International Journal of Innovative Computing, Information and Control
Volume **18**, Number **1**, February **2022**

ICIC International ©2022 ISSN 1349-4198
pp. 121–132

sup-HESITANT FUZZY INTERIOR IDEALS OF SEMIGROUPS AND THEIR sup-HESITANT FUZZY TRANSLATIONS

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Received August 2021; revised November 2021

Abstract. *The main aims of this article are to introduce the concept of a* sup-*hesitant fuzzy interior ideal, which is a generalization of the concepts of a* sup-*hesitant fuzzy ideal, an interval-valued fuzzy interior ideal and a hesitant fuzzy interior ideal, of a semigroup and to study its* sup-*hesitant fuzzy translations. A* sup-*hesitant fuzzy interior ideal is characterized by sets, fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy sets. Further, we discuss the relation between an interior ideal and a generalization of the concepts of the characteristic hesitant fuzzy set and the characteristic interval-valued fuzzy set.*
Keywords: Interior ideal, sup-hesitant fuzzy interior ideal, sup-hesitant fuzzy translation

1. Introduction.

A hesitant fuzzy set, first proposed by Torra and Narukawa [1, 2], is a function from a reference set to a power set of the unit interval. After that several researches were conducted on the generalizations of the concept of hesitant fuzzy sets and application to many algebraic structures, for example, Talee et al. [3] introduced hesitant fuzzy interior ideals in semigroups and investigated their some properties. Jun et al. [4] studied hesitant fuzzy left (right, generalized bi-, bi-, two-sided) ideals of semigroups. Ali et al. [5] introduced the concept of hesitant fuzzy interior ideals in AG-groupoids and gave basic properties. As a generalization of the concepts of hesitant fuzzy bi-ideals and hesitant fuzzy right (resp., left) ideals of semigroups, Muhiuddin et al. [6] introduced the concepts of hesitant fuzzy (m, n)-ideals and hesitant fuzzy $(m, 0)$-ideals (resp., $(0, n)$-ideals) and their properties are studied. Abbasi et al. [7] characterized weakly regular po-semigroups by means of idempotent hesitant fuzzy ideals. Mosrijai et al. [8] introduced the concepts of some sup-hesitant fuzzy UP-ideals of UP-algebras and discussed their properties. Mosrijai and Iampan [9, 10] introduced the concepts of hesitant fuzzy soft sets over UP-algebras and studied some operations. Muhiuddin and Jun [11] introduced sup-hesitant fuzzy subalgebras in BCK/BCI-algebras and investigated their related properties. Muhiuddin et al. [12] introduced sup-hesitant fuzzy ideals in BCK/BCI-algebras

DOI: 10.24507/ijicic.18.01.121

and characterized sup-hesitant fuzzy ideals. Harizavi and Jun [13] introduced the notion of sup-hesitant fuzzy quasi-associative ideal in BCI-algebras. As a generalization of the concepts of an interval-valued fuzzy ideal and a hesitant fuzzy ideal of a ternary semigroup, Julatha and Iampan [14] introduced the concept of sup-hesitant fuzzy ideals and investigated some characterizations of sup-hesitant fuzzy ideals in terms of sets, fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy sets. Jittburus and Julatha [15] introduced a sup-hesitant fuzzy ideal, which is a generalization of the concepts of an interval-valued fuzzy ideal and a hesitant fuzzy ideal, of a semigroup and its sup-hesitant fuzzy translations and sup-hesitant fuzzy extensions. Some properties and characterizations of the sup-hesitant fuzzy ideal and its sup-hesitant fuzzy translations and sup-hesitant fuzzy extensions are discussed via sets, fuzzy sets, hesitant fuzzy sets and interval-valued fuzzy sets. For applications, see [16, 17].

In this paper, as a general concept of a hesitant fuzzy interior ideal, an interval-valued fuzzy interior ideal and a sup-hesitant fuzzy ideal of a semigroup, we introduce a sup-hesitant fuzzy interior ideal which is a generalization of the above concepts. Some characterizations of a sup-hesitant fuzzy interior ideal are given via sets, fuzzy sets, hesitant fuzzy sets and interval-valued fuzzy sets. Also, we discuss relationships between an interior ideal and a generalization of the concepts of the characteristic hesitant fuzzy set and the characteristic interval-valued fuzzy set. Finally, sup-hesitant fuzzy translations and sup-hesitant fuzzy extensions of sup-hesitant fuzzy interior ideals of semigroups are discussed, and their relations are investigated. The results from this article are more extensive than the results of sup-hesitant fuzzy ideals in [15].

2. **Preliminaries.** In this section we first give some basic definitions and results which will be used in this paper.

In what follows, let X be a nonempty set and S a semigroup unless otherwise specified. By an *ideal* of S we mean a nonempty subset I of S such that $SI \subseteq I$ and $IS \subseteq I$. A nonempty subset I of S is called an *interior ideal* of S if $SIS \subseteq I$. A semigroup S is called *regular* if every element $x \in S$, and there exists $y \in S$ such that $x = xyx$.

A *fuzzy set* f [18] in X (or a fuzzy subset of X) is an arbitrary function from X into $[0,1]$ where $[0,1]$ is the unit segment of the real line. A fuzzy subset f of S is called a *fuzzy ideal* of S if it satisfies $\max\{f(x), f(y)\} \leq f(xy)$ for all $x, y \in S$. A fuzzy set f of S is called a *fuzzy interior ideal* of S if it satisfies $f(a) \leq f(xay)$ for all $a, x, y \in S$. Then every fuzzy ideal of a semigroup is a fuzzy interior ideal, and concepts of fuzzy ideals and fuzzy interior ideals of regular semigroups coincide.

Let $\mathcal{D}[0,1]$ denote the family of all closed subintervals of $[0,1]$, that is

$$\mathcal{D}[0,1] = \{[a^-, a^+] | a^-, a^+ \in [0,1] \text{ and } a^- \leq a^+\}.$$

We consider two elements $\bar{a} = [a^-, a^+]$ and $\bar{b} = [b^-, b^+]$ in $\mathcal{D}[0,1]$. We denote $[1,1]$ by $\bar{1}$ and $[0,0]$ by $\bar{0}$. Define the operations $\preceq$, $=$, $\prec$ and rmax in case of two elements in $\mathcal{D}[0,1]$ as follows:

(1) $\bar{a} \preceq \bar{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
(2) $\bar{a} = \bar{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$,
(3) $\bar{a} \prec \bar{b}$ if and only if $\bar{a} \preceq \bar{b}$ and $\bar{a} \neq \bar{b}$,
(4) $\operatorname{rmax}\{\bar{a}, \bar{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}]$.

A mapping $\tilde{A} : X \to \mathcal{D}[0,1]$ is called an *interval-valued fuzzy set* [19] on X, where $\tilde{A}(x) = [A^-(x), A^+(x)]$ for all $x \in X$, A^- and A^+ are fuzzy subsets of X such that $A^-(x) \leq A^+(x)$ for all $x \in X$. An interval-valued fuzzy set $\tilde{A}$ on S is called an *interval-valued fuzzy ideal* [20, 21] of S if it satisfies $\operatorname{rmax}\left\{\tilde{A}(x), \tilde{A}(y)\right\} \preceq \tilde{A}(xy)$ for all $x, y \in S$.

An interval-valued fuzzy set $\tilde{A}$ on S is called an *interval-valued fuzzy interior ideal* [21] of S if it satisfies $\tilde{A}(a) \preceq \tilde{A}(xay)$ for all $a, x, y \in S$. Then an interval-valued fuzzy interior ideal of a semigroup is a generalization of the concept of an interval-valued fuzzy ideal, and concepts of an interval-valued fuzzy ideal and an interval-valued fuzzy ideal of a regular semigroup coincide.

Torra and Narukawa [1, 2] introduced a hesitant fuzzy set on X in terms of a function $\widehat{\vartheta}$ that when applied to X it returns a subset of $[0,1]$, that is, $\widehat{\vartheta} : X \to \mathcal{P}[0,1]$ where $\mathcal{P}[0,1]$ denotes the set of all subset of $[0,1]$. A hesitant fuzzy set $\widehat{\vartheta}$ on S is called a *hesitant fuzzy ideal* [3, 4] of S if it satisfies $\widehat{\vartheta}(x) \cup \widehat{\vartheta}(y) \subseteq \widehat{\vartheta}(xy)$ for all $x, y \in S$. A hesitant fuzzy set $\widehat{\vartheta}$ on S is called a *hesitant fuzzy interior ideal* [22] of S if it satisfies $\widehat{\vartheta}(a) \subseteq \widehat{\vartheta}(xay)$ for all $a, x, y \in S$. Talee et al. [3] showed that every hesitant fuzzy ideal of a semigroup is a hesitant fuzzy interior ideal, and concepts of a hesitant fuzzy ideal and a hesitant fuzzy interior ideal of a regular semigroup coincide.

Let $\widehat{\vartheta}$ be a hesitant fuzzy set on X, λ an element of $[0,1]$ and Φ an element of $\mathcal{P}[0,1]$, define the element $\mathsf{SUP}\,\Phi$ of $[0,1]$ and the subset $\left[\widehat{\vartheta};\Phi\right]$ of X [15] as follows:

$$\mathsf{SUP}\,\Phi = \begin{cases} \sup\Phi & \text{if } \Phi \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\left[\widehat{\vartheta};\Phi\right] = \left\{x \in X \mid \mathsf{SUP}\,\widehat{\vartheta}(x) \geq \mathsf{SUP}\,\Phi\right\}.$$

Then $\mathsf{SUP}\,\tilde{A}(x) = \sup\tilde{A}(x) = A^{+}(x)$ for every interval-valued fuzzy set $\tilde{A}$ on X and for all $x \in X$. The subset $U_{\sup}\left(\widehat{\vartheta};\lambda\right)$ of X, the fuzzy subset $\mathbb{F}_{\widehat{\vartheta}}$ of X and the hesitant fuzzy set $\mathbb{H}\left(\widehat{\vartheta};\Phi\right)$ on X [15] are defined by

$$U_{\sup}\left(\widehat{\vartheta};\lambda\right) = \left\{x \in X \mid \mathsf{SUP}\,\widehat{\vartheta}(x) \geq \lambda\right\},$$

$$\mathbb{F}_{\widehat{\vartheta}} : X \to [0,1],\ x \mapsto \mathsf{SUP}\,\widehat{\vartheta}(x),$$

and

$$\mathbb{H}\left(\widehat{\vartheta};\Phi\right) : X \to \mathcal{P}[0,1],\ x \mapsto \left\{\alpha \in \Phi \mid \mathsf{SUP}\,\widehat{\vartheta}(x) \geq \alpha\right\}.$$

We denote $\mathbb{H}\left(\widehat{\vartheta};\bigcup_{x\in X}\widehat{\vartheta}(x)\right)$ by $\mathbb{H}_{\widehat{\vartheta}}$ and denote $\mathbb{H}\left(\widehat{\vartheta};[0,1]\right)$ by $\mathbb{I}_{\widehat{\vartheta}}$. Then $\mathbb{I}_{\widehat{\vartheta}}$ is an interval-valued fuzzy set on X, and it is easily seen that

$$\widehat{\vartheta}(x) \subseteq \mathbb{H}_{\widehat{\vartheta}}(x) \subseteq \mathbb{I}_{\widehat{\vartheta}}(x)$$

and

$$\mathsf{SUP}\,\widehat{\vartheta}(x) = \mathsf{SUP}\,\mathbb{H}_{\widehat{\vartheta}}(x) = \sup\mathbb{I}_{\widehat{\vartheta}}(x)$$

for all $x \in X$.

In 2021, Jittburus and Julatha [15] studied sup-hesitant fuzzy ideals of semigroups and their sup-hesitant fuzzy translations and extensions as the following.

Definition 2.1. [15] *A hesitant fuzzy set $\widehat{\vartheta}$ on S is called a* sup-*hesitant fuzzy ideal of S related to Φ (briefly, Φ-*sup-*hesitant fuzzy ideal) of S if the set $\left[\widehat{\vartheta};\Phi\right]$ is an ideal of S. We say that $\widehat{\vartheta}$ is a* sup-*hesitant fuzzy ideal of S if $\widehat{\vartheta}$ is a Φ-*sup-*hesitant fuzzy ideal of S for all $\Phi \in \mathcal{P}[0,1]$ when $\left[\widehat{\vartheta};\Phi\right] \neq \emptyset$.*

Theorem 2.1. [15] *Let $\widehat{\vartheta}$ be a hesitant fuzzy set on S. Then the following statements are equivalent.*

(1) $\widehat{\vartheta}$ *is a* sup-*hesitant fuzzy ideal of* S.
(2) $\mathbb{H}_{\widehat{\vartheta}}$ *is a hesitant fuzzy ideal of* S.
(3) $\mathbb{H}_{\widehat{\vartheta}}$ *is a* sup-*hesitant fuzzy ideal of* S.
(4) $\mathbb{I}_{\widehat{\vartheta}}$ *is an interval-valued fuzzy ideal of* S.
(5) $\mathbb{I}_{\widehat{\vartheta}}$ *is a* sup-*hesitant fuzzy ideal of* S.
(6) $\mathbb{I}_{\widehat{\vartheta}}$ *is a hesitant fuzzy ideal of* S.
(7) $\mathbb{F}_{\widehat{\vartheta}}$ *is a fuzzy ideal of* S.
(8) $\mathsf{SUP}\,\widehat{\vartheta}(xy) \geq \max\left\{\mathsf{SUP}\,\widehat{\vartheta}(x), \mathsf{SUP}\,\widehat{\vartheta}(y)\right\}$ *for all* $x, y \in S$.
(9) *The set* $U_{\sup}\left(\widehat{\vartheta}; \lambda\right)$ *is either empty or an ideal of* S *for all* $t \in [0,1]$.
(10) $\mathbb{H}\left(\widehat{\vartheta}; \Phi\right)$ *is a hesitant fuzzy ideal of* S *for all* $\Phi \in \mathcal{P}[0,1]$.

For any subset A of X, the characteristic interval-valued fuzzy set CI_A and the characteristic hesitant fuzzy set CH_A of A on X are defined by

$$\mathrm{CI}_A\colon X \to \mathcal{D}[0,1],\ x \mapsto \begin{cases} \tilde{1} & \text{if } x \in A, \\ \tilde{0} & \text{otherwise,} \end{cases}$$

and

$$\mathrm{CH}_A\colon X \to \mathcal{P}[0,1],\ x \mapsto \begin{cases} [0,1] & \text{if } x \in A, \\ \emptyset & \text{otherwise.} \end{cases}$$

For any subset A of X and $\Phi, \Omega \in \mathcal{P}[0,1]$ with $\mathsf{SUP}\,\Phi < \mathsf{SUP}\,\Omega$, define a map $\chi_A^{(\Phi,\Omega)}$ [14, 15] as follows:

$$\chi_A^{(\Phi,\Omega)}\colon X \to \mathcal{P}[0,1],\ x \mapsto \begin{cases} \Omega & \text{if } x \in A, \\ \Phi & \text{otherwise.} \end{cases}$$

Then $\chi_A^{(\Phi,\Omega)}$ is a hesitant fuzzy set on X, which is called the sup(Φ,Ω)-*characteristic hesitant fuzzy set* of A on X. Note that $\chi_A^{(\emptyset,[0,1])} = \mathrm{CH}_A$ and $\chi_A^{(\tilde{0},\tilde{1})} = \mathrm{CI}_A$.

Theorem 2.2. [15] *For any nonempty subset A of S, the following statements are equivalent.*

(1) A *is an ideal of* S.
(2) CI_A *is an interval-valued fuzzy ideal of* S.
(3) CI_A *is a* sup-*hesitant fuzzy ideal of* S.
(4) CH_A *is a hesitant fuzzy ideal of* S.
(5) CH_A *is a* sup-*hesitant fuzzy ideal of* S.
(6) *If* $\Phi, \Omega \in \mathcal{P}[0,1]$ *and* $\mathsf{SUP}\,\Phi < \mathsf{SUP}\,\Omega$, *then* $\chi_A^{(\Phi,\Omega)}$ *is a* sup-*hesitant fuzzy ideal of* S.

Let $\widehat{\vartheta}$ be a hesitant fuzzy set on X, $\top := 1 - \sup\left\{\mathsf{SUP}\,\widehat{\vartheta}(x) \mid x \in X\right\}$ and $\lambda \in [0, \top]$. A hesitant fuzzy set $\widehat{\theta}$ on X is called a sup-*hesitant fuzzy λ-translation* [11, 15] of $\widehat{\vartheta}$ if $\mathsf{SUP}\,\widehat{\theta}(x) = \mathsf{SUP}\,\widehat{\vartheta}(x) + \lambda$ for all $x \in X$. A hesitant fuzzy set $\widehat{\theta}$ on X is a sup-*hesitant fuzzy extension* [11, 15] of $\widehat{\vartheta}$ if $\mathsf{SUP}\,\widehat{\vartheta}(x) \leq \mathsf{SUP}\,\widehat{\theta}(x)$ for all $x \in X$. Note that (1) every sup-hesitant fuzzy λ-translation of $\widehat{\vartheta}$ is a sup-hesitant fuzzy extension of $\widehat{\vartheta}$, and (2) if $\widehat{\vartheta}_1$ and $\widehat{\vartheta}_2$ are sup-hesitant fuzzy λ-translations of $\widehat{\vartheta}$ and $\widehat{\vartheta}_3$ is a sup-hesitant fuzzy extension of $\widehat{\vartheta}_1$, then $\widehat{\vartheta}_3$ is a sup-hesitant fuzzy extension of $\widehat{\vartheta}_2$.

Definition 2.2. [15] *Let $\widehat{\vartheta}$ and $\widehat{\theta}$ be hesitant fuzzy sets on S. Then $\widehat{\theta}$ is called a* sup-*hesitant fuzzy extension of $\widehat{\vartheta}$ based on an ideal of S (briefly,* sup-*HFI-extension) of $\widehat{\vartheta}$ if the following two assertions are valid.*

(1) $\widehat{\theta}$ *is a* sup-*hesitant fuzzy extension of* $\widehat{\vartheta}$.
(2) *If* $\widehat{\vartheta}$ *is a* sup-*hesitant fuzzy ideal of* S, *then* $\widehat{\theta}$ *is a* sup-*hesitant fuzzy ideal of* S.

Theorem 2.3. [15] *If* $\widehat{\vartheta}$ *is a* sup-*hesitant fuzzy ideal of* S, *then its* sup-*hesitant fuzzy* λ-*translation is a* sup-*HFI-extension of* $\widehat{\vartheta}$ *for all* $\lambda \in [0, \top]$.

3. **Main Results.** In this section, our main results are divided into two parts as follows: 1) sup-hesitant fuzzy interior ideals, and 2) sup-hesitant fuzzy translations.

3.1. sup-hesitant fuzzy interior ideals.

Definition 3.1. *A hesitant fuzzy set* $\widehat{\vartheta}$ *on* S *is called a* sup-*hesitant fuzzy interior ideal of* S *related to* Φ *(briefly,* Φ-sup-*hesitant fuzzy interior ideal of* S*) if the set* $\left[\widehat{\vartheta}; \Phi\right]$ *is an interior ideal of* S. *We say that* $\widehat{\vartheta}$ *is a* sup-*hesitant fuzzy interior ideal of* S *if* $\widehat{\vartheta}$ *is a* Φ-sup-*hesitant fuzzy interior ideal of* S *for all* $\Phi \in \mathcal{P}[0,1]$ *when* $\left[\widehat{\vartheta}; \Phi\right] \neq \emptyset$.

Proposition 3.1. *If* $\Phi, \Omega \in \mathcal{P}[0,1]$, $\widehat{\vartheta}$ *is a* Φ-sup-*hesitant fuzzy interior ideal of* S *and* $\mathsf{SUP}\,\Phi = \mathsf{SUP}\,\Omega$, *then* $\widehat{\vartheta}$ *is an* Ω-sup-*hesitant fuzzy interior ideal of* S.

Proof: It is obvious. □

Lemma 3.1. *Every* sup-*hesitant fuzzy ideal of* S *is a* sup-*hesitant fuzzy interior ideal of* S.

Proof: It is directly obtained from that every ideal of S is an interior ideal of S. □

Theorem 3.1. *Every interval-valued fuzzy ideal of* S *is a* sup-*hesitant fuzzy interior ideal of* S.

Proof: It follows from Lemma 3.1 and every interval-valued fuzzy ideal of S is a sup-hesitant fuzzy ideal of S [15]. □

Theorem 3.2. *Every hesitant fuzzy ideal of* S *is a* sup-*hesitant fuzzy interior ideal of* S.

Proof: It follows from Lemma 3.1 and every hesitant fuzzy ideal of S is a sup-hesitant fuzzy ideal of S [15]. □

By Lemma 3.1 and Theorems 3.1 and 3.2, we have that a sup-hesitant fuzzy interior ideal of S is a generalization of the concepts of a hesitant fuzzy ideal, an interval-valued fuzzy ideal and a sup-hesitant fuzzy ideal of S.

Theorem 3.3. *A hesitant fuzzy set* $\widehat{\vartheta}$ *on a regular semigroup* S *is a* sup-*hesitant fuzzy interior ideal of* S *if and only if* $\widehat{\vartheta}$ *is a* sup-*hesitant fuzzy ideal of* S.

Proof: It is directly obtained from that a nonempty subset I of a regular semigroup S is an interior ideal of S if and only if I is an ideal of S. □

Lemma 3.2. *Every interval-valued fuzzy interior ideal of* S *is a* sup-*hesitant fuzzy interior ideal of* S.

Proof: Assume that $\tilde{A}$ is an interval-valued fuzzy interior ideal of S and $\Phi \in \mathcal{P}[0,1]$ such that $\left[\tilde{A}; \Phi\right] \neq \emptyset$. Let $x, y \in S$ and $a \in \left[\tilde{A}; \Phi\right]$. Then $\sup \tilde{A}(a) \geq \mathsf{SUP}\,\Phi$. Thus

$$\mathsf{SUP}\,\Phi \leq \sup \tilde{A}(a) = A^+(a) \leq A^+(xay) = \sup \tilde{A}(xay)$$

and so $xay \in \left[\tilde{A}; \Phi\right]$. Hence, $\left[\tilde{A}; \Phi\right]$ is an interior ideal of S which implies that $\tilde{A}$ is a Φ-sup-hesitant fuzzy interior ideal of S. Therefore, $\tilde{A}$ is a sup-hesitant fuzzy interior ideal of S. □

Lemma 3.3. *Every hesitant fuzzy interior ideal of S is a* sup-*hesitant fuzzy interior ideal of S.*

Proof: Assume that $\widehat{\vartheta}$ is a hesitant fuzzy interior ideal of S. Let $\Phi \in \mathcal{P}[0,1]$, $x, y \in S$ and $a \in \left[\widehat{\vartheta}; \Phi\right]$. Then $\widehat{\vartheta}(a) \subseteq \widehat{\vartheta}(xay)$ and so

$$\mathsf{SUP}\,\Phi \leq \mathsf{SUP}\,\widehat{\vartheta}(a) \leq \mathsf{SUP}\,\widehat{\vartheta}(xay).$$

Thus, $xay \in \left[\widehat{\vartheta}; \Phi\right]$. Hence, $\left[\widehat{\vartheta}; \Phi\right]$ is an interior ideal of S which implies that $\widehat{\vartheta}$ is a Φ-sup-hesitant fuzzy interior ideal of S. Therefore, $\widehat{\vartheta}$ is a sup-hesitant fuzzy interior ideal of S. $\square$

The following example shows that the converses of Lemma 3.2 and Lemma 3.3 do not hold in general.

Example 3.1. *Let $S = \{a, b, c, d\}$ and define a binary operation $\cdot$ on S as follows:*

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Then S is a semigroup under the binary operation $\cdot$.

(1) *Define a hesitant fuzzy set $\widehat{\vartheta}$ on S by $\widehat{\vartheta}(0) = (0,1)$, $\widehat{\vartheta}(a) = [0,1]$, $\widehat{\vartheta}(b) = \bar{0}$ and $\widehat{\vartheta}(c) = \emptyset$. Then $\widehat{\vartheta}$ is a* sup-*hesitant fuzzy interior ideal of S but not a hesitant fuzzy interior ideal of S because $\widehat{\vartheta}(a) = [0,1] \not\subseteq (0,1) = \widehat{\vartheta}(0) = \widehat{\vartheta}(bac)$.*

(2) *Define an interval-valued fuzzy set $\tilde{A}$ on S by $\tilde{A}(0) = [0,1]$, $\tilde{A}(a) = \bar{1}$ and $\tilde{A}(b) = \tilde{A}(c) = \bar{0}$. Then $\tilde{A}$ is a* sup-*hesitant fuzzy interior ideal of S but not an interval-valued fuzzy interior ideal of S because $\tilde{A}\left(a^3\right) = \tilde{A}(0) = [0,1] \prec \bar{1} = \tilde{A}(a)$.*

By Lemmas 3.2 and 3.3 and Example 3.1, we have that a sup-hesitant fuzzy interior ideal of an arbitrary semigroup S is a generalization of the concepts of a hesitant fuzzy interior ideal and an interval-valued fuzzy interior ideal of S.

Lemma 3.4. *A hesitant fuzzy set $\widehat{\vartheta}$ on S is a* sup-*hesitant fuzzy interior ideal of S if and only if $\mathbb{H}\left(\widehat{\vartheta}; \Phi\right)$ is a hesitant fuzzy interior ideal of S for all $\Phi \in \mathcal{P}[0,1]$.*

Proof: Let $\Phi \in \mathcal{P}[0,1]$ and $a, x, y \in S$. If $\mathbb{H}\left(\widehat{\vartheta}; \Phi\right)(a) = \emptyset$, then it is clear that $\mathbb{H}\left(\widehat{\vartheta}; \Phi\right)(a) \subseteq \mathbb{H}\left(\widehat{\vartheta}; \Phi\right)(xay)$. On the other hand, let $t \in \mathbb{H}\left(\widehat{\vartheta}; \Phi\right)(a)$. Since $a \in \left[\widehat{\vartheta}; \widehat{\vartheta}(a)\right]$ and $\widehat{\vartheta}$ is a sup-hesitant fuzzy interior ideal of S, we have $xay \in \left[\widehat{\vartheta}; \widehat{\vartheta}(a)\right]$. Then

$$\mathsf{SUP}\,\widehat{\vartheta}(xay) \geq \mathsf{SUP}\,\widehat{\vartheta}(a) \geq t \in \Phi,$$

which implies that $t \in \mathbb{H}\left(\widehat{\vartheta}; \Phi\right)(xay)$. Hence, $\mathbb{H}\left(\widehat{\vartheta}; \Phi\right)(a) \subseteq \mathbb{H}\left(\widehat{\vartheta}; \Phi\right)(xay)$. Therefore, $\mathbb{H}\left(\widehat{\vartheta}; \Phi\right)$ is a hesitant fuzzy interior ideal of S.

Conversely, let $\Phi \in \mathcal{P}[0,1]$, $x, y \in S$ and $a \in \left[\widehat{\vartheta}; \Phi\right]$. Then $\mathbb{H}\left(\widehat{\vartheta}; \Phi\right)(a) = \Phi$ and $\mathbb{H}\left(\widehat{\vartheta}; \Phi\right)$ is a hesitant fuzzy interior ideal of S. Thus, $\Phi \subseteq \mathbb{H}\left(\widehat{\vartheta}; \Phi\right)(xay)$. Hence, $\mathsf{SUP}\,\widehat{\vartheta}(xay) \geq \mathsf{SUP}\,\Phi$, which implies that $xay \in \left[\widehat{\vartheta}; \Phi\right]$. Therefore, $\left[\widehat{\vartheta}; \Phi\right]$ is an interior ideal of S, that is $\widehat{\vartheta}$ is a Φ-sup-hesitant fuzzy interior ideal of S. Consequently, we obtain that $\widehat{\vartheta}$ is a sup-hesitant fuzzy interior ideal of S. □

The following theorem, characterizations of a sup-hesitant fuzzy interior ideal of a semigroup are investigated via interval-valued fuzzy sets and hesitant fuzzy sets.

Theorem 3.4. *For any hesitant fuzzy set $\widehat{\vartheta}$ on a semigroup S, the following statements are equivalent.*

(1) $\widehat{\vartheta}$ *is a* sup-*hesitant fuzzy interior ideal of S.*
(2) $\mathbb{H}_{\widehat{\vartheta}}$ *is a hesitant fuzzy interior ideal of S.*
(3) $\mathbb{H}_{\widehat{\vartheta}}$ *is a* sup-*hesitant fuzzy interior ideal of S.*
(4) $\mathbb{I}_{\widehat{\vartheta}}$ *is an interval-valued fuzzy interior ideal of S.*
(5) $\mathbb{I}_{\widehat{\vartheta}}$ *is a* sup-*hesitant fuzzy interior ideal of S.*
(6) $\mathbb{I}_{\widehat{\vartheta}}$ *is a hesitant fuzzy interior ideal of S.*

Proof: (1) ⇒ (2) and (1) ⇒ (6). It follows from Lemma 3.4.
(2) ⇒ (3) and (6) ⇒ (5). It follows from Lemma 3.3.
(4) ⇒ (5). It follows from Lemma 3.2.
(3) ⇒ (1). Let $\Phi \in \mathcal{P}[0,1]$, $x, y \in S$ and $a \in \left[\widehat{\vartheta};\Phi\right]$. Then

$$\mathsf{SUP}\,\mathbb{H}_{\widehat{\vartheta}}(a) = \mathsf{SUP}\,\widehat{\vartheta}(a) \geq \mathsf{SUP}\,\Phi$$

and so $a \in \left[\mathbb{H}_{\widehat{\vartheta}};\Phi\right]$. By assumption (3), we have $\left[\mathbb{H}_{\widehat{\vartheta}};\Phi\right]$ is an interior ideal of S and then $xay \in \left[\mathbb{H}_{\widehat{\vartheta}};\Phi\right]$. Thus

$$\mathsf{SUP}\,\widehat{\vartheta}(xay) = \mathsf{SUP}\,\mathbb{H}_{\widehat{\vartheta}}(xay) \geq \mathsf{SUP}\,\Phi,$$

which implies that $xay \in \left[\widehat{\vartheta};\Phi\right]$. Hence, $\left[\widehat{\vartheta};\Phi\right]$ is an interior ideal of S, that is $\widehat{\vartheta}$ is a Φ-sup-hesitant fuzzy interior ideal of S. Therefore, $\widehat{\vartheta}$ is a sup-hesitant fuzzy interior ideal of S.

(1) ⇒ (4). Let $a, x, y \in S$. Then $a \in \left[\widehat{\vartheta};\widehat{\vartheta}(a)\right]$ and by assumption (1), we have $xay \in \left[\widehat{\vartheta};\widehat{\vartheta}(a)\right]$. Thus, $\mathsf{SUP}\,\widehat{\vartheta}(a) \leq \mathsf{SUP}\,\widehat{\vartheta}(xay)$. Hence,

$$\mathbb{I}_{\widehat{\vartheta}}(a) = \left[0, \mathsf{SUP}\,\widehat{\vartheta}(a)\right] \preceq \left[0, \mathsf{SUP}\,\widehat{\vartheta}(xay)\right] = \mathbb{I}_{\widehat{\vartheta}}(xay).$$

Hence, $\mathbb{I}_{\widehat{\vartheta}}$ is an interval-valued fuzzy ideal of S.
(5) ⇒ (1). It is similar to prove that (3) implies (1), and we omit the proof. □

Theorem 3.5. *Let $\widehat{\vartheta}$ be a hesitant fuzzy set on S. Then the following statements are equivalent.*

(1) $\widehat{\vartheta}$ *is a* sup-*hesitant fuzzy interior ideal of S.*
(2) $\mathbb{F}_{\widehat{\vartheta}}$ *is a fuzzy interior ideal of S.*
(3) $\mathsf{SUP}\,\widehat{\vartheta}(xyz) \geq \mathsf{SUP}\,\widehat{\vartheta}(y)$ *for all $x, y, z \in S$.*

Proof: (1) ⇒ (3). Let $x, y, z \in S$. Then $\Phi = \widehat{\vartheta}(y)$ for some $\Phi \in \mathcal{P}[0,1]$. Thus, $y \in \left[\widehat{\vartheta};\Phi\right]$ and then $\widehat{\vartheta}$ is a Φ-sup-hesitant fuzzy interior ideal of S. Hence, $xyz \in \left[\widehat{\vartheta};\Phi\right]$ and then

$$\mathsf{SUP}\,\widehat{\vartheta}(xyz) \geq \mathsf{SUP}\,\Phi = \mathsf{SUP}\,\widehat{\vartheta}(y).$$

(3) ⇒ (2). It is clear.
(2) ⇒ (1). Let $\Phi \in \mathcal{P}[0,1]$, $x, y \in S$ and $a \in \left[\widehat{\vartheta};\Phi\right]$. Then

$$\mathsf{SUP}\,\widehat{\vartheta}(xay) = \mathbb{F}_{\widehat{\vartheta}}(xay) \geq \mathbb{F}_{\widehat{\vartheta}}(a) = \mathsf{SUP}\,\widehat{\vartheta}(a) \geq \mathsf{SUP}\,\Phi,$$

which implies that $xay \in \left[\widehat{\vartheta}; \Phi\right]$. Hence, $\left[\widehat{\vartheta}; \Phi\right]$ is an interior ideal of S, that is $\widehat{\vartheta}$ is a Φ-sup-hesitant fuzzy interior ideal of S. Therefore, $\widehat{\vartheta}$ is a sup-hesitant fuzzy interior ideal of S. □

For every hesitant fuzzy set $\widehat{\vartheta}$ on a nonempty set X, the hesitant fuzzy set $\widehat{\vartheta}^*$, defined by $\widehat{\vartheta}^*(x) = \left\{1 - \mathsf{SUP}\,\widehat{\vartheta}(x)\right\}$ for all $x \in X$, is said to be the *supremum complement* [8] of $\widehat{\vartheta}$ on X. Then $\mathsf{SUP}\,\widehat{\vartheta}^*(x) = 1 - \mathsf{SUP}\,\widehat{\vartheta}(x)$ for all $x \in X$. For every element λ of $[0,1]$, the set

$$L_{\sup}\left(\widehat{\vartheta}; \lambda\right) = \left\{x \in X \mid \mathsf{SUP}\,\widehat{\vartheta}(x) \leq \lambda\right\}$$

is called a sup-*lower* λ-*level subset* [8] of $\widehat{\vartheta}$.

Theorem 3.6. *Let $\widehat{\vartheta}$ be a hesitant fuzzy set on S. Then $\widehat{\vartheta}^*$ is a* sup-*hesitant fuzzy interior ideal of S if and only if $L_{\sup}\left(\widehat{\vartheta}; \lambda\right)$ is either empty or an interior ideal of S for all $\lambda \in [0,1]$.*

Proof: Let $\lambda \in [0,1]$ and $L_{\sup}\left(\widehat{\vartheta}; \lambda\right) \neq \emptyset$. Choose $\Phi \in \mathcal{P}[0,1]$ such that $\mathsf{SUP}\,\Phi = 1-\lambda$, and we have $\left[\widehat{\vartheta}^*; \Phi\right] = L_{\sup}\left(\widehat{\vartheta}; \lambda\right)$. Since $\widehat{\vartheta}^*$ is a sup-hesitant fuzzy interior ideal of S, we obtain that $L_{\sup}\left(\widehat{\vartheta}; \lambda\right) = \left[\widehat{\vartheta}^*; \Phi\right]$ is an interior ideal of S.

Conversely, let $\Psi \in \mathcal{P}[0,1]$ and $\left[\widehat{\vartheta}^*; \Psi\right] \neq \emptyset$. Choose $\lambda = 1 - \mathsf{SUP}\,\Psi$, and we have $L_{\sup}\left(\widehat{\vartheta}; \lambda\right) = \left[\widehat{\vartheta}^*; \Psi\right]$. By assumption, we obtain that $\left[\widehat{\vartheta}^*; \Psi\right] = L_{\sup}\left(\widehat{\vartheta}; \lambda\right)$ is an interior ideal of S. Hence, $\widehat{\vartheta}^*$ is a Ψ-SUP-hesitant fuzzy interior ideal of S. Therefore, $\widehat{\vartheta}^*$ is a sup-hesitant fuzzy interior ideal of S. □

Theorem 3.7. *A hesitant fuzzy set $\widehat{\vartheta}$ on S is a* sup-*hesitant fuzzy interior ideal of S if and only if $U_{\sup}\left(\widehat{\vartheta}; \lambda\right)$ of S is either empty or an interior ideal of S for all $\lambda \in [0,1]$.*

Proof: Let $\lambda \in [0,1]$ be such that $U_{\sup}\left(\widehat{\vartheta}; \lambda\right) \neq \emptyset$. Choose $\Theta \in \mathcal{P}[0,1]$ such that $\mathsf{SUP}\,\Theta = \lambda$, and we get $\left[\widehat{\vartheta}; \Theta\right] = U_{\sup}\left(\widehat{\vartheta}; \lambda\right)$. Since $\widehat{\vartheta}$ is a sup-hesitant fuzzy interior ideal of S, we see that $U_{\sup}\left(\widehat{\vartheta}; \lambda\right) = \left[\widehat{\vartheta}; \Theta\right]$ is an interior ideal of S.

Conversely, let Θ be an element of $\mathcal{P}[0,1]$ such that $\left[\widehat{\vartheta}; \Theta\right] \neq \emptyset$. Then $U_{\sup}\left(\widehat{\vartheta}; \mathsf{SUP}\,\Theta\right) = \left[\widehat{\vartheta}; \Theta\right]$ and by using assumption, we have that $\left[\widehat{\vartheta}; \Theta\right] = U_{\sup}\left(\widehat{\vartheta}; \mathsf{SUP}\,\Theta\right)$ is an interior ideal of S, that is, $\widehat{\vartheta}$ is a Θ-sup-hesitant fuzzy interior ideal of S. Therefore, $\widehat{\vartheta}$ is a sup-hesitant fuzzy interior ideal of S. □

Corollary 3.1. *For any hesitant fuzzy set $\widehat{\vartheta}$ on a regular semigroup S, the following statements are equivalent.*

(1) *$\widehat{\vartheta}$ is a* sup-*hesitant fuzzy interior ideal of S.*
(2) *$\mathbb{H}_{\widehat{\vartheta}}$ is a hesitant fuzzy ideal of S.*
(3) *$\mathbb{H}_{\widehat{\vartheta}}$ is a* sup-*hesitant fuzzy ideal of S.*
(4) *$\mathbb{I}_{\widehat{\vartheta}}$ is an interval-valued fuzzy ideal of S.*
(5) *$\mathbb{I}_{\widehat{\vartheta}}$ is a* sup-*hesitant fuzzy ideal of S.*
(6) *$\mathbb{I}_{\widehat{\vartheta}}$ is a hesitant fuzzy ideal of S.*
(7) *$\mathbb{F}_{\widehat{\vartheta}}$ is a fuzzy ideal of S.*

(8) $\mathsf{SUP}\,\widehat{\vartheta}(xy) \geq \max\left\{\mathsf{SUP}\,\widehat{\vartheta}(x), \mathsf{SUP}\,\widehat{\vartheta}(y)\right\}$ *for all* $x, y \in S$.

(9) $U_{\sup}\left(\widehat{\vartheta}; \lambda\right)$ *is either empty or an ideal of* S *for all* $\lambda \in [0, 1]$.

(10) $\mathbb{H}\left(\widehat{\vartheta}; \Phi\right)$ *is a hesitant fuzzy ideal of* S *for all* $\Phi \in \mathcal{P}[0, 1]$.

Proof: It follows from Theorems 2.1 and 3.3. □

Theorem 3.8. *Let* A *be a nonempty subset of* S *and* $\Phi, \Omega \in \mathcal{P}[0, 1]$ *be such that* $\mathsf{SUP}\,\Phi < \mathsf{SUP}\,\Omega$. *Then* A *is an interior ideal of* S *if and only if* $\chi_A^{(\Phi,\Omega)}$ *is a* sup-*hesitant fuzzy interior ideal of* S.

Proof: Suppose that $\mathsf{SUP}\,\chi_A^{(\Phi,\Omega)}(xay) < \mathsf{SUP}\,\chi_A^{(\Phi,\Omega)}(a)$ for some $a, x, y \in S$. Then $\chi_A^{(\Phi,\Omega)}(xay) = \Phi$ and $\chi_A^{(\Phi,\Omega)}(a) = \Omega$. Thus, $a \in A$ and $xay \notin A$. Since A is an interior ideal of S, we have $xay \in A$, a contradiction. Hence, $\mathsf{SUP}\,\chi_A^{(\Phi,\Omega)}(xay) \geq \mathsf{SUP}\,\chi_A^{(\Phi,\Omega)}(a)$ for all $a, x, y \in S$. Therefore, it follows from Theorem 3.5 that $\chi_A^{(\Phi,\Omega)}$ is a sup-hesitant fuzzy interior ideal of S.

Conversely, let $a \in A$ and $x, y \in S$. Since $\chi_A^{(\Phi,\Omega)}$ is a sup-hesitant fuzzy interior ideal of S and by using Theorem 3.5, we have

$$\mathsf{SUP}\,\chi_A^{(\Phi,\Omega)}(xay) \geq \mathsf{SUP}\,\chi_A^{(\Phi,\Omega)}(a) = \sup\Omega.$$

Thus, $\chi_A^{(\Phi,\Omega)}(xay) = \Omega$, which implies that $xay \in A$. Hence, A is an interior ideal of S. □

Corollary 3.2. *For any nonempty subset* A *of* S*, the following statements are equivalent.*

(1) A *is an interior ideal of* S.

(2) CI_A *is a* sup-*hesitant fuzzy interior ideal of* S.

(3) CH_A *is a* sup-*hesitant fuzzy interior ideal of* S.

Proof: It follows from Theorem 3.8. □

3.2. **sup-hesitant fuzzy translations.**

Theorem 3.9. *Let* $\widehat{\vartheta}$ *be a* sup-*hesitant fuzzy interior ideal of* S *and* $\lambda \in [0, \top]$. *Then every* sup-*hesitant fuzzy* λ-*translation of* $\widehat{\vartheta}$ *is a* sup-*hesitant fuzzy interior ideal of* S.

Proof: Let $\widehat{\theta}$ be a sup-hesitant fuzzy λ-translation of $\widehat{\vartheta}$. Then for all $a, x, y \in S$, we have

$$\mathsf{SUP}\,\widehat{\theta}(xay) = \mathsf{SUP}\,\widehat{\vartheta}(xay) + \lambda \geq \mathsf{SUP}\,\widehat{\vartheta}(a) + \lambda = \mathsf{SUP}\,\widehat{\theta}(a).$$

By Theorem 3.5, we obtain that $\widehat{\theta}$ is a sup-hesitant fuzzy interior ideal of S. □

Theorem 3.10. *If* $\lambda \in [0, \top]$ *and* $\widehat{\vartheta}$ *is a hesitant fuzzy set on* S *such that its* sup-*hesitant fuzzy* λ-*translation is a* sup-*hesitant fuzzy interior ideal of* S*, then* $\widehat{\vartheta}$ *is a* sup-*hesitant fuzzy interior ideal of* S.

Proof: Let $\lambda \in [0, \top]$ and $\widehat{\theta}$ be both a sup-hesitant fuzzy λ-translation of $\widehat{\vartheta}$ and a sup-hesitant fuzzy interior ideal of S. Then for all $a, x, y \in S$, we have

$$\mathsf{SUP}\,\widehat{\vartheta}(xay) = \mathsf{SUP}\,\widehat{\theta}(xay) - \lambda \geq \mathsf{SUP}\,\widehat{\theta}(a) - \lambda = \mathsf{SUP}\,\widehat{\vartheta}(a)$$

and by using Theorem 3.5, we obtain that $\widehat{\vartheta}$ is a sup-hesitant fuzzy interior ideal of S. □

Theorem 3.11. *Let* $\widehat{\vartheta}$ *be a hesitant fuzzy set on* S *and* $\lambda \in [0, \top]$. *Then every* sup-*hesitant fuzzy* λ-*translation of* $\widehat{\vartheta}$ *is a* sup-*hesitant fuzzy interior ideal of* S *if and only if* $U_{\sup}\left(\widehat{\vartheta}; \alpha - \lambda\right)$ *is either empty or an interior ideal of* S *for all* $\alpha \in [\lambda, 1]$.

Proof: It follows from Theorems 3.5 and 3.10.

Conversely, let $\widehat{\theta}$ be a sup-hesitant fuzzy λ-translation of $\widehat{\vartheta}$. Suppose that there exist $\alpha \in [\lambda, 1]$ and $a, b, x \in S$ such that

$$\mathsf{SUP}\,\widehat{\theta}(axb) < \alpha \leq \mathsf{SUP}\,\widehat{\theta}(x).$$

Then

$$\mathsf{SUP}\,\widehat{\vartheta}(x) = \mathsf{SUP}\,\widehat{\theta}(x) - \lambda \geq \alpha - \lambda.$$

Hence, $x \in U_{\sup}\left(\widehat{\vartheta}; \alpha - \lambda\right)$ and by using assumption, we have $axb \in U_{\sup}\left(\widehat{\vartheta}; \alpha - \lambda\right)$. It follows that $\mathsf{SUP}\,\widehat{\vartheta}(axb) \geq \alpha - \lambda$, i.e., $\mathsf{SUP}\,\widehat{\theta}(axb) \geq \alpha$, it is a contradiction. Thus, $\mathsf{SUP}\,\widehat{\theta}(axb) \geq \mathsf{SUP}\,\widehat{\theta}(x)$ for all $a, b, x \in S$. It follows from Theorem 3.5 that $\widehat{\theta}$ is a sup-hesitant fuzzy interior ideal of S. □

Definition 3.2. *Let $\widehat{\vartheta}$ and $\widehat{\theta}$ be hesitant fuzzy sets on S. Then $\widehat{\theta}$ is called a* sup-*hesitant fuzzy extension of $\widehat{\vartheta}$ based on an interior ideal of S (briefly,* sup-*HFII-extension of $\widehat{\vartheta}$) if the following assertions are valid.*

(1) *$\widehat{\theta}$ is a* sup-*hesitant fuzzy extension of $\widehat{\vartheta}$.*
(2) *If $\widehat{\vartheta}$ is a* sup-*hesitant fuzzy interior ideal of S, then so is $\widehat{\theta}$.*

Theorem 3.12. *If $\widehat{\vartheta}$ is a* sup-*hesitant fuzzy interior ideal of S, then its* sup-*hesitant fuzzy λ-translation is a* sup-*HFII-extension of $\widehat{\vartheta}$ for all $\lambda \in [0, \top]$.*

Proof: It follows from Theorem 3.9. □

Theorem 3.13. *Let $\widehat{\vartheta}$ be a hesitant fuzzy set on a regular semigroup S. The following statements are equivalent.*

(1) *$\widehat{\vartheta}$ is a* sup-*hesitant fuzzy interior ideal of S.*
(2) *Every* sup-*hesitant fuzzy λ-translation of $\widehat{\vartheta}$ is a* sup-*HFII-extension of $\widehat{\vartheta}$ for all $\lambda \in [0, \top]$.*
(3) *Every* sup-*hesitant fuzzy λ-translation of $\widehat{\vartheta}$ is a* sup-*HFI-extension of $\widehat{\vartheta}$ for all $\lambda \in [0, \top]$.*

Proof: It follows from Theorems 2.3, 3.3, and 3.12. □

Theorem 3.14. *Let $\widehat{\vartheta}$ be a* sup-*hesitant fuzzy interior ideal of S and $\lambda_1, \lambda_2 \in [0, \top]$. If $\lambda_1 \geq \lambda_2$, then a* sup-*hesitant fuzzy λ_1-translation of $\widehat{\vartheta}$ is a* sup-*HFII-extension of a* sup-*hesitant fuzzy λ_2-translation of $\widehat{\vartheta}$.*

Proof: It follows from Theorems 3.9, 3.10, and 3.12. □

4. **Conclusions.** In this paper, we have introduced the concept of a sup-hesitant fuzzy interior ideal of a semigroup, which is a generalization of a hesitant fuzzy interior ideal, an interval-valued fuzzy interior ideal and a sup-hesitant fuzzy ideal of a semigroup and examined its some characterizations in terms of sets, fuzzy sets, hesitant fuzzy sets and interval-valued fuzzy sets. Further, we have discussed the relation between an interior ideal and a generalization of the characteristic hesitant fuzzy set and the characteristic interval-valued fuzzy set. Finally, we have discussed sup-hesitant fuzzy translations and sup-hesitant fuzzy extensions of a sup-hesitant fuzzy interior ideal of a semigroup, and investigated their relations. As the notion of interior ideals is a generalization of ideals in semigroups, the results of this study are more extensive than the results of [15].

In the future, we will study a sup-hesitant fuzzy interior ideal in a Γ-semigroup and examine its some characterizations in terms of sets, fuzzy sets, hesitant fuzzy sets and interval-valued fuzzy sets.

Acknowledgment. The authors would like to thank the anonymous referee who provided useful and detailed comments on a previous/earlier version of the manuscript.

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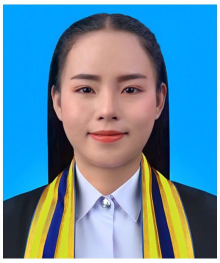

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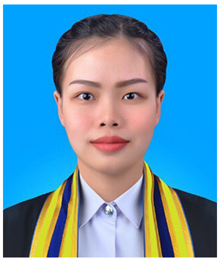

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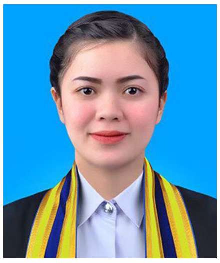

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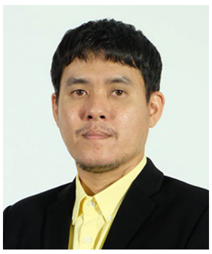

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