}$$

Then, according to the conditions (18) and (19) and in addition to the condition $k_2 > k_1$, the two isoclines (13) and (14) intersect at a unique positive point $(\bar{\bar{y}}, \bar{\bar{w}})$ in the $Int.\mathbb{R}_+^2$ of yw-plane. Thus, if the value of $\bar{\bar{y}}$ is substituted in Equation (12), we get $x(\bar{\bar{y}}) = \bar{\bar{x}}$, which is only positive if the following condition holds:

$$\bar{\bar{y}} < \frac{u_{16}}{u_{15}} \tag{20}$$

Accordingly, the equilibrium point E_2 exists uniquely in the $Int.\mathbb{R}_+^3$ of xyw-space if, in addition to conditions (18)-(20), the isocline $f_1(y, w) = 0$ intersects the y-axis at the positive point $\bar{\bar{k}}_1$.

4) The top-level hunter-free point $E_3 = \left(\bar{\bar{\bar{x}}}, \bar{\bar{\bar{y}}}, \bar{\bar{\bar{z}}}, 0\right)$, represents the extinction of top-level hunter with emphasizing the presence of Lemur animals and mid-level hunter, and this point also simulates a specific biological reality within system (2). It exists uniquely in the interior of the first octant of xyz-space if there is a positive solution to the following set of algebraic Equations (21)-(23):

$$\frac{u_1 y}{x} - \left((u_3 + u_4) + x + \frac{z}{u_2 + x}\right) = 0 \tag{21}$$

$$\frac{u_5x}{y} - \left(u_9 + y + \frac{u_6z}{u_7+y}\right) = 0 \tag{22}$$

$$\frac{u_{10}x}{u_2+x} + \frac{u_{11}y}{u_7+y} - u_{12} = 0 \tag{23}$$

From Equation (23), we have

$$x = \frac{r_4 - r_3y}{r_1 + r_2y} \tag{24}$$

where

$r_1 = u_7(u_{10} - u_{12})$.
$r_2 = (u_{10} + u_{11} - u_{12})$.
$r_3 = u_2(u_{11} - u_{12})$.
$r_4 = u_2u_7u_{12}$.

Substituting Equation (24) for (21) and (22) yields the following two isoclines:

$$g_1(y,z) = u_1y - \left(\frac{(r_4 - r_3y)z}{u_1(r_1+r_2y) + (r_4 - r_3y)}\right) - (u_3+u_4)\left(\frac{r_4 - r_3y}{r_1+r_2y}\right) - \left(\frac{r_4 - r_3y}{r_1+r_2y}\right)^2 = 0 \tag{25}$$

$$g_2(y,z) = u_5\left(\frac{r_4 - r_3y}{r_1+r_2y}\right) - \left(\frac{u_6yz}{u_7+y}\right) - u_9y - y^2 = 0 \tag{26}$$

In Equation (25), when $z \to 0$, we have $y \to y_1$. Here y_1 represents the root of the polynomial Equation (27)

$$B_1y^3 + B_2y^2 + B_3y + B_4 = 0 \tag{27}$$

where

$B_1 = u_1r_2^2$.
$B_2 = 2u_1r_1r_2 + (u_3+u_4)r_2r_3 - r_3^2$.
$B_3 = u_1r_1^2 + (u_3+u_4)r_1r_3 + r_4 - (u_3+u_4)r_2r_4$.
$B_4 = -((u_3+u_4)r_1r_4 + r_4^2)$.

Sequential computing shows that, Equation (27) has a unique positive root y_1 if the following conditions hold

$$B_2 > 0 \text{ or } B_3 < 0 \tag{28}$$

Furthermore, from Equation (26), when $z \to 0$ then $y \to y_2$, where y_2 denotes a positive root of the polynomial Equation (29).

$$C_1y^3 + C_2y^2 + C_3y + C_4 = 0 \tag{29}$$

where

$C_1 = r_2$.
$C_2 = u_9r_2 + r_1$.
$C_3 = u_5r_3 + u_9r_1$.
$C_4 = -u_5r_4$.

Sequential computing shows that Equation (29) has a unique positive root y_2.

From Equation (25), we have $\frac{dy}{dz} = -\left(\frac{\partial g_1}{\partial z}\right)\Big/\left(\frac{\partial g_1}{\partial y}\right)$. So $\frac{dy}{dz} > 0$, if at least one of the following conditions holds

$$\left(\frac{\partial g_1}{\partial z}\right) > 0, \left(\frac{\partial g_1}{\partial y}\right) < 0 \text{ or } \left(\frac{\partial g_1}{\partial z}\right) < 0, \left(\frac{\partial g_1}{\partial y}\right) > 0 \tag{30}$$

Furthermore, from Equation (26) we can see $\frac{dy}{dz} = -\left(\frac{\partial g_2}{\partial z}\right) \Big/ \left(\frac{\partial g_2}{\partial y}\right)$. So $\frac{dy}{dz} < 0$, if at least one of the following conditions holds

$$\left(\frac{\partial g_2}{\partial z}\right) > 0, \left(\frac{\partial g_2}{\partial y}\right) > 0 \text{ or } \left(\frac{\partial g_2}{\partial z}\right) < 0, \left(\frac{\partial g_2}{\partial y}\right) < 0 \tag{31}$$

Then, according to the conditions (30) and (31) and in addition to the condition $y_2 > y_1$, the two isoclines (25) and (26) intersect at a unique positive point $\left(\bar{\bar{\bar{y}}}, \bar{\bar{\bar{z}}}\right)$ in the $Int.\mathbb{R}_+^2$ of yz-plane. So, if the value of $\bar{\bar{\bar{y}}}$ is substituted in Equation (24), we get $x\left(\bar{\bar{\bar{y}}}\right) = \bar{\bar{\bar{x}}}$, which is only positive if the following condition holds

$$\bar{\bar{\bar{y}}} < \frac{r_4}{r_3} \tag{32}$$

Accordingly, the equilibrium point E_3 exists uniquely in the $Int.\mathbb{R}_+^3$ of xyz-space, but only if, in addition to conditions (30)-(32) begin satisfied, the isocline $g_1(y, z) = 0$ intersects the y-axis at the positive point $\bar{\bar{\bar{y}}}_1$.

5) Finally, the positive (coexistence) equilibrium point $E_4 = (x^*, y^*, z^*, w^*)$, represents the presence of all organism of the system (2), and it is considered as one of the essential equilibrium points and an ideal case that simulates the biological reality of an ecological model. It only exists uniquely in the $Int.\mathbb{R}_+^4$ if there is a positive solution to the following set of algebraic Equations (33)-(36)

$$\frac{u_1 y}{x} - \left[(u_3 + u_4) + x + \frac{z}{u_2 + x} + w\right] = 0 \tag{33}$$

$$\frac{u_5 x}{y} - \left[u_9 + y + \frac{u_6 z}{u_7 + y} + u_8 w\right] = 0 \tag{34}$$

$$\frac{u_{10} x}{u_2 + x} + \frac{u_{11} y}{u_7 + y} - (u_{12} + u_{13} w) = 0 \tag{35}$$

$$u_{14} x + u_{15} y - (u_{16} + u_{17} z) = 0 \tag{36}$$

From Equation (36), we have

$$z(x, y) = \frac{u_{14} x + u_{15} y - u_{16}}{u_{17}} \tag{37}$$

Also, from Equation (35) we have

$$w(x, y) = \frac{r_1 x + r_5 xy + r_3 y - r_4}{u_2 u_7 u_{13} + u_7 u_{13} x + u_2 u_{13} y + u_{13} xy} \tag{38}$$

Then, substituting Equations (37) and (38) in Equations (33) and (34) yields the following two isoclines

$$\begin{aligned} L_1(x, y) = u_1 y &- \frac{u_{14} x^2 + u_{16} x - u_{15} xy}{u_{17}(u_2 + x)} - \frac{r_1 x^2 - r_5 x^2 y - r_3 xy + r_4 x}{u_7 u_{13} x + u_2 u_{13} y + u_{13} xy + u_2 u_7 u_{13}} \\ &- (u_3 + u_4) x - x^2 = 0 \end{aligned} \tag{39}$$

$$\begin{aligned} L_2(x, y) = u_5 x &- \frac{u_6 u_{14} xy - u_6 u_{15} y^2 + u_6 u_{16} y}{u_{17}(u_7 + y)} - \frac{u_8 r_1 xy - u_8 r_5 xy^2 - u_8 r_3 y^2 + u_8 r_4 y}{u_7 u_{13} x + u_2 u_{13} y + u_{13} xy + u_2 u_7 u_{13}} \\ &- u_9 y - y^2 = 0 \end{aligned} \tag{40}$$

where r_1, r_3 and r_4 (see for Equation (24)), while $r_5 = u_7 u_{10} + u_2 u_{11} - u_{12}$. From Equation (39), when $y \to 0$ we have $x \to a_1$. Here a_1 represents a positive root of the polynomial Equation (41)

$$I_1 x^2 + I_2 x + I_3 = 0 \tag{41}$$

where

$I_1 = u_7u_{13}u_{17}$.

$I_2 = u_7u_{13}u_{14} + u_7u_{10}u_{17} + u_7u_{13}u_7(u_3 + u_4) + u_2u_7u_{13}u_{17} - u_7u_{12}u_{17}$.

$I_3 = -(u_7u_{13}u_{16} + u_2u_7u_{12}u_{17} - [u_2u_7u_{13}u_{17}(u_3 + u_4)]])$.

Moreover, from Equation (39) we have $\frac{dy}{dx} = -\left(\frac{\partial L_1}{\partial y}\right)\Big/\left(\frac{\partial L_1}{\partial x}\right)$. So $\frac{dy}{dx} < 0$, and hence the isocline (39) is decreasing passing through a_1 if at least one of the following conditions holds

$$\left(\frac{\partial L_1}{\partial y}\right) > 0, \left(\frac{\partial L_1}{\partial x}\right) > 0 \text{ or } \left(\frac{\partial L_1}{\partial y}\right) < 0, \left(\frac{\partial L_1}{\partial x}\right) < 0 \tag{42}$$

Furthermore, from Equation (40), when $y \to 0$, then $x = 0$, since we have $\frac{dy}{dx} = -\left(\frac{\partial L_2}{\partial y}\right)\Big/\left(\frac{\partial L_2}{\partial x}\right)$. So $\frac{dy}{dx} > 0$, and hence the isocline (40) is increasing passing through the origin if at least one of the following conditions holds

$$\left(\frac{\partial L_2}{\partial y}\right) > 0, \left(\frac{\partial L_2}{\partial x}\right) < 0 \text{ or } \left(\frac{\partial L_2}{\partial y}\right) < 0, \left(\frac{\partial L_2}{\partial x}\right) > 0 \tag{43}$$

The two isoclines (39) and (40) will intersect at a unique positive point (x^*, y^*), in the $Int.\mathbb{R}^2_+$ of xy-plane. Then if we substitute the value of x^* and y^* in Equations (37) and (38), we get $z\,(x^*, y^*) = z^*$ and $w\,(x^*, y^*) = w^*$, which may only be positive if the following conditions hold

$$u_{14}x^* + u_{15}y^* > u_{16} \tag{44}$$

$$r_1x^* + r_5x^*y^* + r_3y^* > r_{14} \tag{45}$$

Accordingly, the equilibrium point E_4 exists uniquely in the $Int.\mathbb{R}^4_+$ of $xyzw$-space if, in addition to conditions (42)-(45), the isocline $L_1(x, y) = 0$ intersects the x-axis at the positive point a_1^*.

4. **Stability Analyzing.** After studying all possibilities for the existence of equilibrium points for the system (2), investigating and analyzing the possibility of mathematical stability of these points under specific conditions that used to simulate and match the vision of the biological model becomes a necessity. This analysis enabled us to reach the accurate possible details, and help us maintain the environmental equilibrium for this proposed system.

The local dynamic of system (2) around each of its feasible equilibrium points is discussed analytically using the *linearization technique*. Its Jacobian matrix $J(x, y, z, w)$ at any arbitrary point (x, y, z, w) is represented as $J \equiv DF(x) = (\beta_{ij}) \in \mathbb{R}^{4\times 4}$, with

$$J = (\beta_{ij}) \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \end{bmatrix}$$

This is denoted by, $J_k = J(E_k) = \left(\beta_{ij}^{[k]}\right)$ at the equilibrium E_k, where $k = 0, 1, 2, 3, 4$, and

$\beta_{11}^{[k]} = -\left((u_3 + u_4) + 2x + \frac{u_2z}{(u_2+x)^2} + w\right)$; $\beta_{12}^{[k]} = u_1$; $\beta_{13}^{[k]} = -\left(\frac{x}{u_2+x}\right)$; $\beta_{14}^{[k]} = -x$;

$\beta_{21}^{[k]} = u_2$; $\beta_{22}^{[k]} = -\left(u_9 + 2y + \frac{u_6u_7z}{(u_7+y)^2} + u_8w\right)$; $\beta_{23}^{[k]} = -\left(\frac{u_6y}{u_7+y}\right)$; $\beta_{24}^{[k]} = -u_8y$;

$\beta_{31}^{[k]} = \frac{u_2u_{10}z}{(u_2x)^2}$; $\beta_{32}^{[k]} = \frac{u_7u_{11}z}{(u_7y)^2}$; $\beta_{33}^{[k]} = \frac{u_{10}x}{u_2+x} + \frac{u_{11}y}{u_7+y} - (u_{12} + u_{13}w)$; $\beta_{34}^{[k]} = -u_{13}z$;

$\beta_{41}^{[k]} = u_{14}w$; $\beta_{42}^{[k]} = u_{15}w$; $\beta_{43}^{[k]} = -u_{17}w$; $\beta_{44}^{[k]} = u_{14}x + u_{15}y - (u_{16} + u_{17}z)$.

The local stability analysis around the boundary equilibrium points E_i, $i = 0, 1, 2, 3, 4$ is discussed as in the following.

1) $E_0 = (0, 0, 0, 0)$:

Theorem 4.1. *The equilibrium point E_0 will only be locally and asymptotically stable in the $\mathbb{R}^4_+$ if*

$$u_5 < \frac{u_9(u_3 + u_4)}{u_1} \tag{46}$$

Proof: The Jacobian matrix of system (2) near E_0, is

$$J_0 = \begin{bmatrix} -(u_3 + u_4) & u_1 & 0 & 0 \\ u_5 & -u_9 & 0 & 0 \\ 0 & 0 & -u_{12} & 0 \\ 0 & 0 & 0 & -u_{16} \end{bmatrix} \tag{47}$$

The characteristic equation of J_0 is written in the following form

$$\left[\lambda^2 + A_1\lambda + A_2\right](-u_{12} - \lambda)(-u_{16} - \lambda) = 0 \tag{48}$$

where

$A_1 = u_9(u_3 + u_4)$.

$A_2 = u_9(u_3 + u_4) - u_1u_5$.

It can be seen that all roots of (48) will only have negative real parts if condition (46) holds. Therefore, the equilibrium point E_0 is locally asymptotically stable in $\mathbb{R}^4_+$; otherwise, it will be unstable. □

2) $E_1 = (\bar{x}, \bar{y}, 0, 0)$

Theorem 4.2. *The equilibrium point E_1 will only be locally and asymptotically stable in the $\mathbb{R}^4_+$ if*

$$u_{12} + u_{16} > n_3 + n_4 \tag{49}$$

$$n_3n_4 + u_{12}u_{16} > u_{16}n_3 + u_{12}n_4 \tag{50}$$

in addition to the previous condition (46).

Proof: The Jacobian matrix of system (2) near E_1, can be written as

$$J_1 = \begin{bmatrix} -n_1 & u_1 & \dfrac{-\bar{x}}{u_2 + \bar{x}} & -\bar{x} \\ u_5 & -n_2 & \dfrac{-u_6\bar{y}}{u_7 + \bar{y}} & -u_8\bar{y} \\ 0 & 0 & n_3 - u_{12} & 0 \\ 0 & 0 & 0 & n_4 - u_{16} \end{bmatrix} \tag{51}$$

where

$n_1 = (u_3 + u_4) + 2\bar{x}$.

$n_2 = u_9 + 2\bar{y}$.

$n_3 = \frac{u_{10}\bar{x}}{u_2+\bar{x}} + \frac{u_{11}\bar{y}}{u_7+\bar{y}}$.

$n_4 = u_{14}\bar{x} + u_{15}\bar{y}$.

Then, the characteristic equation of J_1 is

$$\left[\lambda^2 + H_1\lambda + H_2\right]\left[\lambda^2 + H_3\lambda + H_4\right] = 0 \tag{52}$$

where

$H_1 = n_1 + n_2$.

$H_2 = n_1n_2 - u_1u_5$.

$H_3 = u_{12} + u_{16} - (n_3 + n_4)$.
$H_4 = n_3 n_4 + u_{12}u_{16} - (u_{16}n_3 + u_{12}n_4)$.
Therefore, all roots of (52) have negative real parts under the conditions (46), (49) and (50). Then, the equilibrium point E_1 is locally asymptotically stable in $\mathbb{R}^4_+$; otherwise it will be unstable. □

3) $E_2 = (\bar{\bar{x}}, \bar{\bar{y}}, 0, \bar{\bar{w}})$

Theorem 4.3. *The equilibrium point E_2 will only be locally and asymptotically stable $\mathbb{R}^4_+$ if*

$$\bar{\bar{w}} > \max\{\Gamma_1, \Gamma_2\} \tag{53}$$

$$\Gamma_3 > \Gamma_4 \tag{54}$$

$$\Gamma_5 > \Gamma_6 \tag{55}$$

Proof: The Jacobian matrix of system (2) near E_2, can be written as

$$J_2 = \begin{bmatrix} \beta^{[2]}_{11} & \beta^{[2]}_{12} & \beta^{[2]}_{13} & \beta^{[2]}_{14} \\ \beta^{[2]}_{21} & \beta^{[2]}_{22} & \beta^{[2]}_{23} & \beta^{[2]}_{24} \\ 0 & 0 & \beta^{[2]}_{33} & 0 \\ \beta^{[2]}_{41} & \beta^{[2]}_{42} & \beta^{[2]}_{43} & \beta^{[2]}_{44} \end{bmatrix} \tag{56}$$

where
$\beta^{[2]}_{11} = -r_1$; $\beta^{[2]}_{12} = u_1$; $\beta^{[2]}_{13} = -\frac{\bar{\bar{x}}}{u_2+\bar{\bar{x}}}$; $\beta^{[2]}_{14} = -\bar{\bar{x}}$; $\beta^{[2]}_{21} = u_5$; $\beta^{[2]}_{22} = -r_2$; $\beta^{[2]}_{23} = -\frac{u_6\bar{\bar{y}}}{u_7+\bar{\bar{y}}}$;
$\beta^{[2]}_{24} = -u_8\bar{\bar{y}}$; $\beta^{[2]}_{33} = r_3 - r_4$; $\beta^{[2]}_{41} = u_{14}\bar{\bar{w}}$; $\beta^{[2]}_{42} = u_{15}\bar{\bar{w}}$; $\beta^{[2]}_{43} = -u_{17}\bar{\bar{w}}$; $\beta^{[2]}_{44} = r_5 - u_{16}$.
where
$r_1 = (u_3 + u_4) + 2\bar{\bar{x}} + \bar{\bar{w}}$.
$r_2 = u_9 + 2\bar{\bar{y}} + u_8\bar{\bar{w}}$.
$r_3 = \frac{u_{10}\bar{\bar{x}}}{u_2+\bar{\bar{x}}} + \frac{u_{11}\bar{\bar{y}}}{u_7+\bar{\bar{y}}}$.
$r_4 = u_{12} + u_{13}\bar{\bar{w}}$.
$r_5 = u_{14}\bar{\bar{x}} + u_{15}\bar{\bar{y}}$.
Therefore, the characteristic equation of J_2 can be written as

$$\left[\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3\right]((r_3 - r_4) - \lambda) = 0. \tag{57}$$

where
$B_1 = -(r_5 - (r_1 + r_2 + u_{16}))$.
$B_2 = r_1r_2 + u_{16}(r_1 + r_2) + u_{14}\bar{\bar{x}}\bar{\bar{w}} + u_8u_{15}\bar{\bar{y}}\bar{\bar{w}} - (r_5(r_1 + r_2) + u_1u_5)$.
$B_3 = \Gamma_3 - \Gamma_4$.
$\Gamma_3 = (r_1u_8u_{15} + u_1u_8u_{14})\bar{\bar{y}}\bar{\bar{w}} + (u_{14}r_2 + u_5u_{15})\bar{\bar{x}}\bar{\bar{w}} + u_1u_5r_5 + u_{16}r_1r_2$.
$\Gamma_4 = -(u_1u_5u_{16} + r_1r_2r_5)$.
So, either

$$(r_3 - r_4) - \lambda = 0$$

We have $\lambda = (r_3 - r_4) < 0$, under the condition (53)
where
$\Gamma_1 = \frac{1}{3}\left[\frac{u_{10}\bar{\bar{x}}}{u_2+\bar{\bar{x}}} + \frac{u_{11}\bar{\bar{y}}}{u_7+\bar{\bar{y}}} - u_{12}\right]$.
$\Gamma_2 = \frac{(u_{14}-2)\bar{\bar{x}}+(u_{15}-2)\bar{\bar{y}}-(2u_{16}+(u_3+u_4))}{(1+u_8)}$.
$\Gamma_5 = r_5^2(r_1+r_2)+u_1u_5r_5+r_1u_2(r_1+r_2+u_{16})+u_{16}(r_1+r_2)(r_1+r_2+u_{16})+u_{14}\bar{\bar{x}}\bar{\bar{w}}(r_1+r_2+u_{16}) + u_8u_{15}(r_1 + r_2 + u_{16})\bar{\bar{y}}\bar{\bar{w}} + \Gamma_4$.
$\Gamma_6 = \Gamma_3 + r_1r_2r_5 + u_{16}r_5(r_1 + r_2) + u_{14}r_5\bar{\bar{x}}\bar{\bar{w}} + u_8u_{15}r_5\bar{\bar{y}}\bar{\bar{w}}$.
Or,

$$\left[\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3\right] = 0.$$

It is clear that when using the Routh-Hurwitz criterion [9], this equation will only have roots with negative real parts if $B_1 > 0$, $B_3 > 0$, and $B_1B_2 - B_3 > 0$, as long as conditions (53)-(55) hold.

Therefore, the equilibrium point E_2 is locally and asymptotically stable in $\mathbb{R}^4_+$, and is unstable otherwise. □

4) $E_3 = \left(\bar{\bar{\bar{x}}}, \bar{\bar{\bar{y}}}, \bar{\bar{\bar{z}}}, 0\right)$

Theorem 4.4. *The equilibrium point E_3 will only be locally and asymptotically stable in the $\mathbb{R}^4_+$ if*

$$\bar{\bar{\bar{z}}} > \frac{u_{14}\bar{\bar{\bar{x}}} + u_{15}\bar{\bar{\bar{y}}}}{u_{17}} \tag{58}$$

$$u_{12} < c_3 < u_{12} + c_1 + c_2 \tag{59}$$

$$((u_{12} + c_1 + c_2) - c_3)\psi_1 > (c_3 - (u_{12} + c_1 + c_2))\psi_2 + Q_3 \tag{60}$$

Proof: The Jacobian matrix of system (2) near E_3, can be written as

$$J_3 = \begin{bmatrix} \beta^{[3]}_{11} & \beta^{[3]}_{12} & \beta^{[3]}_{13} & \beta^{[3]}_{14} \\ \beta^{[3]}_{21} & \beta^{[3]}_{22} & \beta^{[3]}_{23} & \beta^{[3]}_{24} \\ \beta^{[3]}_{31} & \beta^{[3]}_{32} & \beta^{[3]}_{33} & \beta^{[3]}_{34} \\ 0 & 0 & 0 & \beta^{[3]}_{44} \end{bmatrix} \tag{61}$$

where

$\beta^{[3]}_{11} = -c_1$; $\beta^{[3]}_{12} = u_1$; $\beta^{[3]}_{13} = -\frac{\bar{\bar{\bar{x}}}}{u_2+\bar{\bar{\bar{x}}}}$; $\beta^{[3]}_{14} = -\bar{\bar{\bar{x}}}$; $\beta^{[3]}_{21} = u_5$; $\beta^{[3]}_{22} = -c_2$; $\beta^{[3]}_{23} = -\frac{u_6\bar{\bar{\bar{y}}}}{u_7+\bar{\bar{\bar{y}}}}$; $\beta^{[3]}_{24} = -u_8\bar{\bar{\bar{y}}}$; $\beta^{[3]}_{31} = \frac{u_2u_{10}\bar{\bar{\bar{z}}}}{\left(u_2+\bar{\bar{\bar{x}}}\right)^2}$; $\beta^{[3]}_{32} = \frac{u_7u_{11}\bar{\bar{\bar{z}}}}{\left(u_7+\bar{\bar{\bar{y}}}\right)^2}$; $\beta^{[3]}_{33} = c_3 - u_{12}$; $\beta^{[3]}_{34} = -u_{13}\bar{\bar{\bar{z}}}$; $\beta^{[3]}_{44} = c_4 - c_5$

here

$c_1 = (u_3 + u_4) + 2\bar{\bar{\bar{x}}} + \frac{u_2\bar{\bar{\bar{z}}}}{\left(u_2+\bar{\bar{\bar{x}}}\right)^2}$.

$c_2 = u_9 + 2\bar{\bar{\bar{x}}} + \frac{u_6u_7\bar{\bar{\bar{z}}}}{\left(u_7+\bar{\bar{\bar{y}}}\right)^2}$.

$c_3 = \frac{u_{10}\bar{\bar{\bar{x}}}}{u_2+\bar{\bar{\bar{x}}}} + \frac{u_{11}\bar{\bar{\bar{y}}}}{u_7+\bar{\bar{\bar{y}}}}$.

$c_4 = u_{14}\bar{\bar{\bar{x}}} + u_{15}\bar{\bar{\bar{y}}}$.

$c_5 = u_{16} + u_{17}\bar{\bar{\bar{z}}}$.

Therefore, the characteristic equation of J_3 is written as

$$\left[\lambda^3 + Q_1\lambda^2 + Q_2\lambda + Q_3\right]((c_4 - c_5) - \lambda) = 0 \tag{62}$$

where

$Q_1 = -(c_3 - (u_{12} + c_1 + c_2))$.

$Q_2 = \psi_1 - \psi_2$.

$Q_3 = \frac{c_1u_6u_7u_{11}\bar{\bar{\bar{y}}}\bar{\bar{\bar{z}}}}{\left(u_7\bar{\bar{\bar{y}}}\right)^3} + \frac{u_1u_2u_6u_{10}\bar{\bar{\bar{y}}}\bar{\bar{\bar{z}}}}{\left(u_7+\bar{\bar{\bar{y}}}\right)\left(u_2+\bar{\bar{\bar{x}}}\right)^2} + Q_1 + u_1u_5(c_3 - u_{12})$.

$\psi_1 = \frac{u_6\bar{\bar{\bar{y}}}\left(u_7+u_{11}\bar{\bar{\bar{z}}}\bar{\bar{\bar{x}}}\right)}{\left(u_7\bar{\bar{\bar{y}}}\right)^3+c_1c_2+u_{12}c_1}$.

$\psi_2 = (u_1u_5 + c_1c_3)$.

So, either

$$(c_4 - c_5) - \lambda = 0.$$

We have $\lambda = (c_4 - c_5) < 0$, under the condition (58), or

$$\left[\lambda^3 + Q_1\lambda^2 + Q_2\lambda + Q_3\right] = 0.$$

Using Routh Hurwitz criterion [9], this equation will only have roots with negative real parts if $Q_1 > 0$, $Q_3 > 0$ and $Q_1Q_2 - Q_3 > 0$, as long as conditions (59) and (60) hold.

So, the equilibrium point E_3 will be locally asymptotically stable in $\mathbb{R}^4_+$, and will be unstable otherwise. □

5) $E_4 = (x^*, y^*, z^*, w^*)$

Theorem 4.5. *We assume that the coexistence equilibrium point E_4 exists in $\mathbb{R}^4_+$, and the following conditions are satisfied*

$$p_{12}^2 < \frac{4}{9}\ p_{11}p_{22} \tag{63}$$

$$p_{13}^2 < \frac{4}{9}\ p_{11}p_{33} \tag{64}$$

$$p_{14}^2 < \frac{4}{9}\ p_{11}p_{44} \tag{65}$$

$$p_{23}^2 < \frac{4}{9}\ p_{22}p_{33} \tag{66}$$

$$p_{24}^2 < \frac{4}{9}\ p_{22}p_{44} \tag{67}$$

$$p_{34}^2 > \frac{4}{9}\ p_{33}p_{44} \tag{68}$$

then E_4 is locally and asymptotically stable in $\mathbb{R}^4_+$.

Proof: The Jacobian matrix of system (2) near E_4, can be written as

$$J_4 = \left[\beta_{ij}^{[4]}\right]_{4*4} \tag{69}$$

where

$\beta_{11}^{[4]} = -\left((u_3+u_4) + 2x^* + \frac{u_2 z^*}{(u_2+x^*)^2} + w^*\right)$; $\beta_{12}^{[4]} = u_1$; $\beta_{13}^{[4]} = -\frac{x^*}{u_2+x^*}$; $\beta_{14}^{[4]} = -x^*$;
$\beta_{21}^{[4]} = u_5$; $\beta_{22}^{[4]} = -\left(u_9 + 2y^* + \frac{u_6 u_7 z^*}{(u_7+y^*)^2} + u_8 w^*\right)$; $\beta_{23}^{[4]} = -\frac{u_6 y^*}{u_7+y^*}$; $\beta_{24}^{[4]} = -u_8 y^*$;
$\beta_{31}^{[4]} = \frac{u_2 u_{10} z^*}{(u_2+x^*)^2}$; $\beta_{32}^{[4]} = \frac{u_7 u_{11} z^*}{(u_7+y^*)^2}$; $\beta_{33}^{[4]} = \frac{u_{10}x^*}{u_2+x^*} + \frac{u_{11}y^*}{u_7+y^*} - (u_{12} + u_{13}w^*)$; $\beta_{34}^{[4]} = -u_{13}z^*$;
$\beta_{41}^{[4]} = u_{14}w^*$; $\beta_{42}^{[4]} = u_{15}w^*$; $\beta_{43}^{[4]} = -u_{17}w^*$; $\beta_{44}^{[4]} = u_{14}x^* + u_{15}y^* - (u_{16} + u_{17}z^*)$

Therefore, the linearization of system (2) can be written as

$$\frac{dU}{dt} = J(E_4)U, \text{ where } U = (h_1, h_2, h_3, h_4)^t \text{ and } h_i = x_i - x_i^*$$

Next, we can consider the following function

$$V^{[4]} = \frac{h_1^2}{2} + \frac{h_2^2}{2} + \frac{h_3^2}{2} + \frac{h_4^2}{2}.$$

where $V^{[4]}$: $\mathbb{R}^4_+ \longrightarrow \mathbb{R}$ and $V^{[4]}(E_4) = 0$ with $V^{[4]}(E) \neq 0$, $\forall E \neq E_4$, $E \in \mathbb{R}^4_+$. Hence, it is a positive definite function in $\mathbb{R}^4_+$. Also, the derivative of $V^{[4]}$ with respect to the time (t) is given as in the following:

$$\begin{aligned}\frac{dV^{[4]}}{dt} &= -p_{11}h_1^2 + p_{12}h_1h_2 + p_{13}h_1h_3 + p_{14}h_1h_4 - p_{22}h_2^2 + p_{23}h_2h_3 + p_{24}h_2h_4 \\ &\quad - p_{33}h_3^2 - p_{34}h_3h_4 - p_{44}h_4^2\end{aligned} \tag{70}$$

where

$p_{11} = (u_3+u_4) + 2x^* + \frac{u_2 z^*}{(u_2+x^*)^2} + w^*$; $p_{12} = (u_1+u_5)$; $p_{13} = \frac{u_2u_{10}z^*}{(u_2+x^*)^2} - \left(\frac{x^*}{u_2+x^*}\right)$;
$p_{14} = u_{14}w^* - x^*$; $p_{22} = u_9 + 2y^* + \frac{u_6u_7z^*}{(u_7+y^*)^2} + u_8w^*$; $p_{23} = \frac{u_7u_{11}z^*}{(u_7+y^*)^2} - \left(\frac{u_6y^*}{u_7+y^*}\right)$;
$p_{24} = u_{15}w^* - u_8y^*$; $p_{33} = u_{12} + u_{13}w^* - \left(\frac{u_{10}x^*}{u_2+x^*} + \frac{u_{11}y^*}{u_7+y^*}\right)$; $p_{34} = u_{13}z^* + u_{17}w^*$;
$p_{44} = u_{16} + u_{17}z^* - (u_{14}x^* + u_{15}y^*)$.

According to conditions (63)-(68) we get

$$\frac{dV^{[4]}}{dt} \leqslant -\left[\frac{\sqrt{p_{11}}}{\sqrt{3}}h_1 - \frac{\sqrt{p_{22}}}{\sqrt{3}}h_2\right]^2 - \left[\frac{\sqrt{p_{11}}}{\sqrt{3}}h_1 - \frac{\sqrt{p_{33}}}{\sqrt{3}}h_3\right]^2 - \left[\frac{\sqrt{p_{11}}}{\sqrt{3}}h_1 - \frac{\sqrt{p_{44}}}{\sqrt{3}}h_4\right]^2$$
$$- \left[\frac{\sqrt{p_{22}}}{\sqrt{3}}h_1 - \frac{\sqrt{p_{33}}}{\sqrt{3}}h_3\right]^2 - \left[\frac{\sqrt{p_{22}}}{\sqrt{3}}h_2 - \frac{\sqrt{p_{44}}}{\sqrt{3}}h_4\right]^2 - \left[\frac{\sqrt{p_{33}}}{\sqrt{3}}h_3 + \frac{\sqrt{p_{44}}}{\sqrt{3}}h_4\right]^2$$

where $\frac{dV^{[4]}}{dt} < 0$ and hence, $V^{[4]}$ is Lyapunov function [33]. Thus E_4 is a locally asymptotically stable in $\mathbb{R}^4_+$. □

5. **Discussion.** In this study, we proposed an ecological model for a food web involving a refuge stage-structured. The model consists of three different species of animals: the Lemur animal populations and two types of hunters (the black panthers and hyenas animals). From this model, we can deduce that the intra-species competition among the Lemur animal populations and the substantial difference between the two types of hunters will create struggles between them for survival. It also has shown a proper matching in the results under certain conditions with reality. One of the main aims of this study was to determine how to protect rare species from extinction due to predation and to protect hunter populations by achieving a natural ecological balance. So both hunters and Lemur populations can thrive.

To understand the ecological effects of stage-structured Lemur animals with hunters population interaction, we have analyzed an ecological model that depicts the dynamics of a real food web system, with two different types of functional responses (*Lotka-Volterra* and *Holling type-II*) that characterize the competition between species, respectively. The proposed model consists of four nonlinear autonomous differential equations, which describe the dynamics of four different populations: the immature Lemur animal (X_1), mature Lemur animal (X_2), the mid-level hunter (Y_1), and the top-level hunter (Y_2). The positivity, existence, and boundedness of all solutions of the system were discussed. Besides, the existence of possible equilibrium points and discussion of the stability analysis around these points are also discussed. We observed that there are at most five non-negative equilibrium points in $\mathbb{R}^4_+$.

It is notable that, the trivial point (E_0) always exists and was only locally and asymptotically stable if condition (46) was satisfied. The hunters-free equilibrium point (E_1) exists under condition (8), and it was only a locally and asymptotically stable point if conditions (46), (49) and (50) hold. The mid-level hunter-free equilibrium point (E_2) uniquely exists in $\mathbb{R}^3_+$ if, in addition to conditions (18)-(20) holding, the isocline $f_1(y, w) = 0$ intersects the y-axis at the positive value $\bar{\bar{k}}_1$. Also, it was a locally and asymptotically stable point if conditions (53)-(55) were satisfied. The top-level hunter-free equilibrium point (E_3) uniquely exists in $\mathbb{R}^3_+$ if, in addition to conditions (30)-(32) being satisfied, $g_1(y, z) = 0$ intersects the y-axis at $\bar{\bar{\bar{y}}}_1$. Also, if conditions (58)-(60) were held, then it was locally and asymptotically stable, and vice versa. Moreover, (E_4) uniquely exists in $\mathbb{R}^4_+$ if that conditions (42)-(45) were held, and $L_1(x, y) = 0$ intersected the x-axis at a_1^*. Therefore, it was locally and asymptotically stable if conditions (63)-(68) were satisfied.

6. **Conclusions and Suggestion for Future Work.** Three different species of animals are studied in a food web model designed to preserve species (Lumbar animals) from extinction under objective conditions that mimic reality. It also focuses on preserving environmental diversity by providing a suitable environment for all the organisms under study. Simulations of the shrinking instinct between the organisms under investigation for food, shelter, and survival are discussed. This model can be developed and expanded

to introduce the infectious diseases in some Lemur animals, top and mid-level hunters. It can also be used to maintain the organisms' permanence in some antibiotics (vaccines) in the proposed model.

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